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Roughness in MV-modules

Fereshteh Forouzesh
Dept. of Mathematics, Higher Education Complex of Bam, Kerman, Iran
frouzesh@bam.ac.ir

Mina Mirhosseini
Dept. of Computer Science, Higher Education Complex of Bam, Kerman, Iran
mirhosseini@bam.ac.ir

Abstract

In this paper, we consider an MV-module M over a PMV-algebra A as a universal set and we introduce the notion of a rough A-ideal with respect to an A-ideal of an A-module M, which is an extended notion of an A-ideal in an MV-module M. We also give some properties of the lower and the upper approximations in an A-module. In particular, we study the lower and the upper approximations with respect to fuzzy congruences in MV-modules.

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1 Introduction and Preliminaries

The notion of rough sets was introduced by Pawlak [20]. The relations between rough sets and algebraic systems have been already considered by many mathematicians.

Some authors, for example, Iwinski [16], and Pomykala [22] have studied algebraic properties of rough sets. The lattice theoretical approach has been suggested by Iwinski [16]. Pomykala [22] showed that the set of rough sets forms a stone algebra. Comer [4] presented an interesting discussion of rough sets and various algebras related to the study of algebraic logic, such as Stone algebras and relation algebras. If we substitute an algebraic system instead of the universe set, then a natural question is what will happen. Biswas and Nanda [1] introduced the notion of rough subgroups. Kuroki in [17], introduced the notion of a rough ideal in a semigroup, also see [25]. Davvaz in [5] introduced the notion of rough subrings with respect to an ideal of a ring, also see [6]. Rough modules have been investigated by Davvaz and Mahdavipour [7]. Also Rasouli and Davvaz introduced the notion of roughness in MV-algebras [23].

In 1958, Chang defined the MV-algebras and in 1959 he also proved the completeness theorem which stated the real unit interval [0,1] as a standard model of this logic [2].

In 2003, Di Nola, et.al. introduced the notion of MV-modules over a PMV-algebra and A-ideals in MV-modules [10]. These are structures that naturally correspond to lu-modules over lu-ring [24]. We recall that an lu-ring is a pair (R,u) where $(R, \oplus, \cdot, 0, \leq)$ is an l-ring and u is a strong unit of R (i.e, u is a strong unit of the underlying l-group), with $u \cdot u \leq u$ and l-ring is a structure $(R, +, \cdot, 0, \leq)$ that $(R, +, 0, \leq)$ is an l-group and for any $x, y \in R$, $x \geq 0$ and $y \geq 0$ implies $x \cdot y \geq 0$. Fixing an lu-ring

(R,v), they proved equivalence between the category of lu-modules over (R,v) and the category of MV-modules over $\Gamma(R,v)$. They also proved the natural equivalence between MV-modules and truncated modules [10].

In the present paper, we consider an MV-module over PMV-algebra A as a universal set and we shall introduce the notion of rough A-ideal with respect to an A-ideal of an MV-module, which is an extended notion of an A-ideal in an MV-module. We give some properties of the lower and the upper approximations in an MV-module.

1.1 MV-modules

In this section, we summarize the basic concepts on MV-algebras and MV-modules. For more details on these concepts, we refer the reader to [2], [3]-[12] and [21].

Definition 1. [2] An MV-algebra is a structure $(M, \oplus, ^*, 0)$ where $\oplus$ is a binary operation, *, is a unary operation and 0 is a constant such that the following conditions are satisfied for any $a, b \in M$:

1. $(M, \oplus, 0)$ is an abelian monoid,
2. $(a^*)^* = a$,
3. $0^* \oplus a = 0^*$,
4. $(a^* \oplus b)^* \oplus b = (b^* \oplus a)^* \oplus a$.

Define the constant $1 = 0^*$ and the auxiliary operations $\odot, \vee$ and $\wedge$ by:

$$a \odot b = (a^* \oplus b^*)^*, \quad a \vee b = a \oplus (b \odot a^*), \quad a \wedge b = a \odot (b \oplus a^*).$$

It is shown that $(M, \odot, 1)$ is an abelian monoid and the structure $(M, \vee, \wedge, 0, 1)$ is a bounded distributive lattice [21].

In an MV-algebra M, the Chang distance function is

$$d : M \times M \longrightarrow M, \quad d(a,b) := (a \odot b^*) \oplus (b \odot a^*).$$

Note. An element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. We denote the set of complemented of A by $B(A)$.

Lemma 2. *[21] Let M be an MV-algebra. If $x, y, z, t \in M$ and d is a Chang distance function, then*

1. $x \leq y$ *iff* $y^* \leq x^*$,
2. $x \leq y$, *then* $x \oplus z \leq y \oplus z$ *and* $x \odot z \leq y \odot z$,
3. $(x \vee y)^* = x^* \wedge y^*$, $(x \wedge y)^* = x^* \vee y^*$,
4. $x \oplus x^* = 1$ *and* $x \odot x^* = 0$,
5. *If* $x \leq y$ *and* $z \leq t$, *then* $x \oplus z \leq y \oplus t$,
6. *If* $x \in B(A)$, *then* $x \odot x = x$, $x \oplus x = x$ *and* $x \wedge y = x \odot y$,
7. $x \leq y^* \oplus z$ *if and only if* $x \odot y \leq z$,
8. $d(x, 0) = x$, $d(x, 1) = x$,
9. $d(x^*, y^*) = d(x, y)$,

10. $d(x, y) \leq d(x, z) \oplus d(z, y)$,

11. $d(x \oplus u, y \oplus v) \leq d(x, y) \oplus d(u, v)$.

Lemma 3. *[3] Let M be an MV-algebra. For $x, y \in M$, the following conditions are equivalent*

1. $x^* \oplus y = 1$,

2. $x \odot y^* = 0$.

For any two elements $x, y \in M$, $x \leq y$ iff x and y satisfy the equivalent conditions (1)-(2) in the above lemma.

Definition 4. [2] An ideal of an MV-algebra M is a nonempty subset I of M satisfying the following conditions:

1. If $x \in I$, $y \in M$ and $y \leq x$ then $y \in I$,
2. If $x, y \in I$, then $x \oplus y \in I$.

We denote by $Id(M)$ the set of all ideals of an MV-algebra M.

Definition 5. [3] Let A and B be two MV-algebras. A function $h : A \to B$ is a morphism of MV-algebras if and only if it satisfies the following conditions, for every $x, y \in A$:

1. $h(0) = 0$
2. $h(a \oplus b) = h(a) \oplus h(b)$,
3. $h(a^*) = h(a)^*$.

Lemma 6. *Let M be a linearly ordered MV-algebra and I be an ideal of M. If $x \leq y$ and $[x]_I \neq [y]_I$, for $x, y \in A$, then for each $t \in [x]_I$ and $s \in [y]_I$, $t \leq s$.*

Definition 7. [9] A product MV-algebra (or PMV-algebra, for short) is a structure (A, $\oplus$, *,$\cdot$, 0), where (A, $\oplus$, *, 0) is an MV-algebra and $\cdot$ is a binary associative operation on A such that the following property is satisfied if $x + y$ is defined, then $x \cdot z + y \cdot z$ and $z \cdot x + z \cdot y$ are defined and

$$(x + y) \cdot z = x \cdot z + y \cdot z, \quad z \cdot (x + y) = z \cdot x + z \cdot y,$$

where, $+$ is a partial addition on an MV-algebra A as follows For any $x, y \in M$, $x + y$ is defined if and only if $x \leq y^*$ and in this case,

$$x + y := x \oplus y,$$

the partial addition was defined in [11].

Note. If A is PMV-algebra, then a unit for the product is an element $e \in A$ such that $e \cdot x = x \cdot e = x$ for any $x \in A$. A PMV-algebra that has unity for the product will be called unital.

Theorem 8. *[9] A finite MV-algebra A admits a product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in A$ if and only if A is a Boolean algebra, i.e., $a \oplus a = a$ for any $a \in A$. If it is the case, then $a \cdot b = a \wedge b \in A$.*

Definition 9. [10] Let $(A, \oplus, ^*, \cdot, 0)$ be a PMV-algebra and $(M, \oplus, ^*, 0)$ an MV-algebra. We say that M is a (left) MV-module over A (or, simply, A-module) if there is an external operation:

$$\varphi : A \times M \longrightarrow M, \quad \varphi(\alpha, x) = \alpha x,$$

such that the following properties hold for any $x, y \in M$ and $\alpha, \beta \in A$:

1. If $x + y$ is defined in M, then $\alpha x + \alpha y$ is defined and
$$\alpha(x + y) = \alpha x + \alpha y,$$
2. If $\alpha + \beta$ is defined in A then $\alpha x + \beta x$ is defined in M and
$$(\alpha + \beta)x = \alpha x + \beta x,$$
3. $(\alpha \cdot \beta)x = \alpha(\beta x)$.

We say that M is a unital MV-module if A is a unital PMV-algebra and M is an MV-module over A such that $1_A x = x$ for any $x \in M$.

We will refer to [9, 19] for the basic properties of PMV-algebras. Obviously, a PMV-algebra homomorphism will be an MV-algebra homomorphism which also commutes with the product operation. We shall denote by $\mathcal{PMV}$ the category of product MV-algebras with the corresponding homomorphisms.

In the sequel, an lu-ring will be a pair (R, u) where $(R, \oplus, \cdot, \leq)$ is an l-ring and u is a strong unit of R such that $u \cdot u \leq u$. We imply that the interval $[0, u]$ of an lu-ring (R, u) is closed under the product of R. Thus, if we consider the restriction of $\cdot$ to $[0, u] \times [0, u]$, then the interval $[0, u]$ has a canonical PMV-algebra structure:

$$x \oplus y := (x + y) \wedge u, \quad x^* := u - x, \quad x \cdot y := x \cdot y,$$

for any $0 \leq x, y \leq u$. We shall denote this structure by $[0, u]_R$.

If $\mathcal{UR}$ is the category of lu-rings, whose objects are pairs (R, u) as above and whose morphisms are l-rings homomorphisms which preserve the strong unit, then we get a functor

$$\Gamma : \mathcal{UR} \to \mathcal{PMV},$$

$$\Gamma(R, u) := [0, u]_R, \text{ for any lu-ring } (R, u),$$

$$\Gamma(h) := h \mid_{[0,u]} \text{ for any lu-rings homomorphism h.}$$

In [9] it is proved that Γ establishes a categorical equivalence between $\mathcal{UR}$ and $\mathcal{PMV}$.

Definition 10. [10] Let M and N be two MV-modules over a PMV-algebra A. An A-module homomorphism is an MV-algebra homomorphism $h : M \to N$ such that $h(\alpha x) = \alpha h(x)$, for any $\alpha \in A$ and $x \in M$.

Definition 11. [10] Let M be an A-module. Then ideal $I \subseteq M$ is called an A-ideal if it satisfies the following condition: if $x \in I$ and $\alpha \in A$, then $\alpha x \in I$.

Lemma 12. *[10] If M is an A-module, then the following properties hold for any $x, y \in M$ and $\alpha, \beta \in A$,*

(a) $\alpha x^* \leq (\alpha x)^*$,

(b) $(\alpha x) \odot (\alpha y)^* \leq \alpha(x \odot y^*)$,

(c) $\alpha(x \oplus y) \leq \alpha x \oplus \alpha y$,

(d) If $x \leq y$, *then* $\alpha x \leq \alpha y$,

(e) $(\alpha x)^* = \alpha^* x + (1x)^*$,

(f) $d(\alpha x, \alpha y) \leq \alpha d(x, y)$.

Proposition 13. *[13] Let M be an A-module.*

1. *If $N \subseteq M$ is a nonempty set, then we have $(N] = \{x \in M : x \leq x_1 \oplus \ldots \oplus x_n \oplus \alpha_1 y_1 \oplus \ldots \oplus \alpha_m y_m \quad \text{for some} \quad x_1, \ldots, x_n, y_1, \ldots y_m \in N, \alpha_1, \ldots \alpha_m \in A\}$, where by $(N]$, we mean the ideal generated by N.*

   In particular, for $a \in M$,

$$(a] = \{x \in M : x \leq na \oplus m(\alpha a) \quad \text{for some integer} \quad n, m \geq 0\},$$

2. *If $I_1, I_2 \in Id_A(M)$, then*
   $I_1 \vee I_2 = (I_1 \cup I_2] = \{a \in M : a \leq x_1 \oplus x_2 \quad \text{for some} \quad x_1 \in I_1 \quad \text{and} \quad x_2 \in I_2\}$,

3. *If $x, y \in A$, then $(x \wedge y] \subseteq (x] \cap (y]$.*

1.2 Pawlak approximation spaces

Let θ be an equivalence relation on a set U. The set of the elements of U that are related to $x \in U$, is called the equivalence class of x, and is denoted by $[x]_\theta$. In addition U/θ denote the family of all equivalence classes induced on U by θ. For any $X \subseteq U$, we write X^c to denote the complement of X in U, that is the set $U \setminus X$.

Definition 14. A pair (U, θ) where $U \neq \emptyset$ and θ is an equivalence relation on U, is called an approximation space. The interpretation of rough set is that our knowledge of the objects in U extends only up to a membership in the class of θ, and our knowledge about a subset X of U is limited to the class of θ and their unions.

This leads to the following definition.

Definition 15. For an approximation space (U, θ), by a rough approximation in (U, θ) we mean a mapping $Apr : P(U) \to P(U) \times P(U)$ defined for every $X \in P(U)$ by

$$Apr(X) = (\underline{Apr}(X), \overline{Apr}(X)),$$

where $\underline{Apr}(X) = \{x \in U | [x]_\theta \subseteq X\}$, $\overline{Apr}(X) = \{x \in U | [x]_\theta \cap X \neq \emptyset\}$. $\overline{Apr}(X)$ is called an upper rough approximation of X in (U, θ), while $\underline{Apr}(X)$ is called a lower rough approximation of X in $U, \theta)$.

Definition 16. Given an approximation space (U, θ), a pair $(A, B) \in P(U) \times P(U)$ is called a rough subset in (U, θ) if and only if $(A, B) = Apr(X)$ for some $X \in P(U)$. Note that a rough subset is also called a rough set.

The reader will find a deep study of rough set theory in [1, 5, 6, 7, 16, 17, 20, 23].

Definition 17. Let $Apr(A) = (\underline{Apr}(A), \overline{Apr}(A))$ and $Apr(B) = (\underline{Apr}(B), \overline{Apr}(B))$ be any two rough sets in the approximation space (U, θ). Then

1. $Apr(A) \sqcup Apr(B) = (\underline{Apr}(A) \cup \underline{Apr}(B), \overline{Apr}(A) \cup \overline{Apr}(B))$,
2. $Apr(A) \sqcap Apr(B) = (\underline{Apr}(A) \cap \underline{Apr}(B), \overline{Apr}(A) \cap \overline{Apr}(B))$,
3. $Apr(A) \sqsubseteq Apr(B) \iff Apr(A) \sqcap Apr(B) = Apr(A)$.

When $Apr(A) \sqsubseteq Apr(B)$, we say that $Apr(A)$ is a rough subset of $Apr(B)$.
Thus in the case of rough sets $Apr(A)$ and $Apr(B)$,
$Apr(A) \sqsubseteq Apr(B)$ if and only if $\underline{Apr}(A) \subseteq \underline{Apr}(B)$ and $\overline{Apr}(A) \subseteq \overline{Apr}(B)$.
This property of rough inclusion has all the properties of set inclusion. The rough complement of $Apr(A)$ denoted by $Apr^c(A)$ is defined by

$$Apr^c(A) = (U \setminus \overline{Apr}(A), U \setminus \underline{Apr}(A)).$$

Also, we can define $Apr(A) \setminus Apr(B)$ as follows:

$$Apr(A) \setminus Apr(B) = Apr(A) \sqcap Apr^c(B) = (\underline{Apr}(A) \setminus \overline{Apr}(B), \overline{Apr}(A) \setminus \underline{Apr}(B)).$$

Definition 18. [8] Let (U, θ) be an approximation space and X a non-empty subset of U.

1. If $\underline{Apr}(X) = \overline{Apr}(X)$, then X is called definable.
2. If $\underline{Apr}(X) = \emptyset$, then X is called empty interior.
3. If $\overline{Apr}(X) = U$, then X is called empty exterior.

The lower approximation of X in (U, θ) is the greatest definable set in U contained in X. The upper approximation of X in (U, θ) is the least definable set in U containing X. Therefore we have:

$$\underline{Apr}(X) = \bigcup\{S | S \subseteq X, \text{ S is definable}\},$$

$$\overline{Apr}(X) = \bigcap\{S | X \subseteq S, \text{ S is definable}\}.$$

A rough set X is the family of all subsets of U having the same lower and the same upper approximations of X.

2 Rough A-ideals in MV-modules over PMV-algebras

Throughout this paper M is an MV-module over a PMV-algebra A. We recall that in an MV-algebra M, the Chang distance function is

$$d : M \times M \longrightarrow M, \quad d(a, b) := (a \odot b^*) \oplus (b \odot a^*).$$

Let I be an A-ideal of M. We recall that the relation ρ_I defined by:

$$(x, y) \in \rho_I \quad \text{if and only if} \quad d(x, y) \in I,$$

for any $x, y \in M$, is a congruence with respect to the MV-algebra operations and $(x, y) \in \rho_I$ implies $(\alpha x, \alpha y) \in \rho_I$, for any $\alpha \in A$. Thus, the quotient MV-algebra M/I has a canonical structure of A-module

$$\alpha[x]_I := [\alpha x]_I \quad or \quad \alpha(x/I) := (\alpha x)/I,$$

where $[x]_I = x/I$ is the congruence class of x. $x/I = y/I$ or $x\rho_I y$ if and only if $d(x, y) \in I$ and if $x, y \in M$, then $x/I \leq y/I$ if and only if $x \odot y^* \in I$. Also $a \in x/I$ if and only if $d(a, x) \in I$.

Let I be an A-ideal of an A-module M. Then the quotient group M/I is an A-module with the action of A on M/I given by the well-defined map

$$\alpha(a/I) = (\alpha a)/I, \text{ for all } \alpha \in A,\ a \in M.$$

Let I be an A-ideal of M and X a non-empty subset of M, then the sets

$$\underline{\rho}_I(X) = \underline{Apr}_I(X) = \{x \in M | x/I \subseteq X\} \text{ and } \overline{\rho}_I(X) = \overline{Apr}_I(X) = \{x \in M | x/I \cap X \neq \emptyset\}$$

are called lower and upper approximations of the set X with respect to the A-ideal I. In this case we use the pair (M, I) instead of the approximation space (U, θ).

Now, we give an example of the lower and upper approximations theory applied the MV-module theory.

Example 19. Let $M_2(\mathbb{R})$ be the ring of square matrices of order 2 with real elements and 0 be the matrix with all element 0. If we define the order relation on components $A = (a_{ij})_{i,j=1,2} \geq 0$ iff $a_{ij} \geq 0$ for any i, j, such that $v = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$, then $A = \Gamma(M_2(\mathbb{R}), v)$ is a PMV-algebra. Let $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ be the direct product with the order relation defined on components. If $M = \Gamma(\mathbb{R}^2, u)$ is an MV-algebra, where $u = (1, 1)$, $(x, y)^* = u - (x, y)$, $(x, y) \oplus (z, t) = \min\{u, (x + z, y + t)\}$, and $(x, y) \odot (z, t) = \max\{(0, 0), (x, y) + (z, t) - u\}$, then M is an A-module [10], where the external operation is the usual matrix multiplication

$$(A, (x, y)) \mapsto A \begin{pmatrix} x \\ y \end{pmatrix}.$$

Now, let $I = \{(0, 0)\}$. Then I is an A-ideal of $R \times R$. We consider the maps

$$f(x) = 1/2(\mathrm{sinx} - 4) \text{ and } g(x) = 1/2(\mathrm{sinx} + 4),$$

and suppose that

$$X = \{(x, y) | f(x) \leq y \leq g(x)\}.$$

Then we have $(x, y)/I = \{(a, b) \in M | d((x, y), (a, b)) = 0\} = \{(a, b) \in M | (x, y) = (a, b)\}$. Thus the lower and upper approximations of this set can be calculate in the following way:

$$\underline{Apr}_I(X) = X = \overline{Apr}_I(X)$$

In general, we can prove that:

Remark 20. Let $I = \{0\}$ be a trivial A-ideal of M. Then $\underline{Apr}_I(X) = X = \overline{Apr}_I(X)$, for every non-empty subset X of M. Hence every non-empty subset of M is definiable.

Proof. Note that

$$\begin{aligned}
[x]_I &= \{y \in M : d(y,x) \in I\},\\
&= \{y \in M : x \odot y^* = 0 \text{ and } y \odot x^* = 0\},\\
&= \{y \in M : x \leq y \text{ and } y \leq x\},\\
&= \{y \in M : x = y\},\\
&= \{x\}.
\end{aligned}$$

Hence $\underline{Apr}_I(X) = X = \overline{Apr}_I(X)$. Thus this completes proof. □

Theorem 21. *For every approximation space (M, I) and every subsets $X, Y \subseteq M$, we have:*

1. $\underline{Apr}_I(X) \subseteq X \subseteq \overline{Apr}_I(X)$,
2. $\underline{Apr}_I(\emptyset) = \emptyset = \overline{Apr}_I(\emptyset)$,
3. $\underline{Apr}_I(M) = M = \overline{Apr}_I(M)$,
4. *If* $X \subseteq Y$, *then* $\underline{Apr}_I(X) \subseteq \underline{Apr}_I(Y)$ *and* $\overline{Apr}_I(X) \subseteq \overline{Apr}_I(Y)$,
5. $\underline{Apr}_I(\underline{Apr}_I(X)) = \underline{Apr}_I(X)$,
6. $\overline{Apr}_I(\overline{Apr}_I(X)) = \overline{Apr}_I(X)$,
7. $\overline{Apr}_I(\underline{Apr}_I(X)) = \underline{Apr}_I(X)$,
8. $\underline{Apr}_I(\overline{Apr}_I(X)) = \overline{Apr}_I(X)$,
9. $\underline{Apr}_I(X \cap Y) = \underline{Apr}_I(X) \cap \underline{Apr}_I(Y)$,
10. $\overline{Apr}_I(X \cap Y) \subseteq \overline{Apr}_I(X)\overline{Apr}_I(Y)$,
11. $\underline{Apr}_I(X \cup Y) \supseteq \underline{Apr}_I(X) \cup \underline{Apr}_I(Y)$,
12. $\overline{Apr}_I(X \cup Y) = \overline{Apr}_I(X) \cup \overline{Apr}_I(Y)$.

Proof. The proof is similar to the proof of Theorem 2.1 in [17]. □

The following example shows that the converse of (10) and (11) in the Theorem 21 is not true.

Example 22. Let $M = \{0, a, b, c, d, 1\}$, where $0 < a < c < 1$, $0 < b < d < 1$ and elements of $\{a, b\}$ and $\{c, d\}$ are pairwise incomparable. Define $\oplus$, $\odot$ and $*$ as follows:

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\oplus$	0	a	b	c	d	1
0	0	a	b	c	d	1
a	a	a	c	c	1	1
b	b	c	d	1	d	1
c	c	c	1	1	1	1
d	d	1	d	1	d	1
1	1	1	1	1	1	1

$*$	0	a	b	c	d	1
	1	d	c	b	a	0

Then $(M, \oplus, \odot, *, 0, 1)$ is an MV-algebra [15]. Consider $A = \Gamma(\mathbb{Z}, 1) = \{0, 1\}$, then M is A-module with natural product αx, for any $\alpha \in A$ and $x \in M$. It is clear $I = \{0, a\}$ is an A-ideal of M. Let $X = \{0, a, c\}$ and $Y = \{0, b, 1\}$ are subsets of M. Then the equivalence classes are $[a]_I = [0]_I = \{0, a\}$, $[1]_I = \{1, d\}$, $[b]_I = \{b, c\}$, $[c]_I = \{b, c\}$ and $[d]_I = \{d, 1\}$. Thus we have

$$\begin{aligned}
\underline{Apr}_I(X) &= \{0, a\}, \\
\underline{Apr}_I(Y) &= \emptyset, \\
\underline{Apr}_I(X \cup Y) &= \{0, a, b, c\}, \\
\overline{Apr}_I(X) &= \{0, a, b, c\}, \\
\overline{Apr}_I(Y) &= \{0, a, b, c, d, 1\}, \\
\overline{Apr}_I(X \cap Y) &= \{0, a\}.
\end{aligned}$$

It follows that $\underline{Apr}_I(X \cup Y) \neq \underline{Apr}_I(X) \cup \underline{Apr}_I(Y)$ and $\overline{Apr}_I(X) \cap \overline{Apr}_I(Y) \neq \overline{Apr}_I(X \cap Y)$.

Remark 23. For every approximation space (M, I) and for all $x \in M$, we have

$$\underline{Apr}_I(x/I) = \overline{Apr}_I(x/I)$$

Proof. It follows from Theorem 21 (1) that $\underline{Apr}_I(x/I) \subseteq \overline{Apr}_I(x/I)$. Conversely, let $a \in \overline{Apr}_I(x/I)$. Hence $[a]_I \cap x/I \neq \emptyset$. So there exists $t \in [a]$ and $t \in x/I$.

Now, we only show that $[a \subseteq x/I$. Let $y \in [a]$. Hence $(y, a) \in \rho_I$ and we have $(t, a) \in \rho_I$. We obtain $(y, t) \in \rho_I$ and also we have $(t, x) \in \rho_I$. It follows that $(y, x) \in \rho_I$, that is $y \in x/I$. Thus $[a] \subseteq x/I$. This results $a \in \underline{Apr}_I(x/I)$. □

Remark 24. [21] We recall that if X and Y are non-empty subsets of M, then we have

$$X \vee Y = \{a \in M | a \leq x \oplus y, x \in X, y \in Y\}.$$

Proposition 25. *Let I be an A-ideal of an A-module M and X, Y be non-empty subsets of M. Then*

(i) $\overline{Apr}_I(X \vee Y) \subseteq \overline{Apr}_I(X) \vee \overline{Apr}_I(Y)$. *In the particularly, if M is linearly ordered, then* $\overline{Apr}_I(X \vee Y) = \overline{Apr}_I(X) \vee \overline{Apr}_I(Y)$.

(ii) $\underline{Apr}_I(X) \vee \underline{Apr}_I(Y) \subseteq \underline{Apr}_I(X \vee Y)$.

Proof. The proof is similar to the proofs of Propositions 3.2.1 and 3.2.4 in [23]. □

The following example shows that we can not replace the inclusion symbol $\subseteq$ by an equal sign in Proposition 25 (ii).

Example 26. Let $\Omega = \{1, 2\}$ and $\mathcal{A} = \mathcal{P}(\Omega)$. $\mathcal{A}$ is a PMV-algebra with $\oplus = \cup$ and $\odot = \cdot = \cap$. If we consider $\mathcal{M} = \mathcal{A} = \mathcal{P}(\Omega) = \{\{1\}, \{2\}, \{1, 2\}, \phi\}$, then $\mathcal{M}$ becomes an MV-module over $\mathcal{A}$ with the external operation defined by $AX := A \cap X$ for any $A \in \mathcal{A}$ and $X \in \mathcal{M}$. Consider $X = \{\{1\}\}$ and $Y = \{\{2\}\}$. Obviously, $I = \{\emptyset, \{1\}\}$ is an $\mathcal{A}$-ideal of $\mathcal{M}$. We have $X \vee Y = \mathcal{M}$ and $[\{1\}]_I = \{\emptyset, \{1\}\}$, $[\emptyset]_I = \{\emptyset, \{1\}\}$, $[\{2\}]_I = \{\{2\}, \{1, 2\}\}$ and $[\{1, 2\}]_I = \{\{2\}, \{1, 2\}\}$. Hence we obtain $\underline{Apr}_I(X) = \emptyset$, $\underline{Apr}_I(Y) = \emptyset$. Thus $\underline{Apr}_I(X \vee Y) = \mathcal{M}$ is not a subset of $\underline{Apr}_I(X) \vee \underline{Apr}_I(Y) = \emptyset$.

Lemma 27. *Let I, J be two A-ideals of M such that $I \subset J$ and let X be a non-empty subset of M. Then*

(i) $\underline{Apr}_J(X) \subseteq \underline{Apr}_I(X)$,

(ii) $\overline{Apr}_I(X) \subseteq \overline{Apr}_J(X)$.

Proof. (i) Let $x \in \underline{Apr}_J(X)$. Then $x/I \subseteq x/J \subseteq X$, so $x/I \subseteq X$. Thus $x \in \underline{Apr}_I(X)$. Therefore $\underline{Apr}_J(X) \subseteq \underline{Apr}_I(X)$.

(ii) Let $x \in \overline{Apr}_I(X)$. Then $x/I \cap X \neq \emptyset$. We get $t \in x/I$ and $t \in X$. Hence $d(t, x) \in I \subseteq J$ and $t \in X$. It follows that $d(t, x) \in J$ and $t \in X$. This results $t \in x/J \cap X$. Thus $x/J \cap X \neq \emptyset$. Therefore $x \in \overline{Apr}_J(X)$. □

We recall that an element $a \in A$ is called complemented if there is an element $b \in A$ such that $a \vee b = 1$ and $a \wedge b = 0$. We denote the set of complemented of A by $B(A)$.

Proposition 28. *Let I, J be two A-ideals of an MV-module M and X be a non-empty subset of M.*

(i) *If $X \subseteq B(M)$ or M is linearly ordered, then* $\overline{Apr}_{I\vee J}(X) \subseteq \overline{Apr}_I(X) \vee \overline{Apr}_J(X)$.

(ii) $\underline{Apr}_{I\vee J}(X) \subseteq \underline{Apr}_I(X) \vee \underline{Apr}_J(X)$.

Proof. The proof is similar to the proofs of Propositions 3.2.7 and 3.2.9 in [23]. □

The following examples show that in Proposition 28 (i), (ii), the symbol inclusion can be proper.

Example 29. (i) Consider $\mathcal{M}$ as the MV-module in Example 26. Let $I = \{\emptyset, \{1\}\}$ and $J = \{\emptyset\}$ be two A-ideals of $\mathcal{M}$ and $X = \{\{2\}\} \subseteq B(M)$ be a subset of $\mathcal{M}$. It is easy to check that $[\emptyset]_J = \{\emptyset\}$, $[\{1\}]_J = \{\{1\}\}$, $[\{2\}]_J = \{\{2\}\}$ and $[\{1,2\}]_J = \{\{1,2\}\}$. Hence $\overline{Apr}_I(X) = \{\{2\}, \{1,2\}\}$ and $\overline{Apr}_J(X) = \{\{2\}, \{1,2\}\}$. Thus $\overline{Apr}_{I\vee J}(X) = \{\{2\}, \{1,2\}\}$ and by Remark 24, we have $\overline{Apr}_I(X) \vee \overline{Apr}_J(X) = \mathcal{M}$.

(ii) Consider $\mathcal{M}$ as the MV-module in Example 26. Let $I = \{\emptyset, \{1\}\}$ and $J = \{\emptyset\}$ be two A-ideals of $\mathcal{M}$ and $X = \{\emptyset, \{1\}, \{2\}\}$ be a subset of $\mathcal{M}$. We have $\underline{Apr}_{I\vee J}(X) = \{\emptyset, \{1\}\}$, $\underline{Apr}_I(X) = \{\emptyset, \{1\}\}$ and $\underline{Apr}_J(X) = \{\emptyset, \{1\}, \{2\}\}$. Thus $\underline{Apr}_{I\vee J}(X) \neq \underline{Apr}_I(X) \vee \underline{Apr}_J(X) = \mathcal{M}$.

Lemma 30. *Let I be an A-ideal of an MV-module M and X be a non-empty subset of M. Then X is definable if and only if $\underline{Apr}_I(X) = X$ or $\overline{Apr}_I(X) = X$.*

Proof. The proof is similar to the proof of Lemma 3.2.2 in [23]. □

We recall that X is an MV-subalgebra (for short, subalgebra) of M if and only if X is closed under the MV-operations defined in M.

Proposition 31. *Let M be an A-module and I be an A-ideal of M.*

(i) *If X is an A-ideal of M, then $\overline{Apr}_I(X)$ is a subalgebra too.*

(ii) *In particular, if M is a linearly ordered A-module and J is an A-ideal of M, then $\overline{Apr}_I(J)$ is an A-ideal of M.*

Proof. (i) By Proposition 3.3.3 (i) in [23], we obtain $\overline{Apr}_I(X)$ is a subalgebra of MV-algebra M. It sufficient to show that if $\alpha \in A$ and $x \in \overline{Apr}_I(X)$, then $\alpha x \in \overline{Apr}_I(X)$.

Since $x \in \overline{Apr}_I(X)$, so $x_1 \in [x]_I \cap X$, hence we have $d(x_1, x) \in I$ and $x_1 \in X$. Also by Lemma 12 (f) we deduce that $d(\alpha x_1, \alpha x) \leq \alpha d(x_1, x) \in I$ and since X is A-ideal M, $\alpha x_1 \in X$ and $\alpha x_1 \in [\alpha x]_I$. Thus $\alpha x_1 \in [\alpha x]_I \cap X$. Thus $\alpha x \in \overline{Apr}_I(X)$.

(ii) By Proposition 3.3.3 (ii) in [23] and similar to part (i), we can easily show $\alpha x \in \overline{Apr}_I(J)$, for $\alpha \in A$ and $x \in \overline{Apr}_I(J)$. It can be concluded that $\overline{Apr}_I(J)$ is an A-ideal of M. □

2.1 Rough sets in a quotient MV-module

Let I be an A-ideal of M. It is important to note that the equivalence class X/I containing x plays dual roles. It is a subset of M if considered in relation to the A-module M, and an element of M/I if considered in relation to the quotient MV-module. Therefore the lower and upper approximations can be presented in an equivalent form as shown below:

Let I be an A-ideal of M, and X a non-empty subset of M. Then

$$\underline{\underline{Apr}}_I(X) = \{x/I \in M : x/I \subseteq X\},$$

$$\overline{\overline{Apr}}_I(X) = \{x/I \in M : (x/I) \cap X \neq \emptyset\}.$$

Now, we discuss these sets as subsets of the quotient MV-module M/I.

Proposition 32. *Let I and J be two A-ideals of linearly ordered MV-module M. Then $\overline{\overline{Apr}}_I(J)$ is an A-ideal of M/I.*

Proof. Obviously, $\overline{\overline{Apr}}_I(J)$ is non-empty. Assume that $a/I, b/I \in \overline{\overline{Apr}}_I(J)$ and $\alpha \in A$. Then $a/I \cap J \neq \emptyset$ and $b/I \cap J \neq \emptyset$. So there exist $x \in a/I \cap J$ and $y \in b/I \cap J$. Since J is an A-ideal of M, we have $x \oplus y \in J$ and $\alpha x \in J$. Hence $d(x, a) \in I$ and $d(y, b) \in I$. It follows from Lemma 2 (11) that $d(x \oplus y, a \oplus b) \leq d(x, a) \oplus d(y, b) \in I$. Thus $x \oplus y \in (a \oplus b)/I \cap J$. So $(a/I \oplus b/I) \cap J = (a \oplus b)/I \cap J \neq \emptyset$. Therefore $a/I \oplus b/I \in \overline{\overline{Apr}}_I(J)$.

Now, if $x/I \leq y/I$ and $y/I \in \overline{\overline{Apr}}_I(J)$, then $y/I \cap J \neq \emptyset$. Hence there exists $t \in y/I \cap J$. Since M is linearly ordered MV-algebra, $x \leq y$ or $y \leq x$.

Case 1. If $x \leq y$, then by Lemma 6, for each $s \in [x]_I$, we have $s \leq t$ and since J is an A-ideal, we obtain $s \in J$. Hence $s \in J \cap x/I$, so $x/I \cap J \neq \emptyset$. Thus $x/I \in \overline{\overline{Apr}}_I(J)$.

Case 2. If $y \leq x$, then $y \odot x^* = 0 \in I$, hence $y/I \leq x/I$. So $x/I = y/I$. Thus the proof is complete.

Let $\alpha \in A$ and $x/I \in \overline{\overline{Apr}}_I(J)$. We show that $\alpha(x/I) \in \overline{\overline{Apr}}_I(J)$. Since $x/I \cap J \neq \emptyset$, $x_1 \in [x]_I \cap J$, so we have $d(x_1, x) \in I$, $x_1 \in J$. It follows from Lemma 12 (f) that $d(\alpha x_1, \alpha x) \leq \alpha d(x_1, x) \in I$ and since J is an A-ideal, $\alpha x_1 \in J$ and $\alpha x_1 \in [\alpha x]_I$. Thus $(\alpha x)/I \cap J \neq \emptyset$. Hence $\alpha(x/I) \in \overline{\overline{Apr}}_I(J)$. □

Theorem 33. *Let I, J be two A-ideals of M. Then $\underline{\underline{Apr}}_I(J) \neq \emptyset$ is an A-ideal, when $I \subseteq J$ and J is non-empty interior.*

Proof. Assume that $a/I, b/I \in \underline{\underline{Apr}}_I(J)$ and $\alpha \in A$, then $a \in [a]_I = a/I \subseteq J$ and $b \in [b]_I = b/I \subseteq J$. Let $z \in (a \oplus b)/I$, hence $d(z, a \oplus b) \in J$. Since $a \oplus b \in J$, we obtain $z \in (a \oplus b)/J = J$, thus $(a \oplus b)/I \subseteq J$. It proved that $a/I \oplus b/I \in \underline{\underline{Apr}}_I(J)$.

Now, let $x/I \in \underline{\underline{Apr}}_I(J)$ and $y/I \leq x/I$. We have $x \in [x]_I = x/I \subseteq J$ and since $y \odot x^* \in I \subseteq J$ and $x \in J$, we obtain $y \leq x \vee y = x \oplus (x^* \odot y) \in J$, hence $y \in J$. We must show that $y/I \subseteq J$. Let $z \in [y]_I$, then $d(z, y) \in I \subseteq J$. So $z \in y/J = J$. Hence $z \in J$. Thus $[y]_I = y/I \subseteq J$. Therefore $y/I \in \underline{\underline{Apr}}_I(J)$.

Finally, we show that $\alpha(a/I) \in \underline{\underline{Apr}}_I(J)$. Since $a/I \subseteq J$, we have $a \in J$. Since J is an A-ideal of M, then $\alpha a \in J$. It is sufficient to show that $(\alpha a)/I \subseteq J$. Let $z \in [\alpha a]_I$. Then $d(z, \alpha a) \in I \subseteq J$, this result $z \in [\alpha a]_J = J$. It follows $(\alpha a)/I \subseteq J$. So $\alpha(a/I) \in \underline{\underline{Apr}}_I(J)$. Therefore $\underline{\underline{Apr}}_I(J)$ is an A-ideal of M. □

3 Lower and Upper Approximations with Respect to Fuzzy Congruences

[14] Let M be an A-module. A function θ from $M \times M$ to the unit interval $[0, 1]$ will be called a fuzzy congruence relation on M, if it satisfies the following for $x, y, z \in M$ and $\alpha \in A$:

(C1) $\theta(0, 0) = \theta(x, x)$,

(C2) $\theta(x, y) = \theta(y, x)$,

(C3) $\theta(x, z) \geq \theta(x, y) \wedge \theta(y, z)$,

(C4) $\theta(x \oplus z, y \oplus z) \geq \theta(x, y)$,

(C5) $\theta(x^*, y^*) = \theta(x, y)$,

(C6) $\theta(\alpha x, \alpha y) \geq \theta(x, y)$.

Lemma 34. *[14] If θ is a fuzzy congruence in M, then $\theta(0, 0) \geq \theta(x, y)$, for all $x, y \in M$.*

Let θ and ϕ be two fuzzy relations on M. Then the product $\theta \circ \phi$ is defined by

$$(\theta \circ \phi)(a, b) = \sup_{x \in M}[\min\{\theta(a, x), \phi(x, b)\}]$$

for all $a, b \in M$.

Let θ be a fuzzy congruence relation on M. For each $a \in M$, we define a fuzzy subset θ^a as follows:

$$\theta^a(x) = \theta(a, x)$$

for all $x \in M$. This fuzzy subset θ^a is called a fuzzy congruence class containing $a \in M$. We set

$$M/\theta = \{\theta^a : a \in M\}$$

is called a fuzzy quotient set by θ.

Lemma 35. *Let θ be a fuzzy congruence relation on an A-module M. Then*

$$\theta^{-1}(s) = \{(a, b) \in M \times M : \theta(a, b) = \theta(0, 0) = s\}$$

is a congruence relation on M.

Proof. It is clear that $\theta^{-1}(s)$ is reflexive and symmetric. To prove that $\theta^{-1}(s)$ is transitive, let $(a, b), (b, c) \in \theta^{-1}(s)$. Then $\theta(a, b) = \theta(b, c) = s$. Since θ is a fuzzy congruence relation on M, we have

$$\theta(a, c) \geq \theta(a, b) \wedge \theta(b, c) = s = \theta(0, 0),$$

hence $\theta(a, c) = \theta(0, 0) = s$, so $(a, c) \in \theta^{-1}(s)$, and $\theta^{-1}(s)$ is transitive. Thus $\theta^{-1}(s)$ is an equivalence relation on M.

Now, let $(a, b) \in \theta^{-1}(s)$ and $(c, d) \in \theta^{-1}(s)$. Hence $\theta(a, b) = \theta(c, d) = s$. Since θ is a fuzzy congruence relation on M, we have

$$\theta(c\oplus a, b\oplus d) \geq \theta(c\oplus a, d\oplus a)\wedge\theta(a\oplus d, b\oplus d) \geq \theta(c, d)\wedge\theta(a, b) = \theta(0, 0)\wedge\theta(0, 0) = \theta(0, 0) = s.$$

Hence $\theta(c \oplus a, b \oplus d) = \theta(0, 0)$, so $(a, b) \oplus (c, d) \in \theta^{-1}(s)$.

Let $(a, b) \in \theta^{-1}(s)$. Since θ is a fuzzy congruence relation on M, we have $s = \theta(a, b) = \theta(a^*, b^*)$, this results $(a^*, b^*) \in \theta^{-1}(s)$.

Let $(a, b) \in \theta^{-1}(s)$, and $\alpha \in A$. Then, since θ is a fuzzy congruence relation on M, we have

$$\theta(\alpha a, \alpha b) \geq \theta(a, b) = \theta(0, 0) = s,$$

and so $\theta(\alpha a, \alpha b) = s$. Similarly, we have $(a\alpha, b\alpha) \in \theta^{-1}(s)$. Therefore we obtain $\theta^{-1}(s)$ is a congruence relation on M. □

Theorem 36. *Let θ and ϕ be fuzzy congruence relations on an A-module M. Then $\theta \cap \phi$ is a fuzzy congruence relation on M, and*

$$(\theta \cap \phi)^{-1}(s) = \theta^{-1}(s) \cap \phi^{-1}(s), \text{ where } s = \theta(0, 0) = \phi(0, 0).$$

Proof. It can be easily proved that $\theta \cap \phi$ is a fuzzy congruence relation on M. Let $(a, b) \in (\theta \cap \phi)^{-1}(s)$. Then we have

$$\min\{\theta(a, b), \phi(a, b)\} = (\theta \cap \phi)(a, b) = s,$$

and so

$$\theta(a, b) = \phi(a, b) = s.$$

Thus $(a, b) \in \theta^{-1}(s)$ and $(a, b) \in \phi^{-1}(s)$, and so

$$(a, b) \in \theta^{-1}(s) \cap \phi^{-1}(s).$$

Therefore we obtain that

$$(\theta \cap \phi)^{-1}(s) \subseteq \theta^{-1}(s) \cap \phi^{-1}(s).$$

Conversely, let $(a, b) \in \theta^{-1}(s) \cap \phi^{-1}(s)$. Then

$$(a, b) \in \theta^{-1}(s) \text{ and } (a, b) \in \phi^{-1}(s).$$

Thus we have

$$\theta(a, b) = \phi(a, b) = s.$$

Then we have

$$(\theta \cap \phi)(a, b) = \min\{\theta(a, b), \phi(a, b)\} = \min\{s, s\} = s,$$

and so

$$(a,b) \in (\theta \cap \phi)^{-1}(s).$$

Therefore we have

$$\theta^{-1}(s) \cap \phi^{-1}(s) \subseteq (\theta \cap \phi)^{-1}(s).$$

Thus we obtain that

$$(\theta \cap \phi)^{-1}(s) = \theta^{-1}(s) \cap \phi^{-1}(s).$$ □

Theorem 37. *Let ρ and λ be congruence relations on an A-module M. If X is a non-empty subset of M, then*

$$\overline{(\rho \cap \lambda)}_I(X) \subseteq \overline{\rho}_I(X) \cap \overline{\lambda}_I(X).$$

Proof. Note that $\rho \cap \lambda$ is also a congruence relation on M. Let $c \in \overline{(\rho \cap \lambda)}_I(X)$. Then

$$[c]_{\rho\cap\lambda} \cap X \neq \emptyset.$$

Then there exists an element $a \in [c]_{\rho\cap\lambda} \cap X$. Since $(a,c) \in \rho \cap \lambda$, we have $(a,c) \in \rho$ and $(a,c) \in \lambda$. Thus we have $a \in [c]_\rho$ and $a \in [c]_\lambda$. Since $a \in X$, we have

$$a \in [c]_\rho, \quad a \in X, \text{ and } a \in [c]_\lambda, \quad a \in X.$$

This implies that

$$c \in \overline{\rho}_I(X) \text{ and } c \in \overline{\lambda}_I(X),$$

and so

$$c \in \overline{\rho}_I(X) \cap \overline{\lambda}_I(X).$$

Thus we get

$$\overline{(\rho \cap \lambda)}_I(X) \subseteq \overline{\rho}_I(X) \cap \overline{\lambda}_I(X).$$ □

Theorem 38. *Let ρ and λ be congruence relations on an A-module M. If M is a non-empty subset of M, then*

$$\underline{(\rho \cap \lambda)}_I(X) = \underline{\rho}_I(X) \cap \underline{\lambda}_I(X).$$

Proof.

$$\begin{aligned} c \in \underline{(\rho \cap \lambda)}_I(X) \quad &\Leftrightarrow \quad [c]_{\rho\cap\lambda} \subseteq X, \\ &\Leftrightarrow \quad [c]_\rho \subseteq X \text{ and } [c]_\lambda \subseteq X, \\ &\Leftrightarrow \quad c \in \underline{\rho}_I(X) \text{ and } c \in \underline{\lambda}_I(X), \\ &\Leftrightarrow \quad c \in \underline{\rho}_I(X) \cap \underline{\lambda}_I(X). \end{aligned}$$

Thus we obtain that

$$\underline{(\rho \cap \lambda)}_I(X) = \underline{\rho}_I(X) \cap \underline{\lambda}_I(X).$$ □

Theorem 39. *Let θ and ϕ be fuzzy congruence relations on an A-module M and X a non-empty subset of M, where $s = \theta(0,0) = \phi(0,0)$. Then*

(1) $\underline{(\theta \cap \phi)^{-1}(s)}_I(X) = \underline{(\theta^{-1}(s) \cap \phi^{-1}(s))}_I(X) = \underline{\theta^{-1}(s)}_I(X) \cap \underline{\phi^{-1}(s)}_I(X).$

(2) $\overline{(\theta \cap \phi)^{-1}(s)}_I(X) = \overline{(\theta^{-1}(s) \cap \phi^{-1}(s))}_I(X) \subseteq \overline{\theta^{-1}(s)}_I(X) \cap \overline{\phi^{-1}(s)}_I(X).$

Proof. Those follow from Theorem 36, Theorem 37 and Theorem 38. □

Let α and β be binary relations on an A-module M. Then the product $\alpha\cdot\beta$ of α and β is defined as follows:

$$\alpha\cdot\beta=\{(a,b)\in M\times M:(a,c)\in\alpha \text{ and } (c,d)\in\beta \text{ for some } c\in M\}.$$

Assume α and β are congruence relations on an A-module M. Then, we can easily prove that $\alpha\cdot\beta$ is a congruence if and only if $\alpha\cdot\beta=\beta\cdot\alpha$.

Theorem 40. *Let ρ and λ be congruence relations on a linearly ordered A-module M such that $\rho\cdot\lambda=\lambda\cdot\rho$. If M is an A-module of M and X is an A-ideal of M, then*

$$\overline{\rho}_I(X)\vee\overline{\lambda}_I(X)\subseteq\overline{(\rho\cdot\lambda)}_I(X).$$

Proof. Let c be any element of $\overline{\rho}_I(X)\vee\overline{\lambda}_I(X)$. Then $c\le a\oplus b$ with $a\in\overline{\rho}_I(X)$ and $b\in\overline{\lambda}_I(X)$. Then there exist elements $x,y\in M$ such that

$$x\in[a]_\rho\cap X \text{ and } y\in[b]_\lambda\cap X.$$

Thus $x\in[a]_\rho$, $y\in[b]_\lambda$, and $x,y\in X$. Since X is an A-ideal of M, we have $x\oplus y\in X$. Then $(x,a)\in\rho$ and $(y,b)\in\lambda$, and since ρ and λ are congruence relations, we have

$$(x\oplus y,a\oplus y)\in\rho \text{ and } (a\oplus y,a\oplus b)\in\lambda.$$

Thus we have $(x\oplus y,a\oplus b)\in\rho\cdot\lambda$, and so $x\oplus y\in[a\oplus b]_{\rho\cdot\lambda}$. Therefore we have

$$x\oplus y\in[a\oplus b]_{\rho\cdot\lambda}\cap X,$$

which yields

$$c\le a\oplus b\in(\overline{\rho\cdot\lambda})_I(X).$$

Since by Proposition 31 (ii), $(\overline{\rho\cdot\lambda})_I(X)$ is an A-ideal, we obtain that $c\in(\overline{\rho\cdot\lambda})_I(X)$. Hence we have

$$\overline{\rho}_I(X)\vee\overline{\lambda}_I(X)\subseteq\overline{(\rho\cdot\lambda)}_I(X).$$

□

We note that if θ and ϕ are fuzzy congruence relations, then $\theta\circ\phi$ is a fuzzy congruence relation on M if and only if $\theta\circ\phi=\phi\circ\theta$ (see [14]).

Theorem 41. *Let θ and ϕ be fuzzy congruence relations on an A-module M such that $\theta\circ\phi=\phi\circ\theta$, where $\theta(0,0)=\phi(0,0)$. Then*

$$\theta^{-1}(s)\cdot\phi^{-1}(s)\subseteq(\theta\circ\phi)^{-1}(s).$$

Proof. Let $(a,b)\in\theta^{-1}(s)\cdot\phi^{-1}(s)$. Then there exists an element $c\in M$ such that $(a,c)\in\theta^{-1}(s)$ and $(c,b)\in\phi^{-1}(s)$. Then, since

$$\theta(a,c)=\phi(c,b)=s,$$

we have

$$\begin{aligned}(\theta\circ\phi)(a,b)&=\sup_{x\in M}[\min\{\theta(a,x),\phi(x,b)]\\&\ge\min\{\theta(a,c),\phi(c,b)\}\\&=\min\{s,s\}\\&=s.\end{aligned}$$

and so $(\theta\circ\phi)(a,b)=s$. This implies that $(a,b)\in(\theta\circ\phi)^{-1}(s)$. Thus we obtain that

$$\theta^{-1}(s)\cdot\phi^{-1}(s)\subseteq(\theta\circ\phi)^{-1}(s).$$

□

Remark 42. Let ρ and λ be congruence relations on A-module M. If X and Y are nonempty subsets of M, then the following hold:

(i) $\rho \subseteq \lambda$ implies $\underline{\rho}_I(X) \supseteq \underline{\lambda}_I(Y)$,

(ii) $\rho \subseteq \lambda$ implies $\overline{\rho}_I(X) \subseteq \overline{\lambda}_I(Y)$.

Theorem 43. *Let θ and ϕ be fuzzy congruence relations on an A-module M such that $\theta \circ \phi = \phi \circ \theta$. If X is a nonempty subset of M, then*

(1) $\underline{(\theta^{-1}(s) \cdot \phi^{-1}(s))}_I(X) \supseteq \underline{(\theta \circ \phi)^{-1}(s)}_I(X)$.

(2) $\overline{\theta^{-1}(s) \cdot \phi^{-1}(s))}_I(X) \subseteq \overline{(\theta \circ \phi)^{-1}(s)}_I(X)$.

Proof. Those follow from Theorem 41 and Remark 42 (i), (ii). □

Theorem 44. *Let θ and ϕ be fuzzy congruence relations on a linearly ordered A-module M such that $\theta \circ \phi = \phi \circ \theta$, where $\theta(0,0) = \phi(0,0) = s$. If X is an A-ideal of M, then*

$$(\overline{\theta^{-1}(s)}_I(X)) \vee (\overline{\phi^{-1}(s)}_I(X)) \subseteq \overline{(\theta \circ \phi)^{-1}(s)}_I(X).$$

Proof. Let c be any element of $\overline{\theta^{-1}(s)}_I(X) \vee (\overline{\phi^{-1}(s)}_I(X)$. Then $c \leq a \oplus b$ with $a \in \overline{\theta^{-1}(s)}_I(X)$ and $b \in (\overline{\phi^{-1}(s)}_I(X)$. Then there exist elements $x, y \in M$ such that

$$x \in \theta^a \cap X \text{ and } y \in \phi^b \cap X.$$

This implies that

$$(a, x) \in \theta^{-1}(s) \text{ and } (b, y) \in \phi^{-1}(s),$$

and $x, y \in X$. Then we have

$$\theta(a, x) = \phi(b, y) = s.$$

Since θ and ϕ are fuzzy congruence relations on M, we have and so

$$(\theta \circ \phi)(a \oplus b, x \oplus y) = s.$$

Note that, since X is an A-ideal of M, thus we have

$$x \oplus y \in (\theta \circ \phi)^{a \oplus b} \cap X.$$

This implies that

$$c \leq a \oplus b \in \overline{(\theta \circ \phi)^{-1}(s)}_I(X).$$

Since by Proposition 31(ii), $\overline{(\theta \circ \phi)^{-1}(s)}_I(X)$ is an A-ideal, we obtain that $c \in \overline{(\theta \circ \phi)^{-1}(s)}_I(X)$. We get $(\overline{\theta^{-1}(s)}_I(X)) \vee (\overline{\phi^{-1}(s)}_I(X)) \subseteq \overline{(\theta \circ \phi)^{-1}(s)}_I(X)$. □

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