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On Ultimate Boundedness and Existence of Periodic Solutions of Kind of Third Order Delay Differential Equations

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Abstract
In this paper we first study the problem of uniform ultimate boundedness of a certain third order nonlinear differential equation with delay. Further the existence of periodic solutions for the considered equation are also given, as a consequence of uniform ultimate boundedness results. Finally, some criteria to guarantee the uniform asymptotic stability are derived via the Lyapunov's second method. We also give an example to illustrate our results.

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1 Introduction

Nonlinear differential equations of higher order have been extensively studied with high degree of generality. In particular, boundedness, uniform boundedness, ultimate boundedness, uniform ultimate boundedness and asymptotic behavior of solutions have in the past and also recently been discussed. See for instance Reissig et al. [13], Rouche et al. [19], Yoshizawa [22] and [23]. It is well known that the ultimate boundedness is a very important problem in the theory and applications of differential equations. An effective method for studying the ultimate boundedness of nonlinear differential equations is still the Lyapunov's direct method.

Because of their applications, the existence of periodic solutions of third order differential equations has been also investigated by many researchers in recent years. Besides it is worth-mentioning that there are a few results on the same topic for third order delay differential equations, for example, Chukwu [6], Gui[10], Tunç [21] and Zhu[24].

In 1992, Zhu[24], established some sufficient conditions to ensure the stability, boundedness, ultimate boundedness of the solutions of the following third order non-linear delay differential equation

$$x''' + ax'' + bx' + f(x(t-r)) = e(t). \tag{1.1}$$

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The existence of periodic solutions was also discussed in the case where e(t) is a periodic function.
Recently, in [8], the authors extend results obtained in [24] to the following third order non autonomous differential equation with delay

$$[g(x(t))x'(t)]'' + (h(x(t))x'(t))' + \varphi(x(t))x'(t) + f(x(t-r)) = e(t), \tag{1.2}$$

In this paper, we are concerned with the third order delay differential equation

$$\Big(q(t)\big(g(x(t))x'(t)\big)'\Big)' + a(t)\big(h(x(t))x'(t)\big)' + b(t)\varphi(x(t))x'(t) + c(t)f(x(t-r)) = e(t), \tag{1.3}$$

where $r > 0$ is a fixed delay and $a,\ b,\ c,\ e,\ f,\ g,\ h,$ and φ are continuous functions and depend only on the arguments shown explicitly; $f(0) = 0;\ f'(x), g'(x),\ h'(x),$ and $\varphi'(x)$ exist and are continuous for all x. Our objective here is to extend results obtained in [8] to (1.3). The paper is organized as follows. In section 3 we study the problems of the boundedness and ultimate boundedness of solutions when $e(t) \neq 0$. The assumptions will also give us an opportunity to discuss the existence of periodic solutions of the same equation when $a,\ b,\ c,\ e,\ q,$ are periodic functions. Finally we investigate the asymptotic stability of the zero solution of the delay differential equation (1.3) with $e(t) = 0$. We give an example to illustrate the effectiveness of main results obtained in Section 3.
Clearly the equation discussed by Zhu in [24] is a special case of equation (1.3) when $g(x) = h(x) = \varphi(x) = 1$, $a(x) = a$, and $b(t) = b$, also (1.2) is a special case of (1.3) with $q(t) = 1$.

2 Preliminaries

To describe the main result of this paper, we include some preliminary knowledge on the stability and ultimate boundedness for a general class of nonlinear delay differential system

$$x' = f(t, x_t),\quad x_t(\theta) = x(t+\theta)\ ,\quad -r \le \theta \le 0,\quad t \ge 0, \tag{2.1}$$

where $f : C_H \to \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$, $C_H := \{\phi \in (C[-r, 0],\ \mathbb{R}^n) : \|\phi\| \le H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(\phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Lemma 1. *[12] If there is a continuous functional* $V(t, \phi) : [0, +\infty[\times C_H \to [0, +\infty[$ *locally Lipschitz in* ϕ *and wedges* W_i *such that:*
(i) *If* $W_1(\|\phi\|) \le V(t, \phi)$, $V(t, 0) = 0$ *and* $V'_{(2,1)}(t, \phi) \le 0$.

Then, the zero solution of (2.1) is stable. If in addition $V(t, \phi) \le W_2(\|\phi\|)$. *Then, the zero solution of (2.1) is uniformly stable.*
(ii) *If* $W_1(\|\phi\|) \le V(t, \phi) \le W_2(\|\phi\|)$ *and* $V'_{(2,1)}(t, \phi) \le -W_3(\|\phi\|)$.

Then, the zero solution of (2.1) is uniformly asymptotically stable.

Definition 2. [4] Solutions of (2.1) are uniform ultimate bounded for bound B at $t = 0$ if for each $A > 0$ there is a $K > 0$ such that $\phi \in C_H,\ ||\phi|| < A,\ t \ge K$ imply that $x(t, 0, \phi) < A$.

Lemma 3. *[4] Let* $V(t, \varphi) : \mathbb{R} \times C \to \mathbb{R}$ *be continous and locally Lipschitz in* φ. *If*

$$i)\qquad W_0(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2(\int_{t-r}^{t} W_3(|x(t)|)ds),$$

$$ii)\qquad V'_{(2.1)} \le -W_3(|x(t)|) + M,$$

for some $M > 0$, where $W_i(i = 0, 1, 2, 3)$ are wedges, then the solutions of (2.1) are uniformly bounded and uniformly ultimately bounded for bound B.

If (2.1) is periodic system with period T, we have the following result:

Lemma 4. *[20] Suppose that, for $\alpha > 0$, there exists $L(\alpha) > 0$ such that $|f(t, x_t)| \leq L(\alpha)$, for $t \in [-T, 0]$ and $\|x_t\| \leq \alpha$, and suppose that the solutions of (2.1) are equi-bounded and equi-ultimately bounded for bound B, then there exists a periodic solution of (2.1) of period T.*

3 Main Results

We shall give here some assumptions which will be used on the functions that appeared in equation (1.3). Suppose that there are positive constants $a_0, a_1, b_0, b_1, c_0, c_1, g_0, g_1, h_0, h_1, \varphi_0, \varphi_1, \delta_0, \delta_1, \mu_1$ and μ_2 such that the following conditions are satisfied:

i) $0 < a_0 \leq q(t) \leq a(t) \leq a_1,\ 0 < b_0 \leq b(t) \leq b_1,\ 0 < c_0 \leq c(t) \leq c_1.$

ii) $\int_{-\infty}^{+\infty}(|q'(u)| + |a'(u)| + |b'(u)| + |c'(u)|)du < \infty.$

iii) $0 < g_0 \leq g(x) \leq g_1,\ \ 0 < h_0 \leq h(x) \leq h_1,\ \ 0 < \varphi_0 \leq \varphi(x) \leq \varphi_1.$

iv) $\int_{-\infty}^{+\infty}(|g'(u)| + |h'(u)| + |\varphi'(u)|)du < \infty.$

v) $f(0) = 0, \dfrac{f(x)}{x} \geq \delta_0 > 0\ (x \neq 0)$, and $|f'(x)| \leq \delta_1$ for all x.

vi) $\dfrac{c_1 g_1 \delta_1}{b_0 \varphi_0} < \mu_1 < \dfrac{a_0 h_0}{a_1}$ and

$$\mu_2 = \min\left\{1,\ 2D_2,\ \frac{2g_0^2 D_1}{g_1^2\big(2a_1h_1 + 2 + b_1\varphi_1\big)}\right\}, \text{ where}$$

$$D_1 = \frac{a_0\big(\mu_1 b_0 \varphi_0 - c_1\delta_1 g_1\big)}{g_1^2} > 0,\ D_2 = \frac{a_0h_0 - \mu_1 a_1}{a_1 g_1} > 0,$$

Before stating theorems, let us introduce the following notations:

$$\begin{aligned}
\Theta_1(t) &= \frac{1}{\beta_1}\Big(\frac{1}{g(x(t))}\Big)', \Theta_2(t) = \frac{1}{\beta_1}\Big(\frac{h(x(t))}{g(x(t))}\Big)',\ \Theta_3(t) = \frac{1}{\beta_1}\Big(\frac{\varphi(x(t))}{g(x(t))}\Big)', \\
\Theta_4(t) &= \frac{1}{\beta_1}\Big(\frac{q(t)}{g(x(t))}\Big)',\ \Theta_5(t) = \Big(|q'(t)| + |a'(t)| + |b'(t)| + |c'(t)|\Big), \\
\Theta_6(t) &= \frac{1}{\beta_3}\Big(h(x(t))\Big)',
\end{aligned} \tag{3.1}$$

and

$$\Omega(t) = \int_0^t \Big[|\Theta_1(s)| + |\Theta_2(s)| + |\Theta_3(s)| + |\Theta_4(s)| + (\frac{1}{\beta_1} + \frac{1}{\beta_2})\Theta_5(s) + |\Theta_6(s)|\Big]ds.$$

Also,

$$\begin{aligned}
\gamma_1 &= \max\Big\{\frac{a_1\varphi_1}{2g_0},\ \frac{a_1}{2\mu_1},\ \frac{\mu_1 a_1 h_1}{g_0^2}+\frac{b_1\varphi_1}{2g_0}+\frac{c_1}{2\mu_1}\Big\},\\
\gamma_2 &= \max\Big\{\frac{h_1}{g_0}(\frac{\mu_2+1}{2}+\frac{a_1h_1}{g_0}),\ \frac{a_1\varphi_1}{2g_0},\ \frac{a_1\alpha}{2},\ \frac{\varphi_1 b_1}{2g_0}+\frac{c_1\alpha}{2}\Big\},\ \text{such that } \alpha=\frac{a_0b_0\varphi_0}{c_1g_1a_1},\\
D_3 &= c_0\delta_0-\frac{(1+b_1\varphi_1)}{2}, D_4=\frac{a_0b_0h_0\varphi_0-a_1c_1\delta_1g_1}{g_0^2}>0.
\end{aligned}$$

Now, our main result on the boundedness and ultimate boundedness of (1.3) with $e(t)\neq 0$.

Theorem 5. *If hypotheses (i)-(vi) hold true, and in addition the following conditions are satisfied*

vii) $|e(t)|\leq m$,

viii) $D_3>0$.

Then all solutions of (1.3) are uniformly bounded and uniformly ultimately bounded provided r *satisfies*

$$r<\min\Big\{\ \frac{2D_2-\mu_2}{\delta_1c_1},\ \frac{2D_3}{\delta_1c_1},\ \frac{2g_0^3D_4}{\delta_1c_1[g_0(2+\mu_2)+a_1(\mu_1+h_1)(1+g_0^2)]}\Big\}. \tag{3.2}$$

Proof. We write the equation (1.3) as the following equivalent system

$$\begin{cases}
x'=\dfrac{1}{g(x)}y,\\
y'=\dfrac{1}{q(t)}z,\\
z'=-\dfrac{a(t)h(x)}{q(t)g(x)}z-a(t)\Theta_2(t)y-\dfrac{b(t)\varphi(x)}{g(x)}y-c(t)f(x)+e(t)\\
\qquad +c(t)\displaystyle\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds.
\end{cases} \tag{3.3}$$

Note that the continuity of the functions a, b, c, e, f, g, h, φ, f', g', h', and φ' guarantees the existence of the solutions of (1.3) (see [7], pp.15). It is assumed that the right hand side of the system (3.3) satisfies a Lipschitz condition in $x(t), y(t), z(t)$, and $x(t-r)$. This assumption guarantees the uniqueness of solutions of (1.3) (see [7], pp.15). We shall use as a tool to prove our main results a Lyapunov function $W=W(t,x_t,y_t,z_t)$ defined by

$$W(t,x_t,y_t,z_t)=e^{-\Omega(t)}V(t,x_t,y_t,z_t)=e^{-\Omega(t)}V, \tag{3.4}$$

where

$$V=V_1(t,x_t,y_t,z_t)+V_2(t,x_t,y_t,z_t)+\lambda\int_{-r}^{0}\int_{t+s}^{t}y^2(\xi)d\xi ds,$$

$$\begin{aligned}
V_1 &= \mu_1q(t)c(t)G(x,y)+\frac{\mu_1q(t)}{2}(\frac{a(t)h(t)-\mu_1q(t)}{g^2(x)})y^2+\frac12(z+\frac{\mu_1q(t)}{g(x)}y)^2\\
&\quad+\frac{q(t)}{2}(\frac{\varphi(t)b(t)}{g(x)}-\frac{c(t)\delta_1}{\mu_1})y^2,
\end{aligned} \tag{3.5}$$

$$\begin{aligned} V_2 &= a(t)c(t)h(x)F(x) - \frac{q(t)c^2(t)g(x)}{2b(t)\varphi(x)}f^2(x) + \frac{1}{2}\left(z + \frac{a(t)h(x)}{g(x)}y + \mu_2 x\right)^2 \\ &\quad + \frac{q(t)b(t)\varphi(x)}{2g(x)}\left(y + \frac{c(t)f(x)g(x)}{b(t)\varphi(x)}\right)^2 + \frac{1}{2}\mu_2(1-\mu_2)x^2, \end{aligned} \tag{3.6}$$

such that $F(x) = \int_0^x f(u)du$ and $G(x,y) = F(x) + \frac{1}{\mu_1}f(x)y + \frac{\delta_1}{2\mu_1^2}y^2$. λ is positive constant which will be specified later in the proof. We easily rearrange the above functional V_1 as follows

$$\begin{aligned} V_1 &= \mu_1 q(t)c(t)F(x) + \frac{q(t)b(t)\varphi(x)}{2g(x)}\left(y + \frac{c(t)f(x)g(x)}{b(t)\varphi(x)}\right)^2 - \frac{q(t)c^2(t)g(x)f^2(x)}{2b(t)\varphi(x)} \\ &\quad + \frac{1}{2}(z + \frac{\mu_1 q(t)}{g(x)}y)^2 + \frac{\mu_1 q(t)(a(t)h(x) - \mu_1 q(t))}{2g^2(x)}y^2. \end{aligned} \tag{3.7}$$

Using (i), (iii) and (vi) we have

$$\frac{\mu_1 q(t)(a(t)h(x) - \mu_1 q(t))}{2g^2(x)} \geq \frac{\mu_1 a_0(a_0 h_0 - \mu_1 a_1)}{2g^2(x)} > 0.$$

Thus there exists a positive constant δ_2 such that

$$\frac{1}{2}(z + \frac{\mu_1 q(t)}{g(x)}y)^2 + \frac{\mu_1 q(t)(a(t)h(x) - \mu_1 q(t))}{2g^2(x)}y^2 \geq \delta_2 y^2 + \delta_2 z^2. \tag{3.8}$$

On the other hand, using the assumptions (i), (iii), (v) and (vi) we obtain

$$\begin{aligned} \mu_1 q(t)c(t)F(x) - \frac{q(t)c^2(t)g(x)f^2(x)}{2b(t)\varphi(x)} &\geq \mu_1 q(t)c(t)\int_0^x (1 - c(t)\frac{g(x)f'(u)}{\mu_1 b(t)\varphi(x)})f(u)du \\ &\geq \mu_1 a_1 c_1 \int_0^x (1 - \frac{g_1 c_1 \delta_1}{\mu_1 b_0 \varphi_0})f(u)du \\ &\geq \delta_3 F(x), \end{aligned}$$

where $\delta_3 = \mu_1 a_1 c_1 (1 - \frac{g_1 c_1 \delta_1}{\mu_1 b_0 \varphi_0}) > 0$. Hence, from the last inequality, (3.8) and (3.7),

$$V_1 \geq \delta_3 F(x) + \delta_2 y^2 + \delta_2 z^2. \tag{3.9}$$

Clearly, using hypothesis (v) we have the following estimate

$$V_1 \geq \frac{\delta_3 \delta_0}{2}x^2 + \delta_2 y^2 + \delta_2 z^2. \tag{3.10}$$

By adding and subtracting some terms together with condition (i) we can estimate the functional V_2 above thus

$$\begin{aligned} V_2 &\geq q(t)c(t)H(x,y) + \frac{1}{2}\left(z + \frac{h(x)}{g(x)}y + \mu_2 x\right)^2 + \frac{1}{2}\mu_2(1-\mu_2)x^2 \\ &\quad + \frac{q(t)}{2}\left(\frac{b(t)\varphi(x)}{g(x)} - \alpha c(t)\right)y^2, \end{aligned}$$

where

$$H(x,y) = h(x)F(x) + f(x)y + \frac{\alpha}{2}y^2.$$

From condition (vi) we have $\dfrac{b(t)\varphi(x)}{g(x)} - \alpha c(t) \geq 0$, and $1-\mu_2 \geq 0$, it follows that

$$V_2 \geq q(t)c(t)H(x,y).$$

But

$$\begin{aligned} H(x,y) &= h(x)F(x) + \frac{\alpha}{2}\big(y + \frac{1}{\alpha}f(x)\big)^2 - \frac{1}{2\alpha}f^2(x) \\ &\geq h(x)F(x) - \frac{1}{2\alpha}f^2(x) \\ &\geq \int_0^x \big(h_0 - \frac{\delta_1}{\alpha}\big)f(u)du. \end{aligned}$$

From condition (vi) $H(x,y) \geq 0$. Hence, by (i) we get

$$V_2 \geq a_0c_0H(x,y). \tag{3.11}$$

It is easily seen from (3.6) that

$$\begin{aligned} V_2 &\geq a(t)c(t)h(x)F(x) - \frac{q(t)c^2(t)g(x)}{2b(t)\varphi(x)}f^2(x) \\ &\geq c(t)\big(a_0h_0F(x) - \frac{a_1c(t)g_1}{2b_0\varphi_0}f^2(x)\big) \\ &\geq c_1\int_0^x (a_0h_0 - \frac{a_1g_1c_1\delta_1}{b_0\varphi_0})f(u)du \\ &\geq \delta_4F(x), \end{aligned}$$

where $\delta_4 = c_1\big(a_0h_0 - \dfrac{a_1c_1g_1\delta_1}{b_0\varphi_0}\big) > 0$. Thus from (v) we obtain,

$$V_2 \geq \frac{\delta_4\delta_0}{2}x^2. \tag{3.12}$$

Clearly, from (3.12), (3.10) and the fact that the integral $\int_{-r}^0\int_{t+s}^t y^2(\xi)d\xi ds$ is positive, we deduce that

$$V \geq \delta_2y^2 + \delta_2z^2 + \frac{\delta_5\delta_0}{2}x^2,$$

where $\delta_5 = \delta_3 + \delta_4$. Further simplification of the last inequality gives

$$V \geq k(x^2+y^2+z^2), \tag{3.13}$$

where $k = min\{\delta_2; \dfrac{\delta_5\delta_0}{2}\}$. In view of the hypotheses (i)-(iv) we have

$$\begin{aligned}
\Omega(t) &= \int_0^t \left[|\Theta_1(s)| + |\Theta_2(s)| + |\Theta_3(s)| + |\Theta_4(s)| + (\frac{1}{\beta_1} + \frac{1}{\beta_2})\Theta_5 + |\Theta_6(s)| \right] ds \\
&\le \frac{(1+\varphi_1+h_1+a_1)}{\beta_1} \int_{\sigma_1(t)}^{\sigma_2(t)} \frac{|g'(u)|}{g^2(u)} du + \frac{1}{\beta_1} \int_{\sigma_1(t)}^{\sigma_2(t)} \frac{|\varphi'(u)| + |h'(u)|}{g(u)} du \\
&\quad + \frac{1}{\beta_3} \int_{\sigma_1(t)}^{\sigma_2(t)} |h'(u)|\, du + \frac{1}{\beta_1 g_0} \int_0^t |q'(u)| du \\
&\quad + (\frac{1}{\beta_1} + \frac{1}{\beta_2}) \int_0^t (|q'(u)| + |a'(u)| + |b'(u)| + |c'(u)|) du \\
&\le \frac{(1+\varphi_1+h_1+a_1)}{\beta_1 g_0^2} \int_{-\infty}^{+\infty} |g'(u)|\, du + \frac{1}{\beta_1 g_0} \int_{-\infty}^{+\infty} \big(|\varphi'(u)| + |h'(u)|\big) du \\
&\quad + \frac{1}{\beta_3} \int_{-\infty}^{+\infty} |h'(u)|\, du + \frac{1}{\beta_1 g_0} \int_{-\infty}^{+\infty} |q'(u)| du \\
&\quad + (\frac{1}{\beta_1} + \frac{1}{\beta_2}) \int_{-\infty}^{+\infty} (|q'(u)| + |a'(u)| + |b'(u)| + |c'(u)|) du \le N < \infty,
\end{aligned}$$

where $\sigma_1(t) = \min\{x(0), x(t)\}$, and $\sigma_2(t) = \max\{x(0), x(t)\}$. Therefore we can find a continuous function $W_1(|\Phi(0)|)$ with

$$W_1(|\Phi(0)|) \ge 0 \quad \text{and} \quad W_1(|\Phi(0)|) \le W(t, \Phi).$$

The existence of a continuous function $W_2(\|\phi\|)$ which satisfies the inequality $W(t, \phi) \le W_2(\|\phi\|)$, is easily verified.

For the time derivative of the Lyapunov functional $V(t, x_t, y_t, z_t)$, along the trajectories of the system (3.3), we have

$$V'_{(3.3)} = V'_{1_{(3.3)}} + V'_{2_{(3.3)}} + \lambda r y^2 - \lambda \int_{t-r}^t y^2(\xi) d\xi,$$

where

$$\begin{aligned}
V'_{1_{(3.3)}} &= \mu_1 \big(q(t)c(t)\big)' G(x,y) + \left[\frac{q(t)c(t)g(x)f'(x) - \mu_1 q(t)b(t)\varphi(x)}{g^2(x)} \right] y^2 \\
&\quad + \left[\frac{\mu_1 q(t) - a(t)h(x)}{q(t)g(x)} \right] z^2 - \frac{\mu_1 a(t)q(t)}{2} \frac{h(x)}{g(x)} \Theta_1(t) y^2 \\
&\quad + a(t)\Big(yz + \mu_1 q(t)\big(1 - \frac{1}{g(x)}\big) y^2 \Big) \Theta_2(t) + \frac{q(t)b(t)}{2} \Theta_3(t) y^2 + \mu_1 \Theta_4(t) yz \\
&\quad + \mu_1 \big(a(t)q(t)\big)' \frac{h(x)}{g^2(x)} y^2 + \frac{1}{2} \big(b(t)q(t)\big)' \frac{\varphi(x)}{g(x)} y^2 - \frac{1}{2\mu_1} \big(c(t)q(t)\big)' y^2 \\
&\quad + \mu_1 \frac{q(t)}{g(x)} e(t) y + e(t) z + c(t) \Big(z + \frac{\mu_1 q(t)}{g(x)} y \Big) \int_{t-r}^t \frac{y(s)}{g(x(s))} f'(x(s)) ds.
\end{aligned}$$

In view of conditions (i), (iii) and (v) we get

$$\begin{aligned} V'_{1_{(3.3)}} &\leq \mu_1 A\Theta_5(t)G(x,y) - D_1y^2 - D_2z^2 + \gamma_1\Theta_5(t)y^2 \\ &+\frac{\mu_1 a_1^2}{2}\frac{h_1}{g_0}|\Theta_1(t)|y^2 + \frac{a_1b_1}{2}|\Theta_3(t)|y^2 + \Big(a_1|yz| + \mu_1 a_1^2(1+\frac{1}{g_0})y^2\Big)|\Theta_2(t)| \\ &+\mu_1|\Theta_4(t)||yz| + \mu_1\frac{a_1}{g_0}|y|m + |z|m \\ &+c(t)\left(z + \frac{\mu_1 q(t)}{g(x)}y\right)\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds, \end{aligned}$$

where $A = \max\{a_1, c_1\}$. Using the Schwartz inequality $|uv| \leq \frac{1}{2}(u^2+v^2)$, we obtain

$$\begin{aligned} &\frac{\mu_1 a_1^2}{2}\frac{h_1}{g_0}|\Theta_1(t)|y^2 + \frac{a_1b_1}{2}|\Theta_3(t)|y^2 + \Big(a_1|yz| + \mu_1 a_1^2(1+\frac{1}{g_0})y^2\Big)|\Theta_2(t)| \\ &\qquad +\mu_1|\Theta_4(t)||yz| \leq k_1\Big[|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)|\Big](y^2+z^2), \end{aligned}$$

where $k_1 = max\{\frac{\mu_1}{2}(1+\frac{h_1}{g_0}), \frac{1}{2}(1+\frac{\mu_1}{g_0}), \frac{1}{2}\}$.
From condition (v) and the Schwartz inequality, we obtain the following

$$c(t)\frac{\mu_1 q(t)}{g(x)}y\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds \leq \frac{\delta_1\mu_1 a_1 c_1 r}{2g_0}y^2 + \frac{\mu_1 a_1 c_1\delta_1}{2g_0^3}\int_{t-r}^{t}y^2(\xi)d\xi, \tag{3.14}$$

and

$$c(t)z\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds \leq \frac{\delta_1 c_1 r}{2}z^2 + \frac{\delta_1 c_1}{2g_0^2}\int_{t-r}^{t}y^2(\xi)d\xi.$$

After some rearrangements we get

$$\begin{aligned} V'_{1_{(3.3)}} &\leq \mu_1 A\Theta_5(t)G(x,y) - \left[D_1 - \frac{\mu_1 a_1\delta_1 c_1}{2g_0}r\right]y^2 - \left[D_2 - \frac{\delta_1 c_1 r}{2}\right]z^2 \\ &+\gamma_1\Theta_5(t)y^2 + k_1\Big[|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)|\Big](y^2+z^2) \\ &+\mu_1\frac{a_1}{g_0}|y|m + |z| + \frac{\delta_1 c_1}{2g_0^2}(1+\frac{\mu_1 a_1}{g_0})\int_{t-r}^{t}y^2(\xi)d\xi. \end{aligned} \tag{3.15}$$

In addition,

$$\begin{aligned} V'_{2_{(3.3)}} &= \big(a(t)c(t)\big)'h(x)F(x) + \big(q(t)c(t)\big)'f(x)y + a(t)c(t)\Theta_6(t)F(x) + \mu_2\frac{a(t)h(x)}{g^2(x)}y^2 \\ &+\frac{q(t)c(t)f'(x)g(x) - a(t)b(t)h(x)\varphi(x)}{g^2(x)}y^2 + \frac{\mu_2}{g(x)}\Big(1 - b(t)\varphi(x)\Big)xy + \frac{\mu_2}{g(x)}yz \\ &+\frac{b(t)q(t)}{2}\Theta_3(t)y^2 - \mu_2 c(t)xf(x) + \mu_2 xe(t) + \frac{a(t)h(x)}{g(x)}ye(t) + ze(t) \\ &+a'(t)\frac{h(x)}{g(x)}\big(\mu_2 xy + yz + a(t)\frac{h(x)}{g(x)}y^2\big) + \frac{1}{2}\big(b(t)q(t)\big)'\frac{\varphi(x)}{g(x)}y^2 \\ &+c(t)(\mu_2 x + z + \frac{a(t)h(x)}{g(x)}y)\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds. \end{aligned}$$

We can now proceed analogously to (3.14)

$$\mu_2 c(t)x\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds\leq\frac{\mu_2\delta_1c_1r}{2}x^2+\frac{\mu_2\delta_1c_1}{2g_0^2}\int_{t-r}^{t}y^2(\xi)d\xi,$$

$$\frac{a(t)c(t)h(x)}{g(x)}y\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds\leq\frac{\delta_1a_1c_1h_1r}{2g_0}y^2+\frac{a_1h_1\delta_1c_1}{2g_0^3}\int_{t-r}^{t}y^2(\xi)d\xi,$$

and

$$c(t)z\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds\leq\frac{\delta_1c_1r}{2}z^2+\frac{\delta_1c_1}{2g_0^2}\int_{t-r}^{t}y^2(\xi)d\xi.$$

These estimates and Schwartz inequality imply the following

$$\begin{aligned}
V'_{2_{(3.3)}} \leq\; & \Big[\big(a(t)c(t)\big)'-\big(q(t)c(t)\big)'\Big]h(x)F(x)+\big(q(t)c(t)\big)'H(x,y)\\
&+a(t)c(t)\Theta_6(t)F(x)-\mu_2\Big[c_0\delta_0-\frac{(1+b(t)\varphi(t))}{2}\Big]x^2\\
&+\frac{q(t)c(t)f'(x)g(x)-a(t)b(t)h(x)\varphi(x)}{g^2(x)}y^2\\
&+\frac{\mu_2}{2g^2(x)}\Big(2+b(t)\varphi(x)+2a(t)h(x)\Big)y^2\\
&+\frac{\mu_2}{2}z^2+\frac{b(t)q(t)}{2}\Theta_3(t)y^2+\big(\mu_2|x|+\frac{a(t)h(x)}{g(x)}|y|+|z|\big)m\\
&+a'(t)\frac{h(x)}{g(x)}\big(\mu_2xy+yz+a(t)\frac{h(x)}{g(x)}y^2\big)+\frac{1}{2}\big(b(t)q(t)\big)'\frac{\varphi(x)}{g(x)}y^2\\
&-\frac{\alpha}{2}\big(q(t)c(t)\big)'y^2+\frac{\mu_2\delta_1c_1r}{2}x^2+\frac{\delta_1a_1c_1h_1r}{2g_0}y^2+\frac{\delta_1c_1r}{2}z^2\\
&+\frac{\delta_1c_1}{2g_0^2}\big(\mu_2+\frac{a_1h_1}{g_0}+1\big)\int_{t-r}^{t}y^2(\xi)d\xi.
\end{aligned}$$

It is easy to check that by (i), (iii) and (v) we have

$$\begin{aligned}
\Big[\big(a(t)c(t)\big)'-\big(q(t)c(t)\big)'\Big]h(x)F(x) &\leq \Big[|\big(a(t)c(t)\big)'|+|\big(q(t)c(t)\big)'|\Big]\frac{h_1\delta_1}{2}x^2\\
&\leq \frac{h_1\delta_1B}{2}\Theta_5(t)x^2,
\end{aligned}$$

such that $B=\max\{2a_1,c_1\}$. By conditions (i), (iii) and (v) we have

$$\frac{q(t)c(t)f'(x)g(x)-a(t)b(t)h(x)\varphi(x)}{g^2(x)}\leq\frac{a_1c_1\delta_1g_1-a_0b_0h_0\varphi_0}{g_0^2}<0.$$

Using condition (i) and (iii) again we get

$$\begin{aligned}
V'_{2_{(3.3)}} \le\ & \frac{h_1\delta_1 B}{2}\Theta_5(t)x^2 + A\Theta_5(t)H(x,y) + a_1c_1|\Theta_6(t)|F(x) - \mu_2 D_3 x^2\\
&-\Big[D_4 - \frac{\mu_2}{2g_0^2}\Big(2 + b_1\varphi_1 + 2a_1h_1\Big)\Big]y^2 + \frac{\mu_2}{2}z^2 + \frac{b_1a_1}{2}|\Theta_3(t)|y^2\\
&+\big(\mu_2|x| + \frac{a_1h_1}{g_0}|y| + |z|\big)m + \gamma_2\Theta_5(t)(x^2+y^2+z^2) \qquad (3.16)\\
&+\frac{\mu_2\delta_1c_1r}{2}x^2 + \frac{\delta_1a_1c_1h_1r}{2g_0}y^2 + \frac{\delta_1c_1r}{2}z^2 + \frac{\delta_1c_1}{2g_0^2}\big(\mu_2 + \frac{a_1h_1}{g_0} + 1\big)\int_{t-r}^{t} y^2(\xi)d\xi.
\end{aligned}$$

Combining (3.16), (3.15) and condition (vi) we get

$$\begin{aligned}
V'_{(3.3)} \le\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x)\\
&-\mu_2\Big[D_3 - \frac{\delta_1c_1r}{2}\Big]x^2 - \Big[D_4 - r(\lambda + \frac{\delta_1a_1c_1\mu_1}{2g_0} + \frac{\delta_1a_1c_1h_1}{2g_0})\Big]y^2\\
&-\Big[D_2 - \frac{\mu_2}{2} - \frac{\delta_1c_1r}{2}\Big]z^2 + \mu_2|x|m + \frac{h_1+\mu_1}{g_0}|y|m + 2|z|m\\
&+k_2\Big[|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + \Theta_5(t)\Big](x^2+y^2+z^2)\\
&+\Big(\frac{\delta_1c_1[g_0(2+\mu_2) + a_1(\mu_1+h_1)]}{2g_0^3} - \lambda\Big)\int_{t-r}^{t} y^2(\xi)d\xi,
\end{aligned}$$

where $k_2 = \max\Big\{\gamma_1+\gamma_2,\ \frac{h_1\delta_1B}{2},\ k_1 + \frac{1}{2}\Big\}$.

Choosing $\frac{\delta_1c_1[g_0(2+\mu_2) + a_1(\mu_1+h_1)]}{2g_0^3} = \lambda$, since r and D_3 satisfy (3.2) and condition (viii) respectively, there is $\eta > 0$ such that

$$\begin{aligned}
V'_{(3.3)} \le\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x)\\
&-\eta(x^2+y^2+z^2) + \eta M(|x| + |z| + |y|)\\
&+k_2\big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + \Theta_5(t)\big)(x^2+y^2+z^2), \qquad (3.17)
\end{aligned}$$

where

$$\begin{aligned}
\eta &= \min\Big\{D_4 - r(\lambda + \frac{\delta_1a_1c_1\mu_1}{2g_0} + \frac{\delta_1a_1c_1h_1}{2g_0}),\ D_3 - \frac{\delta_1c_1r}{2},\ D_2 - \frac{\mu_2}{2} - \frac{\delta_1c_1r}{2}\Big\}.\\
M &= \frac{m}{\eta}\max\Big\{2,\ \frac{h_1+\mu_1}{g_0},\ \mu_2\Big\}.
\end{aligned}$$

The above inequality may be written as

$$\begin{aligned}
V'_{(3.3)} \le\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x)\\
&-\frac{\eta}{2}(x^2+y^2+z^2) + k_2\big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)|\big)(y^2+z^2)\\
&-\frac{\eta}{2}[(x-M)^2 + (y-M)^2 + (z-M)^2] + \frac{3\eta}{2}M^2\\
\le\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x) - \frac{\eta}{2}(x^2+y^2+z^2)\\
&+k_2\big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + \Theta_5(t)\big)(x^2+y^2+z^2) + \frac{3\eta}{2}M^2.
\end{aligned}$$

It is easily verified that

$$W'_{(3.3)} = e^{-\Omega(t)}\Big[V'_{(3.3)} - \Big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + (\frac{1}{\beta_1} + \frac{1}{\beta_2})\Theta_5(t) + |\Theta_6(t)|\Big)V\Big].$$

Using the fact that

$$\begin{aligned} G(x,y) &= F(x) + \frac{\delta_1}{2\mu_1^2}\big(y + \frac{\mu_1}{\delta_1}f(x)\big)^2 - \frac{1}{2\delta_1}f^2(x) \\ &\geq F(x) - \frac{1}{2\delta_1}f^2(x) \\ &= \int_0^x (1 - \frac{f'(u)}{\delta_1})f(u)du \geq 0, \end{aligned}$$

since $1 - \dfrac{f'(u)}{\delta_1} \geq 0$. It can be followed from (3.5) and (iii) that there exist $\delta_6 > 0$ such that

$$V_1 \geq \mu_1 a_0 c_0 G(x,y) + \delta_6 y^2 + \delta_6 z^2. \tag{3.18}$$

Combining (3.9) and (3.12) we have

$$V_1 \geq \delta_3 F(x) + \delta_2 y^2 + \delta_2 z^2 \text{ and } V_2 \geq \frac{\delta_4\delta_0}{2}x^2.$$

From (3.11) and (3.18) we get

$$V \geq \mu_1 a_0 c_0 G(x,y) + \delta_6 y^2 + \delta_6 z^2 + a_0 c_0 H(x,y).$$

Hence, by (3.13) and the last inequalities we have the following estimate

$$\begin{aligned} W'_{(3.3)} \leq e^{-\Omega(t)}\Big[V'_{(3.3)} - \Big(&k\big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| \\ &+ \frac{1}{\beta_1}\Theta_5(t)\big)\big(x^2 + y^2 + z^2\big) \\ &+ \frac{1}{\beta_2}\Theta_5(t)\big(\mu_1 a_0 c_0 G(x,y)\delta_6 y^2 + \delta_6 z^2 + a_0 c_0 H(x,y)\big) \\ &+ |\Theta_6(t)|\big(\delta_3 F(x) + \delta_2 y^2 + \delta_2 z^2 + \frac{\delta_4\delta_0}{2}x^2\big)\Big)\Big]. \end{aligned}$$

So choosing $\beta_1 = \dfrac{k}{k_2}$, $\beta_2 = \dfrac{a_0 c_0}{A}$ and $\beta_3 = \dfrac{\delta_3}{a_1 c_1}$ this reduces to

$$W'_{(3.3)}(t, x_t, y_t, z_t) \leq L\left[-\frac{\eta}{2}(x^2 + y^2 + z^2) + \frac{3\eta}{2}^2\right], \quad \text{for some } L > 0.$$

Hence the conclusions of Theorem 5 can be followed from Lemma 3, this completes the proof of Theorem 5 □

The following Theorem being a consequence of Theorem 5 and Lemma 4

Theorem 6. *If hypotheses of Theorem 5 be satisfied and a, b, c, e, q are periodic functions of period T, then there exists at the east periodic solution of system (1.3) with the period T.*

Proof. It only remains to verify using the assumptions of Theorem 5 that the conditions of Lemma 4 follow easily. □

Example 7. We consider the following third order delay differential equation

$$\Big[ln(3+\cos t)\big[(\frac{\cos(x)}{1+x^2}+4)x'(t)\big]'\Big]' + (2ln(5+2\cos t)\big(\frac{\sin x+3e^x+3e^{-x}}{e^x+e^{-x}}x'(t)\big)'$$
$$+(3ln(2+\cos t)+1)(\frac{\sin(x)}{1+x^2}+11)x'(t)$$
$$+(\frac{1}{2}ln(4+\cos t))\left[x(t-r)+\frac{x(t-r)}{1+x^2(t-r)}\right]=3\sin t+5. \tag{3.19}$$

It can be seen that

$$2ln3=a_0\le a(t)=2ln(5+2\cos t)\le 2ln7,\ a'(t)=-\frac{4\sin t}{5+2\cos t},$$
$$1=b_0\le b(t)=3ln(2+\cos t)+1\le 1+3ln3, b'(t)=-3\frac{\sin t}{2+\cos t},$$
$$\frac{ln3}{2}=c_0\le c(t)=\frac{1}{2}ln(4+\cos t)\le\frac{ln5}{2},\ \le c'(t)=\frac{1}{2}\frac{\sin t}{5+\cos t},$$
$$ln2\le q(t)=ln(3+\cos t)\le 2ln2, \le q'(t)=-\frac{\sin t}{3+\cos t},$$
$$50\le\frac{f(x)}{x}=50+\frac{1}{1+x^2}\text{ with } x\neq 0,\ |f'(x)|\le\delta_1=2,\ t\ge 0,$$

Moreover,

$$2\le e(t)=3\sin t+5\le 8,\ 3\le g(x)=\frac{\cos(x)}{1+x^2}+4\le 5,$$
$$10\le\varphi(x)=\frac{\sin(x)}{1+x^2}+11\le 12,\ \frac{5}{2}\le h(x)=\frac{\sin x+3e^x+3e^{-x}}{e^x+e^{-x}}\le\frac{7}{2}.$$

Also, $0.80=\dfrac{c_1g_1\delta_1}{b_0\varphi_0}<\mu_1<\dfrac{a_0h_0}{a_1}=1.41$ and $50=\delta_0>\dfrac{1+\varphi_1b_1}{2c_0}=47.83.$
It is straightforward to verify that

$$\int_{-\infty}^{+\infty}|g'(u)|\,du \le \int_{-\infty}^{+\infty}\left[\left|\frac{\sin u}{1+u^2}\right|+\left|\frac{2u\cos u}{(1+u^2)^2}\right|\right]du \le \pi+2.$$

Similarly,

$$\int_{-\infty}^{+\infty}|\varphi'(u)|\,du \le \int_{-\infty}^{+\infty}\left[\left|\frac{\cos u}{1+u^2}\right|+\left|\frac{2u\sin u}{(1+u^2)^2}\right|\right]du \le \pi+2.$$

$$\int_{-\infty}^{+\infty}|h'(u)|\,du = \int_{-\infty}^{+\infty}\left|\frac{(e^u+e^{-u})\cos u-(e^u-e^{-u})\sin u}{(e^u+e^{-u})^2}\right|du$$
$$\le \int_{-\infty}^{+\infty}\left(\frac{1}{e^u+e^{-u}}+\frac{u}{(e^u+e^{-u})^2}\left(e^u-e^{-u}\right)\right)du=\pi.$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|q'(u)|\,du &= \int_{-\infty}^{+\infty}|\frac{\sin u}{3+\cos u}|du \le \int_{-\infty}^{+\infty}\frac{1}{3+\cos u}du\\ &= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{1}{2+u^2}du = \frac{2}{\sqrt{2}}tan^{-1}(\frac{\pi}{2\sqrt{2}}).\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|a'(u)|\,du &= \int_{-\infty}^{+\infty}|\frac{4\sin u}{5+2\cos u}|du \le \int_{-\infty}^{+\infty}\frac{4}{5+2\cos u}du\\ &= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{8}{7+3u^2}du = \frac{16}{\sqrt{21}}tan^{-1}(\frac{\pi\sqrt{3}}{2\sqrt{7}}).\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|b'(u)|\,du &= 3\int_{-\infty}^{+\infty}|\frac{\sin u}{2+\cos u}|du \le 3\int_{-\infty}^{+\infty}\frac{1}{3+\cos u}du\\ &= 3\int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{1}{2+u^2}du = \frac{4}{\sqrt{3}}tan^{-1}(\frac{\pi}{2\sqrt{3}}).\end{aligned}$$

$$\begin{aligned}\int_{-\infty}^{+\infty}|c'(u)|\,du &= \frac{1}{2}\int_{-\infty}^{+\infty}|\frac{\sin u}{4+\cos u}|du \le \frac{1}{2}\int_{-\infty}^{+\infty}\frac{1}{4+\cos u}du\\ &= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}}\frac{1}{5+3u^2}du = \frac{2}{\sqrt{15}}tan^{-1}(\frac{\pi\sqrt{3}}{2\sqrt{5}}).\end{aligned}$$

Thus all the assumptions of Theorem 5. hold, this shows that every solution of (3.19) is uniformly bounded and uniformly ultimately bounded. Since $a,\ b,\ c,\ e,\ q$ are periodic functions of period 2π, then there exists a periodic solution of (3.19) of period 2π.

For the case $e(t) = 0$, the equation (1.3) is equivalent to the system

$$\begin{cases} x' = \dfrac{1}{g(x)}y,\\ y' = \dfrac{1}{q(t)}z,\\ z' = -\dfrac{a(t)h(x)}{q(t)g(x)}z - a(t)\Theta_2(t)y - \dfrac{b(t)\varphi(x)}{g(x)}y - c(t)f(x) + c(t)\displaystyle\int_{t-r}^{t}\frac{y(s)}{g(x(s))}f'(x(s))ds. \end{cases} \tag{3.20}$$

The following result is introduced.

Corollary 8. *One assumes that all the assumptions (i)-(vi) and (vii) hold. Then the zero solution of equation (1.3) is uniformly asymptotically stable.*

Proof. If $e(t) = 0$, similarly to above proof, the inequality (3.17) becomes

$$\begin{aligned}V'_{(3.20)} \le\ & A\Theta_5(t)\big(\mu_1 G(x,y) + H(x,y)\big) + a_1c_1|\Theta_6(t)|F(x) - \eta(x^2+y^2+z^2)\\ & + k_2\big(|\Theta_1(t)| + |\Theta_2(t)| + |\Theta_3(t)| + |\Theta_4(t)| + \Theta_5(t)\big)(x^2+y^2+z^2),\end{aligned}$$

Hence

$$W'_{(3.20)}(t,x_t,y_t,z_t) \le L\left[-\eta(x^2+y^2+z^2)\right], \quad \text{for some } \ L>0.$$

Thus, all the conditions of Lemma 1 are satisfied. This shows that the zero solution of equation (1.3) is uniformly asymptotically stable. □

4 Conclusions

Liapunov's method has proved to be a popular and useful technique in the study of the stability and boundedness of solutions of higher order non-linear differential equations. In this paper we examine the boundedness and ultimate boundedness of solutions for certain third order non-linear non-autonomous differential equations with delay. Sufficient conditions were obtained for the existence of at least one periodic solution of the equation. Finally, we investigate the asymptotic stability of the zero solution of the same equation for the case $e(t) = 0$.

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