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A note and a short survey on supporting lines of compact convex sets in the plane

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Abstract
After surveying some known properties of compact convex sets in the plane, we give two rigorous proofs of the general feeling that supporting lines can be *slide-turned* slowly and continuously. Targeting a wide readership, our treatment is elementary on purpose.

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1 Motivation

Nowadays, there is a growing interest in the combinatorial properties of convex sets, usually, in compact convex sets. A large part of the papers belonging to this field go back to Erdős and Szekeres [15]; see, for example, Dobbins, Holmsen, and Hubard [12] and [13], Pach and Tóth [24] and [25], and their references. Recently, besides combinatorists and geometers, algebraists are also interested in compact convex sets; see, for example, Adaricheva [1], Adaricheva and Bolat [2], Adaricheva and Nation [4], Czédli [8], [9], and [10], Czédli and Kincses [11], and Richter and Rogers [26]. The interest of algebraists is explained by the fact that *antimatroids*, introduced by Korte and Lovász [17] and [18], and the dual concept of *abstract convex geometries*, introduced by Edelman and Jamison [14], have close connections to lattice theory. These connections are surveyed in Adaricheva and Czédli [3], Adaricheva and Nation [4], Czédli [7], and Monjardet [21]. Finally, there are other types of combinatorial investigations of convex sets; the most recent is, perhaps, Novick [23].

One of the most important concepts related to planar convex sets is that of *supporting lines.* Most of the papers mentioned above rely, explicitly or implicitly, on the properties of these lines. We guess that not only the experts of advanced analysis of convex sets and functions are interested in the above papers; at least, this is surely true in case of the first author of the present paper. However, it is quite difficult to explain to or understand by all the interested readers in a short, easy-to-follow, but rigorous way that why one of the most useful property of compact convex sets holds. This property, which seems to

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be absent in the literature, will be formulated in Theorem 1. This theorem is the "note" occurring in the title.

This motivates the aim of this short paper: even if Theorem 1 could be proved in a shorter way by using advanced tools of Analysis and even if it states what is expected by geometric intuition, we are going to give a rigorous proof for it. Actually, we give two different proofs. We believe that if other statements for planar compact convex sets like (2.6) deserve proofs that are easy to reference, then so does this theorem. Note that Czédli [10] exemplifies why the present paper is expected to be useful in further research: while the first version, `arXiv:1611.09331v1`, of [10] spends a dozen of pages on properties of supporting lines, its second version needs only few lines and a reference to the present paper. Also, we exemplify the use of Theorem 1 by an easy corollary, which is a well known but we have not found a rigorous proof for it.

2 A short survey

A *compact* subset of the plane $\mathbb{R}^2$ is a topologically closed bounded subset. The *boundary* of H will be denoted by ∂H. A subset H of $\mathbb{R}^2$ is *convex*, if for any two points $X, Y \in \mathbb{R}^2$, the closed line segment $[X, Y]$ is a subset of H. In this section, H will stand for a compact convex set. Even if this is not always repeated, we always assume that a convex set is nonempty. Each line ℓ gives rise to two *closed halfplanes*; their intersection is ℓ. Usually, unless otherwise is stated explicitly, we assume that ℓ is a *directed line*; then we can speak of the *left and right halfplanes* determined by ℓ. Points or sets in the left halfplane are *on the left* of ℓ; being on the right is defined analogously. If H is on the left of ℓ such that $H \cap \ell = \emptyset$, then H is *strictly on the left* of ℓ. The *direction* of a directed line ℓ will be denoted by $\operatorname{dir}(\ell) \in [-\pi, \pi)$. It is understood modulo 2π, whence we could also consider $\operatorname{dir}(\ell)$ an element of $[0, 2\pi)$. Furthermore, denoting the unit circle $\{\langle x, y\rangle : x^2 + y^2 = 1\}$ by C_{unit}, we will often say that $\operatorname{dir}(\ell) \in C_{\text{unit}}$. Following the convention of Yaglom and Boltyanskiĭ [31], if H is on the left of ℓ and $\ell \cap H \neq \emptyset$, then ℓ is a *supporting line* of H. Clearly, for a supporting line ℓ of H, $\ell \cap H = \ell \cap \partial H \neq \emptyset$. We know from Yaglom and Boltyanskiĭ [31, page 8] that parallel to each line ℓ, a compact convex set with nonempty interior has exactly two supporting lines. Hence, without any stipulation on the interior,

$$\text{for every } \alpha \in C_{\text{unit}}\text{, a compact convex set has exactly one supporting line of direction } \alpha. \tag{2.1}$$

Note at this point that, by definition, a *curve* is the range $\operatorname{Range}(g)$ of a continuous function g from an interval I of positive length to $\mathbb{R}^n$ for some $n \in \{2, 3, 4, \dots\}$. If $x_1 \neq x_2 \Rightarrow g(x_1) \neq g(x_2)$ except possibly for the endpoints of I, then $\operatorname{Range}(g)$ is a *simple curve*. A *Jordan curve* is a homeomorphic *planar* image of a circle of nonzero radius, that is, a Jordan curve is a *simple closed curve in the plane*. A curve is *rectifiable* if the lengths of its inscribed polygons form a bounded subset of $\mathbb{R}$. The following statement is known, say, from Latecki, Rosenfeld, and Silverman [19, Thm. 32] and Topogonov [30, page 15]; see also [32].

$$\text{For a compact convex } H \subseteq \mathbb{R}^2 \text{ with nonempty interior, } \partial H \text{ is a rectifiable Jordan curve.} \tag{2.2}$$

For $P \in \partial H$, there are two possibilities; see, for example, Yaglom and Boltyanskiĭ [31, page 12]. First, if there is exactly one supporting line through P,

$$\text{then } P \text{ is a } \textit{regular point} \text{ of } \partial H \text{ and the curve } \partial H \text{ is } \textit{smooth} \text{ at } P. \tag{2.3}$$

Second, if there are at least two distinct supporting lines ℓ_1 and ℓ_2 through P, then P is a *corner* of ∂H (or of H). In both cases, a supporting line ℓ containing P is called the *last semitangent* of H through P if for every small positive ε, there is an $\varepsilon' \in (0, \varepsilon)$ such that the line obtained from ℓ by rotating it around P *forward* (that is, counterclockwise) by ε' degree is not a supporting line. The *first semitangent* is defined similarly. The first and the last semitangents coincide iff $P \in \partial H$ is a regular point. For $P \in \partial H$,

ℓ_P^- and ℓ_P^+ will denote the first semitangent and the last semitangent through P, respectively. When they coincide, $\ell_P := \ell_P^- = \ell_P^+$ will stand for the *tangent line* through P. (2.4)

Let us emphasize that no matter if $P \in \partial H$ is a regular point or a vertex,

there exists a supporting line through P; in particular, both ℓ_P^- and ℓ_P^+ exist and they are uniquely determined. (2.5)

Besides Yaglom and Boltyanskiǐ [31], this folkloric fact is also included, say, in Boyd and Vanderberghe [6, page 51]. We note but will not use the fact that every line separating P and the interior of H is a supporting line through P. As an illustration for (2.5), some supporting lines of H are given in Figure 1. If ℓ_i is the supporting line denoted by i in the figure, then $\ell_1 = \ell_{P_1}$ is a tangent line, $\ell_{P_2}^- = \ell_2$ is the first semitangent through P_2, and $\ell_{P_2}^+ = \ell_4$ is the last semitangent through the same point. We know from, say, Borwein and Vanderwerff [5, 2.2.15 in page 42], Yaglom and Boltyanskiǐ [31, page 110], or even from [32], that the boundary ∂H of a compact convex set $H \subseteq \mathbb{R}^2$ can have $\aleph_0$ many corners. This possibility, which is not so easy to imagine, also justifies that we are going to give a rigorous proof for our theorem. Next, restricting ourselves to the compact case and to the plane, we recall the *strict separation theorem* as follows.

If $H_1, H_2 \subseteq \mathbb{R}^2$ are *disjoint* compact convex set, then there exists a directed line ℓ such that H_1 is strictly on the left and H_2 is strictly on the right of ℓ. (2.6)

This result follows, for example, from Subsection 2.5.1 in Boyd and Vandenberghe [6] plus the fact that the *distance* $\mathrm{dist}(H_1, H_2)$ of H_1 and H_2 is positive in this case.

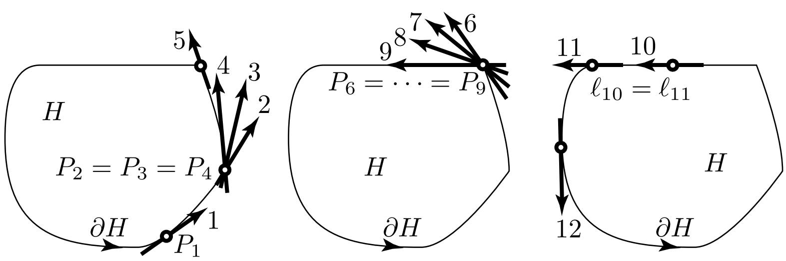

Figure 1. Supporting lines

3 A note and its corollary

Given a compact convex set H, visual intuition tells us that any supporting line can be continuously transformed to any other supporting line. We think of this transformation as a *slow*, *continuous* progression in time. For example, in Figure 1, ℓ_{i+1} comes, after some time, later than ℓ_i, for $i \in 1, \ldots, 11$. While continuity makes a well-known mathematical sense, a comment on slowness is appropriate here. By *slowness* we shall mean rectifiability, because this is what guarantees that running the process with a constant speed, it will terminate. Therefore, since rectifiability is an adjective of curves, we are going to associate a simple closed rectifiable curve with H such that the progression is described by moving along this curve forward. The only problem with this initial idea is that, say, ℓ_{11} cannot follow ℓ_{10}, because they are the same supporting lines. Therefore, we consider pointed supporting lines. A *pointed supporting line* of H is a pair $\langle P, \ell\rangle$ such that $P \in \partial H$ and ℓ is a supporting line of H through P. The transition from ℓ_i to ℓ_{i+1} will be called *slide-turning.* Of course, the $\langle P_i, \ell_i\rangle$, for $i \in \{1, \dots, 12\}$, represent only twelve snapshots of a continuous progression. In order to capture the progression mathematically, note that each pointed supporting line $\langle P, \ell\rangle$ of H is determined uniquely by the point $\langle P, \mathrm{dir}(\ell)\rangle \in \mathbb{R}^4$. To be more precise, define the following *cylinder*

$$\mathrm{Cyl} := \mathbb{R}^2 \times C_{\mathrm{unit}} = \{\langle x, y, z, t\rangle \in \mathbb{R}^4 : z^2 + t^2 = 1\} \subseteq \mathbb{R}^4. \tag{3.1}$$

As the crucial concept of this section, the *slide curve* of H is

$$\mathrm{Sli}(H) := \{\langle P, \mathrm{dir}(\ell)\rangle : \langle P, \ell\rangle \text{ is a pointed supporting line of } H\}; \tag{3.2}$$

it is a subset of Cyl. Although $\mathrm{Sli}(H)$ looks only a set at present, it will soon turn out that it is a curve. Actually, the main result of the paper says the following.

Theorem 1. *For every nonempty compact convex set $H \subseteq \mathbb{R}^2$,* $\mathrm{Sli}(H)$ *is a* rectifiable *simple closed curve.*

In order to exemplify the usefulness of this theorem, we state a corollary. Although it is well known, we have not found a rigorous proof for it.

Corollary 2. *If $H_1, H_2 \subseteq \mathbb{R}^2$ are disjoint compact convex sets with nonempty interiors, then they have exactly four non-directed supporting lines in common.*

The stipulation on the interior above can be relaxed but then we have to speak of *directed* supporting lines.

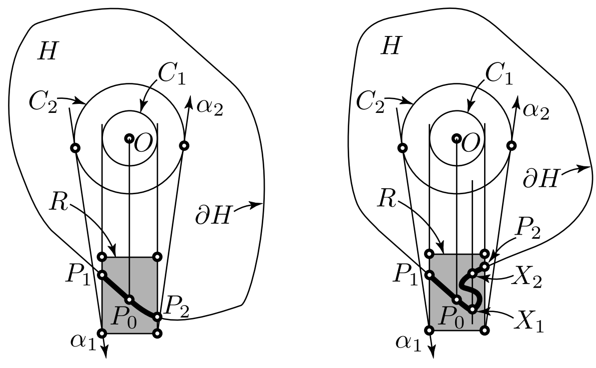

Figure 2. Reducing the problem to functions

4 Proofs

First proof of Theorem 1. We can assume that the interior of H is nonempty, because otherwise H is a line segment, possibly a singleton segment, and the statement trivially holds. In order to reduce the task to functions rather than convex sets, let P_0 be an arbitrary point of ∂H. Pick a point O in the interior of H, and choose a coordinate system such that both P_0 and O are on the y-axis and O is above P_0; see on the left of Figure 2. For a positive u, let C_1 and C_2 be the circles of radii u and $2u$ around O; we can assume that u is so small that C_2 is in the interior of H. Let A be the intersection of ∂H and the closed strip S between the two vertical tangent lines of C_1. In the figure, A is the thick arc of ∂H between P_1 and P_2. Let

$$\begin{aligned}&\mathrm{Sli}_A(H) := \{\langle P, \mathrm{dir}(\ell)\rangle : P \in A,\ \langle P, \mathrm{dir}(\ell)\rangle \in \mathrm{Sli}(H)\},\\ &\text{and similarly for future other arcs of } \partial H.\end{aligned} \tag{4.1}$$

Since the distance of O and the complement set of H is positive, we can assume that u is so small that the grey-filled rectangle containing A in the figure is strictly below C_2. (We have some freedom to choose the upper and lower edges of this rectangle.) Let $\alpha_1, \alpha_2 \in C_{\mathrm{unit}}$ be the directions of the external common supporting lines of C_2 and this rectangle, see the figure. Note that if we consider C_{unit} the interval $[-\pi, \pi)$, then $\alpha_1 = -\alpha_2$. The presence of C_2 within H guarantees the second half of the following observation:

$$\begin{aligned}&0 < \alpha_2 < \pi \text{ and for every supporting line } \ell \text{ of } H\\ &\text{that contains a point of } A,\ -\alpha_2 \le \mathrm{dir}(\ell) \le \alpha_2.\end{aligned} \tag{4.2}$$

We claim that

$$A \text{ is the graph of a convex function } f\colon [-u, u] \to \mathbb{R}. \tag{4.3}$$

By the convexity of H and (2.2), every vertical line in the strip S intersects A. Suppose, for a contradiction, that U is not the graph of a function. Then a vertical line in S intersects A in at least two distinct points, X_1 and X_2. Let, say, X_2 be above X_1; see on the right of the figure. Then X_2 is in the interior of the convex hull of $\{X_1\} \cup C_2$, whereby it is in the interior rather than on the boundary of H. This contradiction shows that f is a function. It is convex, since so is H. This proves (4.3). Clearly, the same consideration shows that

$$\text{each ray starting from } O \text{ intersects } \partial H \text{ exactly once.} \tag{4.4}$$

For a real-valued function $f\colon \mathbb{R} \to \mathbb{R}$ and x_0 in the interior of its domain, the left derivative $\lim_{x\to x_0-}(f(x)-f(x_0))/(x-x_0)$ and the right derivative of f at x_0 are denoted by $f'_-(x_0)$ and $f'_+(x_0)$, respectively. By a theorem of Stolz [29], see also Niculescu and Persson [22, Theorem 1.3.3], if f is convex in the open interval $(-u, u)$, then

$$\begin{aligned}&\text{for all } x, x_1, x_2 \in (-u, u), \text{ both } f'_-(x) \text{ and } f'_+(x) \text{ exist},\\ &f'_-(x) \le f'_+(x), \text{ and } x_1 < x_2 \text{ implies that } f'_+(x_1) \le f'_-(x_2).\end{aligned} \tag{4.5}$$

Recall that a function g from a subset of $\mathbb{R}^k$ to $\mathbb{R}^n$ is *Lipschitz* (or f is a *Lipschitz function* or f is *Lipschitzian*) if there exists a positive constant L such that $\mathrm{dist}(g(x), g(x')) \le L \cdot \mathrm{dist}(x, x')$ holds for all x and x' in the domain of g. Since f is convex, we know from Rockafellar [27, Theorems 10.1, 10.4, and 24.1] that

$$\begin{aligned}&\text{in } (-u, u),\ f \text{ is Lipschitz},\ f'_- \text{ is continuous from}\\ &\text{the left, and } f'_+ \text{ is continuous from the right.}\end{aligned} \tag{4.6}$$

Note that if a function is Lipschitz in an interval, then it is uniformly continuous there. From now on, we consider f only in the open interval $(-u, u)$ and we fix a positive $v \in (0, u)$, For $x_0 \in (-u, u)$, the *subdifferential* is defined as the interval

$$\begin{aligned}f^{(\mathrm{sub})}(x_0) &= \{d \in \mathbb{R} : \forall x \in (-u, u),\ f(x) \ge f(x_0) + d(x - x_0)\}\\ &= [f'_-(x_0), f'_+(x_0)];\end{aligned} \tag{4.7}$$

see Niculescu and Persson [22, Section 1.5]. As a consequence of (4.5), the subdifferential is a *dissipative* set-valued function, that is,

$$\begin{aligned}&\text{for } x_1, x_2 \in (-u, u), \text{ if } x_1 < x_2,\ d_1 \in f^{(\mathrm{sub})}(x_1),\\ &\text{and } d_2 \in f^{(\mathrm{sub})}(x_2), \text{ then } d_1 \le d_2.\end{aligned} \tag{4.8}$$

Consider the set

$$D := \{\langle x, d\rangle : x \in [-v, v] \text{ and } d \in f^{(\mathrm{sub})}(x)\} \subseteq \mathbb{R}^2 \tag{4.9}$$

with the (strict) lexicographic ordering

$$\langle x_1, d_1\rangle <^{\mathrm{lex}} \langle x_2, d_2\rangle \overset{\mathrm{def}}{\iff} (x_1 < x_2, \text{ or } x_1 = x_2 \text{ and } d_1 < d_2). \tag{4.10}$$

We define a function

$$t\colon D \to \mathbb{R} \ \text{ by } \ t(x, d) = x + d. \tag{4.11}$$

Note that $t(x, d)$ is a short form of $t(\langle x, d\rangle)$. Recall that the *Manhattan distance* of $\langle x_1, d_1\rangle$ and $\langle x_2, d_2\rangle$ in $\mathbb{R}^2$ is defined as $\mathrm{d_M}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle) := |x_1 - x_2| + |d_1 - d_2|$. It has the usual properties of a distance function. It follows from (4.5) that, for $\langle x_1, d_1\rangle$ and $\langle x_2, d_2\rangle$ in D (rather than in $\mathbb{R}^2$),

$$\text{if } \langle x_1, d_1\rangle \le^{\mathrm{lex}} \langle x_2, d_2\rangle, \text{ then } \mathrm{d_M}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle) = t(x_2, d_2) - t(x_1, d_1); \tag{4.12}$$

that is, for points of D, the Manhattan distance is derived from the function t. Let $\mathrm{dist}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle)$ stand for the Euclidean distance $((x_1 - x_2)^2 + (d_1 - d_2)^2))^{1/2}$; in $\mathbb{R}^4$, it is understood analogously. For the sake of a later reference, we note in advance that for $x^{(i)}, d^{(i)} \in \mathbb{R}^2$, the *Manhattan distance* in $\mathbb{R}^4$ is understood as

$$\mathrm{d_M}(\langle x^{(1)}, d^{(1)}\rangle, \langle x^{(2)}, d^{(2)}\rangle) := \mathrm{dist}(x^{(1)}, x^{(2)}) + \mathrm{dist}(d^{(1)}, d^{(2)}). \tag{4.13}$$

It is well known and easy to see that, for all $\langle x_1, d_1\rangle, \langle x_1, d_1\rangle$ in $\mathbb{R}^2$, and even in $\mathbb{R}^4$ if $x_1, x_2, d_1, d_2 \in \mathbb{R}^2$,

$$\mathrm{dist}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle) \le \mathrm{d_M}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle) \le 2 \cdot \mathrm{dist}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle). \tag{4.14}$$

It follows from (4.12) and the second half of (4.14) that t is a Lipschitz function (with Lipschitz constant 2). Since $\mathrm{d_M}(-,-)$ is a distance function, (4.12) yields that t is injective. Actually, it is bijective as a $D \to \mathrm{Range}(t)$ function. Thus, it has an inverse function, $t^{-1}\colon \mathrm{Range}(t) \to D$, which is also bijective. In order to see that the function t^{-1} is also a Lipschitz function, let $y_i = t(x_i, d_i) = x_i + d_i \in \mathrm{Range}(t)$, for $i \in \{1, 2\}$. Since $\mathrm{dist}(-,-)$ is a symmetric function, we can assume that $\langle x_1, d_1\rangle \le^{\mathrm{lex}} \langle x_2, d_2\rangle$. We can also assume that $d_1 \le d_2$; either because $x_1 = x_2$ and then we can interchange the subscripts 1 and 2, or because $x_1 < x_2$ and (4.8) applies. With these assumptions, let us compute:

$$\begin{aligned}\mathrm{dist}(y_1, y_2) &= |y_2 - y_1| = |x_2 + d_2 - (x_1 + d_1)| = |x_2 - x_1 + d_2 - d_1|\\ &= x_2 - x_1 + d_2 - d_1 = |x_1 - x_2| + |d_1 - d_2| = \mathrm{d_M}(\langle x_1, d_1\rangle, \langle x_2, d_2\rangle).\end{aligned}$$

Hence, using the second part of (4.14), it follows that the function t^{-1} is Lipschitz (with Lipschitz constant 2). So, we can summarize that

$$\begin{aligned}&t\colon D \to \mathrm{Range}(t) \text{ and } t^{-1}\colon \mathrm{Range}(t) \to D \text{ are reciprocal bijections}\\ &\text{and both of them are Lipschitz; in short, } t \text{ is } \textit{bi-Lipschitzian.}\end{aligned} \tag{4.15}$$

Next, let $w_1 = t(-v, f'_-(-v))$ and $w_2 = t(v, f'_+(v))$. We claim that

$$\mathrm{Range}(t) = [w_1, w_2]. \tag{4.16}$$

In order to see the easier inclusion, assume that $\langle x, d\rangle \in D$. Using (4.8) and (4.10), we obtain that $\langle -v, f'_-(-v)\rangle \le^{\mathrm{lex}} \langle x, d\rangle \le^{\mathrm{lex}} \langle v, f'_+(v)\rangle$. Thus, since (4.12) yields that t is order-preserving, we conclude that $w_1 \le t(x, d) \le w_2$, that is, $\mathrm{Range}(t) \subseteq [w_1, w_2]$. In order to show the converse inclusion, assume that $s \in [w_1, w_2]$. We need to find an $\langle x_0, d_0\rangle \in D$ such that $s = t(x_0, d_0)$, that is, $s = x_0 + d_0$. Define

$$\begin{aligned} x^- &:= \sup\{x : \text{there is a } d \text{ such that } \langle x, d\rangle \in D \text{ and } x + d \le s\},\\ x^+ &:= \inf\{x : \text{there is a } d \text{ such that } \langle x, d\rangle \in D \text{ and } x + d \ge s\}. \end{aligned} \tag{4.17}$$

Since $t(-v, f'_-(-v)) = w_1 \le s \le w_2 = t(v, f'_+(v))$, the sets occurring in (4.17) are nonempty. Hence, both x^- and x^+ exist and we have that $x^-, x^+ \in [-v, v]$. Suppose, for a contradiction, that $x^+ < x^-$. Then $x^- = 3\varepsilon + x^+$ for a positive ε. By (4.17), which defines x^- and x^+, we can pick $\langle x^\dagger, d^\dagger\rangle, \langle x^\ddagger, d^\ddagger\rangle \in D$ such that $x^\dagger \in (-\varepsilon + x^-, x^-]$, $t(x^\dagger, d^\dagger) = x^\dagger + d^\dagger \le s$, $x^\ddagger \in [x^+, \varepsilon + x^+)$, and $t(x^\ddagger, d^\ddagger) = x^\ddagger + d^\ddagger \ge s$. In particular, $x^\dagger + d^\dagger \le x^\ddagger + d^\ddagger$. However, since $x^\ddagger < x^\dagger$, the dissipative property from (4.8) gives that $d^\ddagger \le d^\dagger$, whereby $x^\dagger + d^\dagger \ge x^\dagger + d^\ddagger > x^\ddagger + d^\ddagger$, contradicting $x^\dagger + d^\dagger \le x^\ddagger + d^\ddagger$. This proves that $x^- \le x^+$. Next, suppose for a contradiction that $x^- < x^+$. Let $x^* := (x^- + x^+)/2$, and pick a $d^* \in f^{(\mathrm{sub})}(x^*)$. Since $x^* + d^* \le s$ would contradict the definition of x^-, we have that $x^* + d^* > s$, which contradicts the definition of x^+. This excludes the case $x^- < x^+$. So we have that $x^- = x^+$, and we let $x_0 := x^- = x^+$. Clearly, for all x and the corresponding d in the upper line of (4.17), $x + f'_-(x) \le x + d \le s$. Hence, the left continuity formulated in (4.6) gives that $t(x_0, f'_-(x_0)) = x_0 + f'_-(x_0) = x^- + f'_-(x^-) \le s$. Similarly, $t(x_0, f'_+(x_0)) = x_0 + f'_+(x_0) = x^+ + f'_+(x^+) \ge s$. So $x_0 + f'_-(x_0) \le s \le x_0 + f'_+(x_0)$, whereby (4.7) gives a $d_0 \in [f'_-(x_0), f'_+(x_0)]$ such that $s = x_0 + d_0 = t(x_0, d_0)$. This proves (4.16).

It is well known (and evident) that, with self-explanatory domains,

> the composition of two bi-Lipschitzian functions is bi-Lipschitzian. Thus, a bi-Lipschitzian function maps a rectifiable simple curve to a rectifiable simple curve. (4.18)

Before utilizing (4.18), we need some preparations. Let $Q_1 = \langle -v, f(-v)\rangle$ and $Q_2 = \langle v, f(v)\rangle$; they are points on the arc A before and after P_0, respectively. Let B be the sub-arc of A (and of ∂H) from Q_1 to Q_2, and note that P_0 is in the interior of B. Let $f^*\colon [-v, v] \to B$ be the function defined by $f^*(x) := \langle x, f(x)\rangle$. Using (4.6) and the relation between the Euclidean and the Manhattan distance functions, see (4.14), it follows that f^* is Lipschitz. This fact implies trivially that f^* is bi-Lipschitzian. So is the arctangent function on $[-v, v]$. Therefore, it follows in a straightforward way from (4.14) that the Cartesian (or categorical) product function

> $\langle f^*, \arctan\rangle\colon D \to \mathrm{Sli}_B(H)$, defined by $\langle x, d\rangle \mapsto \langle f^*(x), \arctan(d)\rangle$, where $\mathrm{Sli}_B(H)$ is defined in (4.1), is bi-Lipschitzian. (4.19)

The line segment $[w_1, w_2]$ is clearly a simple rectifiable curve. So is D by (4.11), (4.15), (4.16), and (4.18). Hence, (4.18) and (4.19) yield that $\mathrm{Sli}_B(H)$ is a simple rectifiable curve. Finally, since $P_0 \in \partial H$ was arbitrary and since the endpoints of B can be omitted from B, we obtain that ∂H can be covered by a set $\{B_i : i \in I\}$ of open arcs such that the $\mathrm{Sli}_{B_i}(H) \subseteq \mathrm{Cyl}$ are simple rectifiable curves. Clearly, the $\mathrm{Sli}_{B_i}(H)$ cover $\mathrm{Sli}(H)$. Since

∂H is compact, we can assume that I is finite. Therefore, $\mathrm{Sli}(H)$ is covered by finitely many open simple rectifiable curves. Furthermore, (4.4) yields that each of these open curves overlaps with its neighbors. Thus, we conclude the validity of Theorem 1. $\square$

In the following proof, the argument leading to (4.20) can be extracted from the more general approach of Kneser [16] and Stachó [28]. For the planar case and for the reader's convenience, it is more convenient to prove (4.20) directly.

Second proof of Theorem 1. Define $H^{+1} := \{P \in \mathbb{R}^2 : \mathrm{dist}(P, H)) \le 1\}$. First, we prove that H^{+1} is a compact convex set. Let Q be a limit point of H^{+1} and suppose, for a contradiction, that $Q \notin H^{+1}$. This means that $\mathrm{dist}(Q, H) = 1 + 3\varepsilon$ for a positive $\varepsilon \in \mathbb{R}$. Take a sequence $(P_n : n \in \mathbb{N})$ of points in H^{+1} such that $\lim_{n\to\infty} P_n = P$. For each $n \in N$, pick a point $Q_n \in H$ such that $\mathrm{dist}(P_n, Q_n) \le 1$. Since H is compact, the sequence $(Q_n : n \in \mathbb{N})$ has a convergent subsequence. Deleting members if necessary, we can assume that $(Q_n : n \in \mathbb{N})$ itself converges to a point Q of H. Take a sufficiently large $n \in \mathbb{N}$ such that $\mathrm{dist}(P, P_n) < \varepsilon$ and $\mathrm{dist}(Q_n, Q) < \varepsilon$. Then $1 + 3\varepsilon = \mathrm{dist}(P, Q) \le \mathrm{dist}(P, P_n) + \mathrm{dist}(P_n, Q_n) + \mathrm{dist}(Q_n, Q) \le \varepsilon + 1 + \varepsilon = 1 + 2\varepsilon$ is a contradiction. Hence, H^{+1} is closed, whereby it is obviously compact. In order to show that it is convex, let $X, Y \in H^{+1}$ and let $\lambda \in (0, 1)$; we need to show that $Z := (1 - \lambda)X + \lambda Y \in H^{+1}$. The containments $X \in H^{+1}$ and $Y \in H^{+1}$ are witnessed by some $X_0, Y_0 \in H$ such that $\mathrm{dist}(X, X_0) \le 1$ and $\mathrm{dist}(Y, Y_0) \le 1$. Since H is convex, $Z_0 := (1 - \lambda)X_0 + \lambda Y_0 \in H$. The vectors $\vec{a} := X - X_0$ and $\vec{b} := Y - Y_0$ are of length at most 1, and it suffices to show that so is $\vec{c} := Z - Z_0$. Since $(\vec{a}, \vec{b}) \le ||\vec{a}|| \cdot ||\vec{b}|| \le 1$, we have that

$$\begin{aligned}(\vec{c}, \vec{c}) &= ((1 - \lambda)\vec{a} + \lambda\vec{b}, (1 - \lambda)\vec{a} + \lambda\vec{b}) \\ &= (1 - \lambda)^2(\vec{a}, \vec{a}) + \lambda^2(\vec{b}, \vec{b}) + 2\lambda(1 - \lambda)(\vec{a}, \vec{b}) \\ &\le (1 - \lambda)^2 + \lambda^2 + 2\lambda(1 - \lambda) = 1.\end{aligned}$$

Hence, $\mathrm{dist}(Z, Z_0) = ||\vec{c}|| \le 1$, and H^{+1} is convex. Thus, (2.2) gives that

$$\partial H^{+1} \text{ is rectifiable Jordan curve.} \tag{4.20}$$

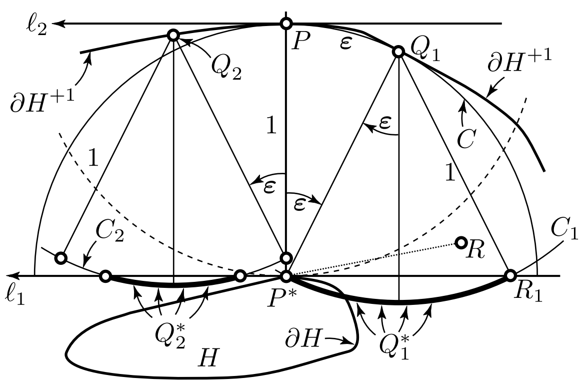

Figure 3. Illustration for the second proof

Clearly, $\partial H^{+1} = \{X : \mathrm{dist}(X, H) = 1\} = \{X : \mathrm{dist}(X, \partial H) = 1\}$. Define the following relation

$$\rho := \{\langle P, P^*\rangle \in \partial H^{+1} \times \partial H : \mathrm{dist}(P, P^*) = 1\}$$

between ∂H^{+1} and ∂H; see Figure 3. Let $\langle P, P^*\rangle \in \rho$ as in the figure. The coordinate system is chosen so that P and P^* determine a vertical line and P is above P^*. Through P^* and P, let ℓ_1 and ℓ_2 be the lines of direction π; they are perpendicular to $[P^*, P]$. We claim that

$$\ell_1 \text{ is a supporting line of } H. \tag{4.21}$$

Suppose to the contrary that ℓ_1 is not a supporting line and pick a point $R \in H$ strictly on the right of ℓ_1; see the figure. Since $P \in \partial H^{+1}$, $\mathrm{dist}(P, H) = 1$, whereby R cannot be inside the dotted circle of radius 1 around P. However, since this circle touches ℓ_1 at P^*, the line segment $[P^*, R]$, which is a subset of H by convexity, has a point inside the dotted circle. This contradicts $\mathrm{dist}(P, H) = 1$ and proves (4.21). From (4.21), it follows that if $\langle P, Q\rangle \in \rho$, then $Q = P^*$. Hence,

$$\begin{aligned}&f\colon \partial H^{+1} \to \mathrm{Sli}(H), \text{ defined by } f(P) = \langle P^*, \mathrm{dir}(\ell^*)\rangle \in \mathrm{Sli}(H) \iff\\ &\langle P, P^*\rangle \in \rho,\ \ell^* \text{ is a supporting line, and } \ell^* \text{ is perpendicular to } [P, P^*]\end{aligned}$$

is a mapping. Trivially,

$$\begin{aligned}&g\colon \mathrm{Sli}(H) \to \partial H^{+1}, \text{ defined by } g(\langle P^*, \mathrm{dir}(\ell^*)\rangle) = P \iff\\ &\mathrm{dir}([P^*, P]) = \mathrm{dir}(\ell^*) - \pi/2 \text{ and } \mathrm{dist}(P, P^*) = 1,\end{aligned} \tag{4.22}$$

is also a mapping. Moreover f and g are reciprocal bijections. Recall from Luukkainen [20, Definition 2.14] that a function $\tau\colon X \to Y$ is *Lipschitz in the small* if there are $\delta > 0$ and $L \geq 0$ such that $\mathrm{dist}(\tau(x_1), \tau(x_2)) \leq L \cdot \mathrm{dist}(x_1, x_2)$ for all $x_1, x_2 \in X$ with $\mathrm{dist}(x_1, x_2) \leq \delta$. We know from [20, 2.15] that every bounded function with this property is Lipschitz. We are going to show that f and g are Lipschitz in the small, witnessed by $\delta = 1/5$ and $L = 9$, because then $g = f^{-1}$, (4.18), and (4.20) will imply the theorem. (Note that $\delta = 1/5$ and $L = 9$ are convenient but none of them is optimal.)

First, we deal with f. Assume that $Q_1 \in \partial H^{+1}$ such that $\gamma := \mathrm{dist}(P, Q_1) < \delta = 1/5$; see Figure 3. The angle $\boldsymbol{\varepsilon} := \angle(PP^*Q_1)$, which is the length of the circular arc from P to Q_1, is close to γ in the sense that

$$\text{both } \boldsymbol{\varepsilon}/\gamma \text{ and } \gamma/\boldsymbol{\varepsilon} \text{ are in the interval } (99/100, 101/100); \tag{4.23}$$

this is shown by easy trigonometry since both $\sin(1/5)/(1/5)$ and $(1/5/)\sin(1/5)$ are in the open interval on the right of (4.23). Let C and C_1 be the circles of radius 1 around P^* and Q_1, respectively. Since $\mathrm{dist}(Q_1, H) = 1$, Q_1 is not in the interior of (the disk determined by) C. Also, since ℓ_1 is a supporting line of H, we have that ℓ_2 is a supporting line of H^{+1} and Q_1 cannot be strictly on the right (that is, above) ℓ_2. So either Q_1 is on the circle C, or it is above C but not above ℓ_2 (but then we write Q_2 instead of Q_1 in the figure). Denote $f(Q_1)$ by $\langle Q_1^*, \mathrm{dir}(\ell_1^*)\rangle$. Clearly, Q_1^* is on the thick arc of C_1 from P^* to R_1, as indicated in the figure. The length of this arc is $2\boldsymbol{\varepsilon}$, whence $\mathrm{dist}(P^*, Q_1^*) \leq 2\boldsymbol{\varepsilon}$. Since ℓ_1^* is perpendicular to $[Q_1, Q_1^*]$ and Q_1^* is on the thick arc of C_1, we have that $\mathrm{dist}(\mathrm{dir}(\ell^*), \mathrm{dir}(\ell_1^*)) \leq \boldsymbol{\varepsilon} \leq 2\boldsymbol{\varepsilon}$. So the Manhattan distance $\mathrm{d_M}(\langle P^*, \mathrm{dir}(\ell^*)\rangle, \langle Q_1^*, \mathrm{dir}(\ell_1^*)\rangle)$, see (4.13), is at most $4\boldsymbol{\varepsilon}$. Hence, (4.14) and (4.23) yield that $\mathrm{dist}(f(P), f(Q_1) \leq 9 \cdot \mathrm{dist}(P, Q_1)$. The other case, represented by Q_2, follows from the fact that $\mathrm{dist}(P^*, Q_2^*)$ and $\mathrm{dist}(\mathrm{dir}(\ell^*), \mathrm{dir}(\ell_2^*))$ are smaller than the respective distances in the previous case. This shows that f is Lipschitz in the small.

Next, we deal with g. Assume that $\langle P^*, \mathrm{dir}(\ell^*)\rangle$ and $\langle P_1^*, \mathrm{dir}(\ell_1^*)\rangle$ are in $\mathrm{Sli}(H)$ and their distance, γ, is less than δ. With the auxiliary point $\langle P^*, \mathrm{dir}(\ell_1^*)\rangle \in \mathbb{R}^4$, which need not be in $\mathrm{Sli}(H)$, we have that $\mathrm{dist}(\langle P^*, \mathrm{dir}(\ell^*)\rangle, \langle P^*, \mathrm{dir}(\ell_1^*)\rangle) \leq \gamma$ and $\mathrm{dist}(\langle P^*, \mathrm{dir}(\ell_1^*)\rangle, \langle P_1^*, \mathrm{dir}(\ell_1^*)\rangle) \leq \gamma$. Although the auxiliary point is not in the domain of g in general, we can extend the domain of g to this point by (4.22). Since the secants of the unit circles are shorter than the corresponding circular arcs, whose lengths equal the corresponding central angles, it follows that $\mathrm{dist}(g(\langle P^*, \mathrm{dir}(\ell^*)\rangle), g(\langle P^*, \mathrm{dir}(\ell_1^*)\rangle)) \leq \gamma$. Since parallel shifts are distance-preserving, $\mathrm{dist}(g(\langle P^*, \mathrm{dir}(\ell_1^*)\rangle), g(\langle P_1^*, \mathrm{dir}(\ell_1^*)\rangle)) = \gamma$. Hence, the triangle inequality yields that $\mathrm{dist}(g(\langle P^*, \mathrm{dir}(\ell^*)\rangle), g(\langle P_1^*, \mathrm{dir}(\ell_1^*)\rangle)) \leq 2\gamma \leq 9\delta$. Thus, g is also Lipschitz in the small, as required. This completes the second proof of Theorem 1. □

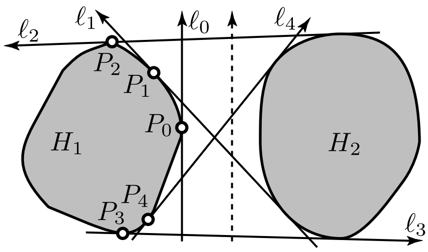

Figure 4. Illustration for Corollary 2

Proof of Corollary 2. By (2.6), we have a directed line, the dotted one in Figure 4, such that H_1 is strictly in the left and H_2 is strictly on the right of this line. By (2.1), we can take a $\langle P_0, \mathrm{dir}(\ell_0)\rangle \in \mathrm{Sli}(H_1)$ such that ℓ_0 and the dotted line have the same direction. For $0 < L \in \mathbb{R}$, let

$$L \cdot C_{\text{unit}} \text{ denote the circle } \{\langle x, y\rangle : x^2 + y^2 = (L/(2\pi))^2\} \text{ of perimeter } L.$$

Since $\mathrm{Sli}(H_1)$ is a rectifiable simple closed curve by Theorem 1, we can let L be its perimeter. Let

$$\{h(t) : t \in L \cdot C_{\text{unit}}\} \text{ be a parameterization of } \mathrm{Sli}(H_1) \tag{4.24}$$

such that $\langle P_0, \mathrm{dir}(\ell_0)\rangle = h(t_0)$. We think of the parameter t as the *time* measured in seconds. While the time t is slowly passing, $\langle P(t), \mathrm{dir}(\ell(t))\rangle$ is slowly and continuously moving forward along $\mathrm{Sli}(H_1)$, and the directed supporting line $\langle P(t), \ell(t)\rangle$ is *slide-turning* forward, slowly and continuously. Since H_2 is compact, the distance $\mathrm{dist}(\ell(t), H_2)$ is always witnessed by a pair of points in $\ell(t) \times H_2$, and this distance is a continuous function of t. At $t = t_0$, this distance is positive and H_2 is on the right of $\ell_0 = \ell(t_0)$. Slide-turn this pointed supporting line around H_1 forward during L seconds; that is, make a full turn around $\mathrm{Sli}(H_1)$. By continuity, in the chronological order listed below, there are

1. a last $t = t_1$ such that H_2 is still on the right of $\ell(t)$ (this t_1 exists, because it is the first value of t where $\mathrm{dist}(\ell(t), H_2) = 0$),

2. a first $t = t_2$ such that H_2 is on the left of $\ell(t)$,

3. a last $t = t_3$ such that H_2 is still on the left of $\ell(t)$,

4. a first $t = t_4$ such that H_2 is on the right of $\ell(t)$.

In Figure 4, $h(t_i) = \langle P(t_i), \mathrm{dir}(\ell(t_i))\rangle$ is represented by $\langle P_i, \ell_i\rangle$. Clearly, $\ell_1, \ldots, \ell_4$ is the list of all common supporting lines and these lines are pairwise disjoint. □

References

[1] K. Adaricheva, Representing finite convex geometries by relatively convex sets, *European J. of Combinatorics* **37** (2014), 68–78.

[2] K. Adaricheva and M. Bolat, Representation of convex geometries by circles on the plane, `arXiv:1609.00092v1`, last accessed December 2016.

[3] K. Adaricheva and G. Czédli, Note on the description of join-distributive lattices by permutations, *Algebra Universalis* **72** (2014), 155–162.

[4] K. Adaricheva and J. B. Nation, *Convex geometries*, in G. Grätzer and F. Wehrung (eds.) "Lattice Theory: Special Topics and Applications", vol. 2, Birkhäuser, Cham, 2015.

[5] J. M. Borwein and J. D. Vanderwerff, "Convex Functions: Constructions, Characterizations and Counterexamples", Encyclopedia of Mathematics and its Applications **109**. Cambridge University Press, Cambridge, 2010.

[6] S. Boyd and L. Vandenberghe, "Convex optimization", Cambridge University Press (2004),

[7] G. Czédli, Coordinatization of join-distributive lattices, *Algebra Universalis* **71** (2014), 385–404.

[8] G. Czédli, Finite convex geometries of circles, *Discrete Math.* **330** (2014), 61–75.

[9] G. Czédli, An easy way to a theorem of Kira Adaricheva and Madina Bolat on convexity and circles, `arXiv:1610.02540`, last accessed December 2016.

[10] G. Czédli Characterizing circles by a convex combinatorial property, `arXiv:1611.09331`, last accessed December 2016.

[11] G. Czédli and J. Kincses, Representing convex geometries by almost-circles, `arXiv:1608.06550`, last accessed December 2016.

[12] M. G. Dobbins, A. Holmsen and A. Hubard, The Erdős-Szekeres problem for non-crossing convex sets, *Mathematika* **60**, (2014) 463–484.

[13] M. G. Dobbins, A. Holmsen and A. Hubard, Regular systems of paths and families of convex sets in convex position, *Trans. Amer. Math. Soc.* **368** (May 2016) 3271–3303.

[14] P. H. Edelman and R. E. Jamison, The theory of convex geometries, *Geom. Dedicata* **19** (1985), 247–271.

[15] P. Erdős and G. Szekeres, A combinatorial problem in geometry, *Compositio Mathematica* **2** (1935), 463–470.

[16] M. Kneser, Über den Rand von Parallelkörpern (in German), *Math. Nachr.* **5** (1951), 241–251.

[17] B. Korte and L. Lovász, *Mathematical structures underlying greedy algorithms*, Proc. int. FCT-Conf., Szeged, Hungary 1981 "Fundamentals of computation theory", Springer Lect. Notes Comput. Sci. **117**, Amsterdam, 1981, 205–209.

[18] B. Korte and L. Lovász, Structural properties of greedoids. *Combinatorica* **3** (1983), 359–374.

[19] L. Latecki, A. Rosenfeld and R. Silverman, Generalized convexity: CP_3 and boundaries of convex sets. *Pattern Recognition* **28** (1995), 1191–1199.

[20] J. Luukkainen, Rings of functions in Lipschitz topology, *Ann. Acad. Sci. Fenn. Ser. A I Math.* **4** (1979), 119–135.

[21] B. Monjardet, A use for frequently rediscovering a concept, *Order* **1** (1985), 415–417.

[22] C. P. Niculescu and L.-E. Persson, "Convex functions and their applicatcions. A contemporary approach", Springer, New York, 2006.

[23] M. Novick, On the number of directions determined by the common tangents to a family of pairwise disjoint convex sets in the plane, *Discrete Comput. Geom.* **53** (2015), 261–275.

[24] J. Pach and G. Tóth, A generalization of the Erdős–Szekeres Theorem to disjoint convex sets, *Discrete and Computational Geometry* **19** (1998), 437–445.

[25] J. Pach and G. Tóth, Erdős–Szekeres-type theorems for segments and noncrossing convex sets, *Geom. Dedicata* **81** (2000), 1–12.

[26] M. Richter and L. G. Rogers, Embedding convex geometries and a bound on convex dimension, *Discrete Mathematics*, to appear, `arXiv:1502.01941`, last accessed December 2016.

[27] R. T. Rockafellar, "Convex Analysis", Princeton Mathematical Series **28**, Princeton University Press, Princeton, N.J. 1970.

[28] L. L. Stachó, On the volume funtion of parallel sets, *Acta Sci. Math. (Szeged)* **38** (1976), 365–374.

[29] O. Stolz, "Grunzüge der Differential und Integralrechnung", Vol. 1, Teubner, Leipzig, 1893.

[30] V. A. Toponogov, "Differential Geometry of Curves and Surfaces, A Concise Guide", Birkhäuser, Boston, 2006.

[31] I. M. Yaglom and V. G. Boltyanskiĭ, "Convex Figures" (English translation), Holt, Rinehart and Winston Inc., New York, 1961.

[32] "Convex curve", `https://en.wikipedia.org/wiki/Convex_curve`, last accessed December 2016.