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A note on Euclid's Theorem concerning the infinitude of the primes

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Abstract

We present another elementary proof of Euclid's Theorem concerning the infinitude of the prime numbers. This proof is "geometric" in nature and it employs very little beyond the concept of "proportion."

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Euclid's Theorem ([2], Book IX, Proposition 20) establishes the existence of infinitely many prime numbers. It has been one of the cornerstones of mathematical thought. More than a dozen different proofs of this result, with many clever simplifications and variants, have been published over the past two millennia (for lists of proofs and good discussions of their historical relevance, see [1], [3], [4] and [6]). A decade ago, in [5], we gave a short direct proof of Euclid's Theorem that has received a surprising amount of attention. Here we would like to present another idea, not quite as simple as the first one, but perhaps equally fundamental. It makes use of the ancient concept of proportion, the theory of which was perfected by Pythagoras, Eudoxus and finally Euclid himself (a fact demonstrated by the results summarized in Book V of his *Elements* [2]).

We rephrase the problem slightly. The question we ask is: Why cannot products of powers of a finite number of primes cover the entire set $\mathbb{N}$?

We investigate the factorization geometrically and consider the canonical representation as an operation (on exponents) in two dimensions, with single prime powers representing what we will call the "vertical" and their products the "horizontal" dimensions.

Vertical Dimension. For a fixed prime number p, and $0 \le i \le m$, there are $m+1$ positive integers that can be written in the form p^i, the largest of which is p^m. Since, clearly, $m+1 \le (1+1)^m = 2^m \le p^m$, many integers are not of this form; so for the proportion $\nabla(p^m)$ of these powers (up to p^m) we not only have $\nabla(p^m) < 1$, for all $m > 1$ (as well as $\nabla(p^m) \to 0$, as $m \to \infty$), but also $\nabla(p^m) > \nabla(p^{m+1})$, because

$$\frac{m+1}{p^m} > \frac{m+2}{p^{m+1}} \iff 1 - \frac{1}{m+2} > \frac{1}{p}. \tag{1}$$

Thus, considered vertically, the proportions are monotonically decreasing.

Horizontal Dimension. Recall that a function $f : \mathbb{N} \to \mathbb{C}$ is called multiplicative, if $f(1) = 1$ and $f(ab) = f(a)f(b)$, for all $a, b \in \mathbb{N}$ with $\gcd(a, b) = 1$. A critically important property of the proportions ∇ is their multiplicativity. For all $k \geq 2$, let us define

$$\nabla(p_1^{m_1} \cdots p_k^{m_k}) \ := \ \frac{\#\{n = p_1^{a_1} \cdots p_k^{a_k} \ : \ 0 \leq a_j \leq m_j, \text{ for } 1 \leq j \leq k\}}{p_1^{m_1} \cdots p_k^{m_k}},$$

then, for all permutations of exponents m_i, we have

$$\nabla(p_1^{m_1} \cdots p_k^{m_k}) = \nabla(p_1^{m_1}) \cdots \nabla(p_k^{m_k}) < 1. \tag{2}$$

In other words, the multiplicativity of ∇ implies the horizontal monotonicity.

Combining these two monotonic orthogonal trends is enough to prove the infinitude of the prime numbers. This is because the vertical dimension is (trivially) infinite, and (1) implies an ever-increasing sparseness of integers represented by a given prime power; while from the monotonicity property of (2) it follows that the same will remain true upon any finite composition of such powers, and therefore only an infinite horizontal dimension could possibly compensate for the growing deficit and create a complete cover of $\mathbb{N}$, guaranteed by the unique factorization theorem.

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References

[1] Dickson, L. E., "History of the theory of numbers." Vol 1, Chapter XVIII, 2nd ed., Chelsea Publishing, New York, 1992.

[2] Euclid, "Elements." Alexandria, c. 300 BC (for a modern edition, see Heath, T. L., "Thirteen Books of Euclid's Elements." Dover, New York, 1956).

[3] Narkiewicz, W., "The development of prime number theory. From Euclid to Hardy and Littlewood." Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2000.

[4] Ribenboim, P., "The little book of bigger primes." Springer-Verlag, New York, 2004.

[5] Saidak, F., A new proof of Euclid's theorem, *Amer. Math. Monthly* **113** (2006), no. 10, 937–938.

[6] Sierpiński, W., "Elementary theory of numbers", 2nd Edition, Polish Sci. Pub., Warsaw, 1988.