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One and two-step new hybrid methods for the numerical solution of first order initial value problems

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Abstract

In this paper, one and two-step high order hybrid methods are presented. By adding the off-step points y_{n+v}, $(0 < \nu < 1)$, in the right hand side of the classical hybrid methods, we will discuss about the zero-stability, consistency and convergence of introduced procedures. The numerical experimentation showed that our method is considerably more efficient compared to well known methods used for the numerical solution of first order initial value problems.

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1 Introduction

Consider the initial value problems for a single first order ordinary differential equation

$$y' = f(x, y), \quad y(a) = \eta. \tag{1.1}$$

Initial value problems occur frequently in applications. The numerical solution of these kind of problems is a central task in all simulation environments for mechanical, electrical, chemical systems. There are special purpose simulation programs for application in these fields, which often require from their users a deep understanding of the basic properties of the underlying numerical methods [13, 14, 15]. Kopal in 1955, believe [10] that extrapolation and substitution methods' can be regarded as two extreme ways for a construction of numerical solutions of ordinary differential equations leaving a vast no man's land in between, the exploration of which has barely as yet begun. In this context 'extrapolation methods' means method of linear multistep type and 'substitution methods' means method of Runge-Kutta type. From discussion in some papers and books on the relative merits of linear multistep and Runge-Kutta methods, it emerged that the former class of methods, though generally the more efficient in terms of accuracy and weak stability properties for a given number of functions evaluations per step, suffered the disadvantage of requiring additional starting values and special procedures for changing steplength. These difficulties would be reduced, without sacrifice, if we could lower the stepnumber of the linear multistep methods without reducing their order. The difficulty here lies in satisfying the essential condition of zero-stability. This 'zero-stability barrier' was circumvented by the introduction, in 1964-5, of modified linear multistep formula which incorporate a function evaluation at on off-step point. Such formula, simultaneously proposed by Gragg and Stetter [6], Butcher [1], and Gear [4] were christened 'hybrid' by the last author an apt name since, whilst retaining certain linear multistep characteristics, hybrid methods share with Runge-Kutta methods the property of utilizing data at points other than the step points. Thus, we may regard the introduction of hybrid formulae as an important step into the no man's land described by Kopal.

The k-step classical hybrid methods formula [7, 8, 11] are as follows

$$\sum_{j=0}^{k} \alpha_j y_{n+j} = h \sum_{j=0}^{k} \beta_j f_{n+j} + h\beta_v f_{n+v} \tag{1.2}$$

where $\alpha_k = +1$, α_0 and β_0 are not both zero, $v \notin \{0, 1, \ldots, k\}$, and also $f_{n+v} = f(x_{n+v}, y_{n+v})$. These methods are similar to linear multistep methods in predictor-corrector mode, but with one essential modification: an additional predictor is introduced at an off-step point. This means that the final (corrector) stage has an additional derivative approximation to work from. This greater generality allows the consequences of the Dahlquist barrier [3], to be avoided and it is actually possible to obtain convergent k-step methods with order $2k + 1$ up to $k = 7$. Even higher orders are available if two or more off-step points are used. The three independent discoveries of this approach were reported in [2, 3, 4, 5, 9, 12, 15, 16]. Although a flurry of activity by other authors followed, these methods have never been developed to the extent that they have been implemented in general purpose software. Recall that the formula (1.2) is zero-stable if no root of the polynomial $\rho(\xi) = \sum_{j=0}^{k} \alpha_j \xi^j$ has modulus greater than one and if every root with modulus one is simple. Thus Gragg and Stetter's results showed that [6], with certain exceptions, we can utilize both of new parameters v and β_v we have introduced, to raise the order of (1.2) to two above attained by linear multistep methods having the same right-hand side and the same value for k'. In this paper by utilizing parameter v in term y_{n+v}, in the right-hand side of (1.2), we prove that zero-stability property is hold.

2 k-step high order hybrid methods

For the numerical solution of the first order initial value problem (1.1), we introduce the new hybrid methods of the form

$$y_{n+1} = \sum_{j=1}^{k} a_j y_{n-j+1} + \sum_{j=1}^{v} b_j y_{n-\theta_j+1} + h \sum_{j=0}^{k} c_j f_{n-j+1} + h \sum_{j=1}^{v} d_j f_{n-\theta_j+1} \tag{2.1}$$

where a_j, b_j, c_j, d_j, $0 < \theta_j < k$ such that $\theta_j \notin \{0, 1, 2, \cdots, k\}$, $j = 1, 2, \ldots, v$ are $(2k+3v+1)$ arbitrary parameters. Formula (2.1) can only be used if we know the values of the solution $y(x)$ and $y'(x)$ at k successive points. These k values will be assumed to be given. Further, if $c_0 = 0$, this equation is refereed to as an explicit or predictor formula since y_{n+1} occurs only on one side of the equation. Also if $c_0 \neq 0$, the equation is referred to as an implicit or corrector formula since y_{n+1} occurs in both sides of the equation. In other words the unknown y_{n+1} cannot be calculated directly since it is contained within y'_{n+1}. Now with the difference equation (2.1), we can associate the difference operator L defined next.

Definition 1. Let the differential equation (1.1) have a unique solution $y(x)$ on $[a, b]$ and suppose that $y(x) \in C^{(p+1)}[a, b]$ for $p \geq 1$. Then the deference operator L for the method of (2.1) can be written as

$$\begin{aligned} L[y(x),h] &= y(x+h) - \sum_{j=1}^{k} a_j y(x+(1-j)h) - h\sum_{j=0}^{k} c_j y'(x+(1-j)h) \\ &- \sum_{j=1}^{v}\Big[b_j y(x+(1-\theta_j)h) + hd_j y'(x+(1-\theta_j)h)\Big] \end{aligned} \tag{2.2}$$

Definition 2. For the method (2.1), we define the functions $\rho(\xi)$ and $\sigma(\xi)$ as

$$\rho(\xi) = \xi^k - \sum_{j=1}^{k} a_j \xi^{k-j} - \sum_{j=1}^{v} b_j \xi^{k-\theta_j}, \quad \sigma(\xi) = \sum_{j=0}^{k} c_j \xi^{k-j} \tag{2.3}$$

and these functions so called the first and second characteristic functions, respectively.

We can assume that the functions $\rho(\xi)$ and $\sigma(\xi)$ have no common factors since, otherwise, (2.1) can be reduced to an equation of lower order. In order that the difference equation (2.1) should be useful for numerical integration, it is necessary that (2.1) be satisfied with high accuracy by the solution of the differential equation $y' = f(x,y)$, when h is small for an arbitrary function $f(x,y)$. This imposes restrictions on the coefficients a_j and b_j. We assume that the function $y(x)$ has continuous derivatives of sufficiently high order. We firstly use the Taylor series expansion to determine all the coefficients of (2.1), which can be written as

$$\begin{aligned} L[y(x),h] &= \sum_{i=0}^{\infty} \frac{h^i}{i!} y^{(i)}(x_n) - \sum_{j=1}^{k} a_j \Big[y(x_n) + \frac{(1-j)h}{1!} y^{(1)}(x_n) \\ &+ \frac{(1-j)^2h^2}{2!} y^{(2)}(x_n) + \cdots + \frac{(1-j)^q h^q}{q!} y^{(q)}(x_n) + \cdots\Big] \\ &- \sum_{j=1}^{v}\Big[b_j\Big(y(x_n) + \frac{(1-\theta_j)h}{1!} y^{(1)}(x_n) + \frac{(1-\theta_j)^2h^2}{2!} y^{(2)}(x_n) + \cdots \\ &+ \frac{(1-\theta_j)^q h^q}{q!} y^{(q)}(x_n) + \cdots\Big) - hd_j\Big(y'(x_n) + \frac{(1-\theta_j)h}{1!} y^{(2)}(x_n) \\ &+ \frac{(1-\theta_j)^2h^2}{2!} y^{(3)}(x_n) + \cdots + \frac{(1-\theta_j)^q h^q}{q!} y^{(q+1)}(x_n) + \cdots\Big)\Big] \\ &- \sum_{j=0}^{k} hc_j\Big[y'(x_n) + \frac{(1-j)h}{1!} y^{(2)}(x_n) + \frac{(1-j)^2h^2}{2!} y^{(3)}(x_n) + \cdots \\ &+ \frac{(1-j)^q h^q}{q!} y^{(q+1)}(x_n) + \cdots\Big]. \end{aligned} \tag{2.4}$$

Therefor, we have

$$\begin{aligned} L[y(x),h] &= \Big[1-\sum_{j=1}^{k}a_j-\sum_{j=1}^{v}b_j\Big]y(x_n) \\ &+ \Big[1-\sum_{j=1}^{k}(1-j)a_j-\sum_{j=1}^{v}\big((1-\theta_j)b_j+d_j\big)-\sum_{j=0}^{k}c_j\Big]hy'(x_n) \\ &+ \Big[\frac{1}{2!}-\sum_{j=1}^{k}\frac{(1-j)^2a_j}{2!}-\sum_{j=1}^{v}\Big[\frac{(1-\theta_j)^2b_j}{2!}+\frac{(1-\theta_j)d_j}{1!}\Big] \\ &- \sum_{j=0}^{k}\frac{(1-j)c_j}{1!}\Big]h^2y^{(2)}(x_n)+\cdots+\Big[\frac{1}{q!}-\sum_{j=1}^{k}\frac{(1-j)^qa_j}{q!} \\ &- \sum_{j=1}^{v}\Big[\frac{(1-\theta_j)^qb_j}{q!}+\frac{(1-\theta_j)^{(q-1)}d_j}{(q-1)!}\Big] \\ &- \sum_{j=0}^{k}\frac{(1-j)^{(q-1)}c_j}{(q-1)!}\Big]h^qy^{(q)}(x_n)+\cdots. \end{aligned}$$

Then we get

$$L[y(x),h]=C_0y(x_n)+C_1hy^{(1)}(x_n)+\cdots+C_qh^qy^{(q)}(x_n)+\cdots, \tag{2.5}$$

where

$$\begin{aligned} C_q &= \frac{1}{q!}-\sum_{j=1}^{k}\frac{(1-j)^q}{q!}a_j-\sum_{j=1}^{v}\Big[\frac{(1-\theta_j)^q}{q!}b_j+\frac{(1-\theta_j)^{(q-1)}}{(q-1)!}d_j\Big] \\ &- \sum_{j=0}^{k}\frac{(1-j)^{(q-1)}}{(q-1)!}c_j \end{aligned}$$

Definition 3. The linear multistep hybrid method (2.1) are said to be of order p if

$$C_0=C_1=C_2=\cdots=C_p=0, \qquad C_{p+1}\neq 0$$

thus for any function $y(x)\in C^{(p+2)}$ and for some nonzero constant C_{p+1}, we have

$$L[y(x),h]=-C_{p+1}h^{p+1}y^{(p+1)}(x_n)+O(h^{p+2}) \tag{2.6}$$

where $C_{p+1}/\sigma(1)$ is called the error constant.

In particular, $L[y(x),h]$ vanishes identically when $y(x)$ is polynomial whose degree is less than or equal to p.

Lemma 4. *The linear multistep hybrid method* (2.1) *is consistent if and only if*

$$\rho(1)=0, \qquad \rho'(1)=\sigma(1)+\sum_{j=1}^{k}d_j \tag{2.7}$$

Proof. We know that the general linear multistep methods are consistent if and only if they have the order of $p\geq 1$. This implies $C_0=C_1=0$. Therefore by a simple calculation, we get (2.7). □

Definition 5. The linear multistep hybrid method (2.1) is said to be consistent if it has the order of $p\geq 1$.

2.1 One-step new hybrid methods with one off-Step point

Upon choosing $k = v = 1$ in (2.1), we get

$$y_{n+1} = a_1 y_n + b_1 y_{n-\theta_1+1} + h(c_0 f_{n+1} + c_1 f_n) + h d_1 f_{n-\theta_1+1}, \tag{2.8}$$

where a_1, b_0, b_1, c_0, c_1, and $0 < \theta_1 < 1$ are 6 arbitrary parameters. In order to implement such a formula, a special predictor to estimate $y_{n-\theta_1+1}$ is necessary, we suppose that θ_1 is free parameter and by substituting $C_i = 0$, $i = 0, 1, 2, 3, 4$, we have

$$\begin{aligned}
C_0 &= 1 - a_1 - b_1 = 0 \\
C_1 &= 1 - \Big((1-\theta_1)b_1 + d_1\Big) - (c_0 + c_1) = 0 \\
C_2 &= \frac{1}{2!} - \frac{1}{2!}\Big[(1-\theta_1)^2 b_1 + 2(1-\theta_1)d_1\Big] - c_0 = 0 \\
C_3 &= \frac{1}{3!} - \frac{1}{3!}\Big[(1-\theta_1)^3 b_1 + 3(1-\theta_1)^2 d_1\Big] - \frac{1}{2!}c_0 = 0 \\
C_4 &= \frac{1}{4!} - \frac{1}{4!}\Big[(1-\theta_1)^4 b_1 + 4(1-\theta_1)^3 d_1\Big] - \frac{1}{3!}c_0 = 0
\end{aligned}$$

Now if we consider θ_1 is free parameter, we have

$$a_1 = \frac{\theta_1^3(\theta_1 - 2)}{(\theta_1 - 1)^3(\theta_1 + 1)}, \quad b_1 = \frac{2\theta_1 - 1}{(\theta_1 - 1)^3(\theta_1 + 1)}, \quad c_0 = \frac{\theta_1}{2(\theta_1 + 1)} \tag{2.9}$$

$$c_1 = \frac{\theta_1^3}{2(\theta_1 - 1)^2(\theta_1 + 1)}, \quad d_1 = \frac{\theta_1}{2(\theta_1 - 1)^2(\theta_1 + 1)}, \tag{2.10}$$

and its local truncation error is

$$\begin{aligned}
E &= \Big[\frac{1}{5!} - \frac{1}{5!}(1-\theta_1)^5 b_1 - \frac{1}{4!}(1-\theta_1)^4 d_1 - \frac{1}{4!}c_0\Big] h^5 y^{(5)}(\xi) \\
&= \frac{-\theta_1^3}{240(\theta_1 + 1)} h^5 y^{(5)}(\xi).
\end{aligned} \tag{2.11}$$

Theorem 6. *Any methods derived from* (2.8)*, under Lemma 4 conditions, are zero-stable.*

Proof. For this propose, we show that the function $\rho(\xi) = \xi - a_1 - b_1\xi^{1-\theta_1}$ has no roots other than $\xi_1 = 1$. Let $1 - \theta_1 = \nu$ then obviously $0 < \nu < 1$, and with conditions of Lemma 2.4, we can write first characteristic function $\rho(x)$ as $\rho(x) = x - a_1 - (1 - a_1)x^\nu$. Obviously $\xi_1 = 1$ is principal root of $\rho(x)$. If we suppose ρ has a root $\alpha > 1$ then ρ' must have a root β such that $1 < \beta < \alpha$. Therefore

$$\rho'(\beta) = 0 \quad \Longrightarrow 1 - \nu b_1 \beta^{\nu-1} = 0 \quad \Longrightarrow \nu b_1 \beta^{\nu-1} = 1 \quad \Longrightarrow \beta^{1-\nu} = \nu b_1$$

now since $\beta > 1$ then $\nu b_1 > 1$ hence $\nu > \frac{1}{b_1} > 1$ and this is a contradiction. Now suppose ρ has a root $0 < \alpha < 1$. then ρ' must have a root β such that $0 < \alpha < \beta < 1$. Therefore

$$\rho(\alpha) = 0 \quad \Longrightarrow \alpha - a_1 - b_1\alpha^\nu = 0 \quad \Longrightarrow b_1\alpha^\nu = \alpha - a_1 \tag{2.12}$$

But $\rho'(\beta) = 0$ then

$$\beta^{1-\nu} = \nu b_1 \tag{2.13}$$

and from (2.12) we can write

$$\nu b_1 \alpha^\nu = \nu(\alpha - a_1) \tag{2.14}$$

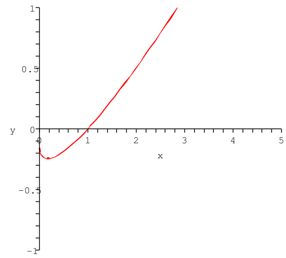

Figure 1. The first characteristic function $\rho(\xi)$, for one-step new hybrid method with $\theta_1 = \frac{1}{3}$

Therefore from (2.13) and (2.14) we have $\alpha^{\nu}\beta^{1-\nu} = \nu(\alpha - a_1)$. Now since $0 < \alpha, \beta < 1$ then $\nu(\alpha - a_1) < 1$ therefore $\alpha - a_1 > 1$, this means that $\alpha > 1 + a_1$ and this is a contradiction since a_1 is positive. Similarly we can show that ρ can not has negative root and this completes the proof. □

Theorem 7. *Any methods derived from* (2.8), *under Lemma 4 and Theorem 2 conditions are convergent.*

Proof. As is known, the necessary and sufficient conditions for linear multistep methods to be convergent are that they must be consistent and zero-stable. Then according to the Lemma 2.4 and Theorem 2.6, all methods generated from (2.8), are convergent. □

If we take $\theta_1 = \frac{1}{2}$, we have

$$a_1 = 1 \quad , \quad c_0 = \frac{1}{6}, \quad c_1 = \frac{1}{6}, \quad d_1 = \frac{2}{3}, \quad b_1 = 0, \tag{2.15}$$

and the method is

$$y_{n+1} = y_n + \frac{h}{6}(f_n + 4f_{n+\frac{1}{2}} + f_{n+1}), \tag{2.16}$$

which is Milne-Simpson rule, as known as, the implicit one-step classical hybrid method of order 4 and its local truncation error is

$$E = -\frac{1}{2880}h^5 y^{(5)}(\xi), \quad \xi \in (x_n, x_{n+1}).$$

By choosing $\theta_1 = \frac{1}{3}$, we have

$$a_1 = \frac{5}{32}, \quad c_0 = \frac{1}{8}, \quad c_1 = \frac{1}{32}, \quad d_1 = \frac{9}{32}, \quad b_1 = \frac{27}{32}, \tag{2.17}$$

hence the method is

$$y_{n+1} = \frac{5}{32}y_n + \frac{27}{32}y_{n+\frac{2}{3}} + \frac{h}{32}(f_n + 9f_{n+\frac{2}{3}} + 4f_{n+1}), \tag{2.18}$$

which is the implicit one-step new hybrid method of order 4. The first characteristic function $\rho(\xi)$, for this method is $\rho(\xi) = \xi - \frac{27}{32}\xi^{\frac{2}{3}} - \frac{5}{32}$, which has only one root $\xi_1 = 1$, so this method is zero-stable, and the figure of this function is shown in Figure 1. Moreover its local truncation error is $E = -\frac{1}{8640}h^5 y^{(5)}(\xi)$.

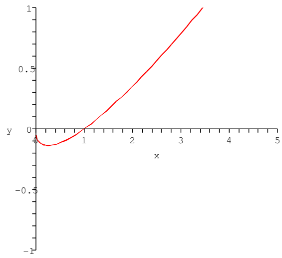

Figure 2. The first characteristic function $\rho(\xi)$, for one-step new hybrid method with $\theta_1 = \frac{1}{4}$

If we apply the Routh-Hurwitz criterion to investigate the weak stability of (2.15), the Routh-Hurwitz criterion will be clearly satisfied if and only if $\overline{h} \in (-3.41, 0)$ which required interval of absolute stability, this means that the interval of absolute stability of our new method is $(-3.41, 0)$. Similarly if we take $\theta_1 = \frac{1}{4}$ we have

$$a_1 = \frac{7}{135}, \quad b_0 = \frac{1}{10}, \quad b_1 = \frac{1}{90}, \quad c_1 = \frac{8}{45}, \quad c_2 = \frac{128}{135}, \tag{2.19}$$

then the method is

$$y_{n+1} = \frac{1}{135}(7y_n + 128y_{n+\frac{3}{4}}) + \frac{h}{90}(f_n + 16f_{n+\frac{3}{4}} + 9f_{n+1}), \tag{2.20}$$

that is implicit one-step new hybrid method of order 4. The first characteristic function $\rho(\xi)$, for this method is $\rho(\xi) = \xi - \frac{128}{135}\xi^{\frac{3}{4}} - \frac{7}{135}$, which has only one root $\xi_1 = 1$, so this method is zero-stable, and the figure of this function is shown in Figure 2. Moreover its local truncation error is $E = -\frac{1}{19200}h^5 y^{(5)}(\xi)$. Using Routh-Hurwitz criterion, its interval of absolute stability is $(-3.6, 0)$.

2.2 Two-step new hybrid methods with one off-step point

Upon choosing $k = 2$ and $v = 1$ in (3), we get

$$y_{n+1} = a_1 y_n + a_2 y_{n-1} + b_1 y_{n-\theta_1+1} + h(c_0 f_{n+1} + c_1 f_n + c_2 f_{n-1}) + h d_1 f_{n-\theta_1+1}, \tag{2.21}$$

where a_1, a_2, b_1, c_0, c_1, c_2, d_1 and $0 < \theta_1 < 1$ are 8 arbitrary parameters. In order to implement such a formula, a special predictor to estimate $y_{n-\theta_1+1}$ is necessary, we supose that θ_1 is free parameter and by substituting $C_i = 0$, $i = 0, 1, \ldots, 6$, we have

$$\begin{aligned}
C_0 &= 1 - a_1 - a_2 - b_1 = 0 \\
C_1 &= 1 + a_2 - \Big((1-\theta_1)b_1 + d_1\Big) - (c_0 + c_1 + c_2) = 0 \\
C_2 &= \frac{1}{2!} - \frac{1}{2!}\Big[a_2 + (1-\theta_1)^2 b_1 + 2(1-\theta_1)d_1\Big] - (c_0 - c_2) = 0 \\
C_3 &= \frac{1}{3!} - \frac{1}{3!}\Big[-a_2 + (1-\theta_1)^3 b_1 + 3(1-\theta_1)^2 d_1\Big] - \frac{1}{2!}(c_0 + c_2) = 0 \\
C_4 &= \frac{1}{4!} - \frac{1}{4!}\Big[a_2 + (1-\theta_1)^4 b_1 + 4(1-\theta_1)^3 d_1\Big] - \frac{1}{3!}(c_0 - c_2) = 0 \\
C_5 &= \frac{1}{5!} - \frac{1}{5!}\Big[-a_2 + (1-\theta_1)^5 b_1 + 5(1-\theta_1)^4 d_1\Big] - \frac{1}{4!}(c_0 + c_2) = 0 \\
C_6 &= \frac{1}{6!} - \frac{1}{6!}\Big[a_2 + (1-\theta_1)^6 b_1 + 6(1-\theta_1)^5 d_1\Big] - \frac{1}{5!}(c_0 - c_2) = 0,
\end{aligned}$$

now if we consider θ_1 is free parameter, we have

$$a_1 = -\frac{8\theta_1^3}{(3\theta_1+2)(\theta_1-1)^3}, \qquad a_2 = \frac{\theta_1^3(3\theta_1-8)}{(3\theta_1+2)(\theta_1-2)^3}, \tag{2.22}$$

$$b_1 = \frac{8(3\theta_1^2-6\theta_1+2)}{(3\theta_1+2)(\theta_1-1)^3(\theta_1-2)^3}, \tag{2.23}$$

$$c_0 = \frac{\theta_1}{3\theta_1+2}, \quad c_1 = \frac{4\theta_1^3}{(\theta_1-1)^2(3\theta_1+2)}, \quad c_2 = \frac{\theta_1^3}{(\theta_1-2)^2(3\theta_1+2)} \tag{2.24}$$

$$d_1 = \frac{4\theta_1}{(3\theta_1+2)(\theta_1-1)^2(\theta_1-2)^2}, \tag{2.25}$$

and its local truncation error is

$$\begin{aligned} E &= \frac{1}{7!}\left[1+a_2-\Big[(1-\theta_1)^7 b_1 + 7(1-\theta_1)^6 d_1 + 7(c_0+c_2)\Big]\right] h^7 y^{(7)}(\xi) \\ &= -\frac{\theta_1^3}{1260(3\theta_1+2)} h^7 y^{(7)}(\xi). \end{aligned} \tag{2.26}$$

Theorem 8. *Any methods derived from* (2.21)*, under Lemma 4 conditions, are zero-stable.*

Proof. Proving this theorem is similar to theorem 2.6. □

Theorem 9. *Any methods derived from* (2.21)*, under Lemma 4 and theorem 6 conditions are convergent.*

Proof. Proving this theorem is similar to theorem 7. □

If we take $\theta_1 = \frac{1}{2}$, we have

$$a_1 = \frac{16}{7}, \quad a_2 = \frac{13}{189}, \quad b_1 = -\frac{256}{189}, \quad d_1 = \frac{64}{63},$$
$$c_0 = \frac{1}{7}, \quad c_1 = \frac{4}{7}, \quad c_2 = \frac{1}{63}$$

and the method is

$$\begin{aligned} y_{n+1} &= \frac{1}{189}\Big(432 y_n + 13 y_{n-1} - 256 y_{n+\frac{1}{2}}\Big) \\ &+ \frac{h}{63}(9 f_{n+1} + 36 f_n + f_{n-1} + 64 f_{n+\frac{1}{2}}), \end{aligned} \tag{2.27}$$

which is the implicit two-step hybrid method of order 6. The first characteristic function $\rho(\xi)$, for this method is $\rho(\xi) = \xi^2 - \frac{432}{189}\xi + \frac{256}{189}\xi^{\frac{1}{2}} - \frac{13}{189}$, that has three roots, the principal root of which is $\xi_1 = 1$, and also $|\xi_i| < 1$, $i = 2, 3$. So this method is zero-stable, and figure of this function is shown in Figure 3. Moreover its local truncation error is $E = -\frac{1}{35280} h^7 y^{(7)}(\xi)$, $\xi \in (x_{n-1}, x_{n+1})$. In the numerical experiment for (2.27), one obtains two more unknowns, $y_{n+\frac{1}{2}}$ and $y'_{n+\frac{1}{2}}$, to be solved beside y_{n+1}. For this propose, Gear [4] has used the differentiation formula given by

$$y_{n+\frac{1}{2}} = y_{n-1} + \frac{h}{8}(9 f_n + 3 f_{n-1}),$$

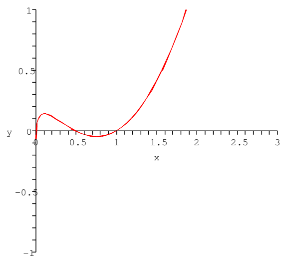

Figure 3. The first characteristic function $\rho(\xi)$, for two-step new hybrid method with $\theta_1 = \frac{1}{2}$

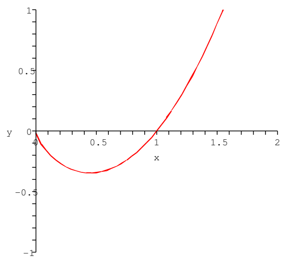

Figure 4. First characteristic function $\rho(\xi)$, for two-step new hybrid method with $\theta_1 = \frac{1}{3}$

and to calculate $y'_{n+\frac{1}{2}}$, the authors [15] have used the L-stable differentiation formula given by

$$y'_{n+\frac{1}{2}} = \frac{3}{4h}\Big(y_{n+1} - y_n\Big) - \frac{3}{4}\left(y'_n - \frac{4}{3}y'_{n+1}\right) - \frac{3h}{4}\left(\frac{1}{2}y''_n + \frac{2}{3}y''_{n+1}\right).$$

By selecting $\theta_1 = \frac{1}{3}$, we have

$$a_1 = \frac{1}{3}, \quad a_2 = \frac{7}{375}, \quad b_1 = \frac{81}{125}, \quad d_1 = \frac{9}{25}$$

$$c_0 = \frac{1}{9}, \quad c_1 = \frac{1}{9}, \quad c_2 = \frac{1}{225},$$

hence the method is

$$\begin{aligned} y_{n+1} &= \frac{1}{375}\Big(125y_n + 7y_{n-1} + 243y_{n+\frac{2}{3}}\Big) \\ &+ h(\frac{1}{9}f_{n+1} + \frac{1}{9}f_n + \frac{1}{225}f_{n-1} + \frac{9}{25}f_{n+\frac{2}{3}}), \end{aligned} \tag{2.28}$$

is the implicit two-step new hybrid method of order 6. The first characteristic function $\rho(\xi)$, for this method is $\rho(\xi) = \xi^2 - \frac{125}{375}\xi - \frac{243}{375}\xi^{\frac{2}{3}} - \frac{7}{375}$, that has only one root $\xi_1 = 1$. So this method is zero-stable, and the figure of this function is shown in Figure 4. Moreover its local truncation error is $E = -\frac{1}{102060}h^7 y^{(7)}(\xi)$. If we apply the Routh-Hurwitz criterion to investigation the weak stability of (2.27), the Routh-Hurwitz criterion is clearly satisfied if and only if $\overline{h} \in (-5.21, 0)$ which required interval of absolute stability, this means that the interval of absolute stability of our new method is $(-5.21, 0)$.

x_i	Runge-Kutta	New Method (17)
1.0	0	0
2.2	-0.001373	-0.153994×10^{-7}
3.4	-0.000321	-0.933694×10^{-9}
4.6	0.000121	-0.140638×10^{-9}
5.8	0.000058	$-0.334977 \times 10^{-10}$
7.0	0.000033	$-0.105402 \times 10^{-10}$
$\vdots$	$\vdots$	$\vdots$
25.0	0.000001	$-0.462995 \times 10^{-14}$

Table 1. Absolute errors for the example 10, with $h = 0.1$, are calculated for comparison among four methods: four stage Runge-Kutta method and our new method (2.27).

x_i	Runge-Kutta	New Method (17)
1.0	0	0
2.2	-0.001373	-0.402936×10^{-9}
3.4	-0.000321	$-0.253444 \times 10^{-10}$
4.6	0.000121	$-0.387989 \times 10^{-11}$
5.8	0.000058	$-0.932727 \times 10^{-12}$
7.0	0.000033	$-0.295256 \times 10^{-12}$
$\vdots$	$\vdots$	$\vdots$
25.0	0.000001	$-0.132385 \times 10^{-15}$

Table 2. Absolute errors for the example 10, with $h = 0.025$, are calculated for comparison among four methods: four stage Runge-Kutta method and our new method (2.27).

3 Numerical Example

In this section we present some numerical results to compare our new new hybrid methods with that of other multistep methods.

Example 10. Consider the initial value problem

$$\begin{cases} y' = -5xy^2 + \frac{5}{x} - \frac{1}{x^2}, \\ y(1) = 1. \end{cases}$$

The theoretical solution of this initial value problem is $y(x) = \frac{1}{x}$. The numerical results when $h = 0.1$ are given in table 1 and as the calculations with $h = 0.025$ displayed in table 2. We compared the results of our new hybrid methods and four stage Runge-Kutta method on this problem with $h = 0.1$ and $h = 0.025$.

Example 11. Consider the initial value problem

$$\begin{cases} y' = 4x\sqrt{y}, \\ y(1) = 1. \end{cases}$$

The theoretical solution of this initial value problem is $y(x) = (1 + x^2)^2$, and our new methods, for this problem are exact.

x	y_i	New Method (2.27)
1	y_1	-1.608986324
	y_2	1.011070494
3	y_1	-1.646634125
	y_2	.9609611194
5	y_1	-1.682511516
	y_2	.9180799893
10	y_1	-1.765934684
	y_1	.8330908005

Table 3. Results of example 12, with $\mu = 65$, which are convergent for the stiff problem Van der Pol's equation.

T	h	Y	Error of (2.27)	Error of Wu's Method in [17]
50	0.05	y_1	3.312e-16	1.97e-15
		y_2	8.625e-12	2.02e-11

Table 4. Comparison of the absolute errors in the approximations obtained using the new method (2.27) and the sixth-order method of Wu et al. [17] for Example 3.4.

Example 12. Consider the van der Pol's equation

$$\begin{cases} y_1' = y_2, \\ y_2' = \mu^2((1-y_1^2)y_2 - y_1), \end{cases}$$

with initial value $y(0) = (2,0)^T$. We choose $\mu = 65$. We present the numerical solution of this problem using the new hybrid method (2.27) at some selected points in Table 3.

Example 13. Consider the stiff initial value problem

$$\begin{cases} y_1' = -1002y_1 + 1000y_2^2, \\ y_2' = y_1 - y_2(1+y_2), \\ y_1(0) = 1, \quad y_2(0) = 1. \end{cases}$$

With the exact solution $y_1 = \exp(-2t)$ and $y_2 = \exp(-t)$. This equation has been solved numerically for $T = 50$ using exact starting values and the Wu's method. In the numerical experiment, we take the step lengths $h = 0.05$. In Table 4, we present the absolute errors at the end-point.

Example 14. Consider the stiff problem

$$\begin{cases} y_1' = -20y_1 - 0.25y_2 - 19.75y_3, \\ y_2' = 20y_1 - 20.25y_2 + 0.25y_3, \\ y_3' = 20y_1 - 19.75y_2 - 0.25y_3, \\ y_1(0) = 1, \quad y_2(0) = 0 \quad y_3(0) = -1. \end{cases}$$

T	h	Y	Error of (2.27)	Error of Wu's Method in [17]
50	0.005	y_1	5.26e-21	1.38e-20
		y_2	5.26e-21	1.38e-20
		y_3	5.26e-21	1.38e-20
100	0.1	y_1	6.35e-32	3.57e-31
		y_2	6.35e-32	3.57e-31
		y_3	6.35e-32	3.57e-31

Table 5. Comparison of the absolute errors in the approximations obtained using the new method (2.27) and the sixth-order method of Wu et al. [17] for Example 13.

With the exact solution

$$\begin{cases} y_1 = \frac{\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)+\sin(20t))\right]}{2}, \\ y_2 = \frac{\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)-\sin(20t))\right]}{2}, \\ y_3 = \frac{-\left[\exp(-0.5t)+\exp(-20t)\times(\cos(20t)-\sin(20t))\right]}{2}. \end{cases}$$

This equation has been solved numerically for $T = 50$ and $T = 100$ using exact starting values and the Wu's method. In the numerical experiment, we take the step lengths $h = 0.005$ and $h = 0.1$. In Table 5, we present the absolute errors at the end-point.

Example 15. Consider the stiff problem

$$\begin{cases} y_1' = -0.1y_1 - 49.9y_2, \\ y_2' = -50y_2, \\ y_3' = 70y_2 - 120y_3, \\ y_1(0) = 2, \quad y_2(0) = 1 \quad y_3(0) = 2. \end{cases}$$

With the exact solution

$$\begin{cases} y_1 = e^{-0.1t} + e^{-50t}, \\ y_2 = e^{-50t}, \\ y_3 = e^{-50t} + e^{-120t}. \end{cases}$$

This equation has been solved numerically for $T = 0.1$ and $T = 0.18$ using exact starting values and the Wu's method. In the numerical experiment, we take the step lengths $h = 0.001$ and $h = 0.01$. In Table 6, we present the absolute errors at the end-point.

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T	h	Y	Error of (2.27)	Error of Wu's Method in [17]
0.1	.001	y_1	2.36e-9	1.75e-7
		y_2	6.89e-10	3.59e-8
		y_3	7.21e-10	3.72e-8
0.18	.01	y_1	3.26e-8	1.64e-5
		y_2	7.26e-9	2.79e-7
		y_3	9.26e-9	2.79e-7

Table 6. Comparison of the absolute errors in the approximations obtained using the new method (2.27) and the sixth-order method of Wu et al. [17] for Example 3.6.

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