

# New sufficient conditions for starlikeness of certain integral operators involving Bessel functions

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## Abstract

The purpose of the present paper is to investigate a new integral operator associated with Bessel function. Some sufficient conditions are derived for this integral operator belonging to various subclasses of starlike functions under certain conditions.

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## 1 Introduction

Let  $\mathcal{A}$  denote the class of functions  $f$  of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1.1)$$

which are analytic in the open unit disc  $\mathbb{U} = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$  and satisfy the normalization condition  $f(0) = f'(0) - 1 = 0$ . Further, we denote by  $S$  the subclass of  $\mathcal{A}$  consisting of functions of the form (1.1) which are also univalent in  $U$ . A function  $f(z)$  in  $\mathcal{A}$  is said to be starlike of order  $\delta$  ( $0 \leq \delta < 1$ ) if the following condition is satisfied

$$\Re \left\{ \frac{z f'(z)}{f(z)} \right\} > \delta, \quad (z \in \mathbb{U}), \quad (1.2)$$

we denote by  $S^*(\delta)$  the class of starlike functions of order  $\delta$ . Clearly  $S^*(\delta) \subset S^*(0) = S^*(0 \leq \delta < 1)$  and  $S^* \subset S$ .

The Bessel function of the first kind of order  $\nu$  is defined by the infinite series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{n! \Gamma(n+\nu+1)}, \quad (1.3)$$

where  $\Gamma$  stands for the Euler-Gamma function  $z \in \mathbb{C}$  and  $\nu \in \mathbb{R}$ . Recently, Szasz and Kupan [17] investigated the univalence of the normalized Bessel function of first kind  $g_\nu : \mathbb{U} \rightarrow \mathbb{C}$  defined by

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}). \quad (1.4)$$

Baricz and Frasin [3] have obtained the sufficient condition for the univalence of the various integral operators involving Bessel functions of the first kind of order  $\nu$ .

Recently, Frasin [5] introduced the following integral operator involving the normalized Bessel function of the first kind

$$F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n}(z) = \int_0^z \prod_{i=1}^n \left( \frac{g_{\nu_i}(t)}{t} \right)^{\alpha_i} dt, \quad (1.5)$$

and obtain several sufficient condition for this operator to be convex and strongly convex of given order in the open disc  $\mathbb{U}$ . Recently analogous to these result Porwal and Breaz [15] studied the sufficient condition for the operator defined by (1.5) for certain class of univalent functions. Very recently, Mishra and Panigrahi [9] obtain some sufficient conditions for starlikeness of certain integral operator. In this paper, motivated with the above mentioned work and work of Kumar [6] we obtain some sufficient condition for the operator defined by (1.5) is in the class  $S^*$ .

To prove our main results we shall require the following lemmas:

**Lemma 1.** ([17]) Let  $\nu > \frac{-5+\sqrt{5}}{4}$  and consider the normalized Bessel function of the first kind  $g_\nu : \mathbb{U} \rightarrow \mathbb{C}$  defined by

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}),$$

where  $J_\nu$  stands for Bessel function of the first kind, then the following inequality hold for all  $z \in \mathbb{U}$ .

$$\left| \frac{z g'_\nu(z)}{g_\nu(z)} - 1 \right| \leq \frac{\nu+2}{4\nu^2+10\nu+5}.$$

**Lemma 2.** ([16]) If  $f \in \mathcal{A}$  satisfies

$$\Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} < \frac{\delta+1}{2(\delta-1)}, \quad (z \in \mathbb{U}),$$

for some  $2 \leq \delta < 3$ ,  
or

$$\Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} < \frac{5\delta-1}{2\delta+1}, \quad (z \in \mathbb{U}),$$

for some  $1 \leq \delta \leq 2$ , then  $f \in S^*$ .

**Lemma 3.** ([16]) If  $f \in \mathcal{A}$  satisfies

$$\Re \left\{ 1 + \frac{z f''(z)}{f'(z)} \right\} > -\frac{\delta+1}{2\delta(\delta-1)}, \quad (z \in \mathbb{U}),$$

for some  $\delta \leq -1$ ,  
or

$$\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \frac{3\delta + 1}{2\delta(\delta + 1)}, \quad (z \in \mathbb{U}),$$

for some  $\delta > 1$ , then  $f \in S^* \left( \frac{\delta+1}{2\delta} \right)$ .

Throughout this paper, we frequently use the notation  $F_{\nu_1, \dots, \nu_n, \alpha_1, \dots, \alpha_n}(z) = F_{\nu_i, \alpha_i}(z)$ .

## 2 Main Results

**Theorem 4.** Let  $n$  be a natural number such that  $\nu_1, \nu_2, \dots, \nu_n > \left( \frac{-5+\sqrt{5}}{4} \right)$ . Consider the function  $g_{\nu_i} : \mathbb{U} \rightarrow \mathbb{C}$  defined by

$$g_{\nu_i}(z) = 2^{\nu_i} \Gamma(\nu_i + 1) z^{1-\frac{\nu_i}{2}} J_{\nu_i}(z^{\frac{1}{2}}).$$

Let  $\nu = \min \{ \nu_1, \nu_2, \dots, \nu_n \}$  and suppose that the inequality

$$\frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i \leq \frac{3 - \delta}{2(\delta - 1)} \quad (2.1)$$

is satisfied. Then the function  $F_{\nu_i, \alpha_i}(z)$  defined by (1.5) is in the class  $S^*$  for some  $2 \leq \delta \leq 3$ .

*Proof.* First we observe that, since for all  $i \in \{1, 2, \dots, n\}$ , we have  $g_{\nu_i} \in \mathcal{A}$  i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i, \alpha_i}(z) = \prod_{i=1}^n \left( \frac{g_{\nu_i}(z)}{z} \right)^{\alpha_i}.$$

Taking logarithmic differentiation

$$\begin{aligned} \frac{F''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} &= \sum_{i=1}^n \alpha_i \left( \frac{g'_{\nu_i}(z)}{g_{\nu_i}(z)} - \frac{1}{z} \right) \\ 1 + \frac{zF''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} &= 1 + \sum_{i=1}^n \alpha_i \left( \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right) \\ \Re \left\{ 1 + \frac{zF''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right\} &= 1 + \sum_{i=1}^n \alpha_i \Re \left( \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right) \\ &\leq 1 + \sum_{i=1}^n \alpha_i \left| \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right| \\ &\leq 1 + \sum_{i=1}^n \alpha_i \left( \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \right). \end{aligned} \quad (2.2)$$

For all  $z \in \mathbb{U}$  and  $\nu_1, \nu_2, \dots, \nu_n > \frac{-5+\sqrt{5}}{4}$ . Since the function  $\phi : \left( \frac{-5+\sqrt{5}}{4}, \infty \right) \rightarrow \mathbb{R}$ , defined by,

$$\phi(x) = \frac{x + 2}{4x^2 + 10x + 5}$$

is decreasing and consequently for all  $i \in \{1, 2, \dots, n\}$ . We have

$$\frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 2} \leq \frac{\nu + 2}{4\nu^2 + 10\nu + 5}.$$

Using this result, inequality (2.7) can be written as

$$\Re \left\{ 1 + \frac{zF''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right\} \leq 1 + \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i.$$

Since

$$\begin{aligned} 1 + \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i &< \frac{\delta + 1}{2(\delta - 1)}. \\ \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i &< \frac{\delta + 1}{2(\delta - 1)} - 1 \\ &= \frac{\delta + 1 - 2\delta + 2}{2(\delta - 1)} \\ &= \frac{3 - \delta}{2(\delta - 1)} \end{aligned}$$

Hence from Lemma 2  $F_{\nu_i, \alpha_i}(z) \in S^*$  for some  $2 \leq \delta \leq 3$ .

Thus the proof of Theorem 4 is established.  $\square$

**Theorem 5.** Let  $n$  be a natural number and  $\nu_1, \nu_2, \dots, \nu_n > \left(\frac{-5 + \sqrt{5}}{4}\right)$ . Consider the function  $g_{\nu_i} : \mathbb{U} \rightarrow \mathbb{C}$  defined by

$$g_{\nu_i}(z) = 2^{\nu_i} \Gamma(\nu_i + 1) z^{1 - \frac{\nu_i}{2}} J_{\nu_i}(z^{\frac{1}{2}}).$$

Let  $\nu = \max\{\nu_1, \nu_2, \dots, \nu_n\}$  be a positive real numbers and suppose that the inequality

$$\frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i \leq \frac{2\delta^2 - \delta + 1}{2\delta(\delta - 1)}, \quad (2.3)$$

is satisfied then  $F_{\nu_i, \alpha_i}(z)$  defined by (1.5) is in the class  $S^*(\frac{\delta+1}{2\delta})$  for some  $\delta \leq -1$ .

*Proof.* First we observe that since for all  $i \in \{1, 2, 3, \dots, n\}$ , we have  $g_{\nu_i} \in \mathcal{A}$  i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i, \alpha_i}(z) = \prod_{i=1}^n \left( \frac{g_{\nu_i}(z)}{z} \right)^{\alpha_i}.$$

Taking logarithmic differentiation,

$$\Re \left\{ 1 + \frac{zF''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right\} = \sum_{i=1}^n \alpha_i \Re \left( \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right) + 1.$$

From Lemma 1, we have

$$\left| \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right| \leq \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5}.$$

Since  $\Re(z) \leq |z|$

$$\begin{aligned} \Re \left\{ 1 - \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} \right\} &\leq \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \\ \Re \left\{ \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} \right\} &\geq 1 - \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \\ \Re \left\{ 1 + \frac{zF''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right\} &\geq \sum_{i=1}^n \alpha_i \left( 1 - \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \right) + 1 - \sum_{i=1}^n \alpha_i \\ &= - \left( \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \right) \sum_{i=1}^n \alpha_i + 1 \\ &\geq - \frac{\delta + 1}{2\delta(\delta - 1)} \\ - \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i &\geq -1 - \frac{\delta + 1}{2\delta(\delta - 1)} \\ \frac{\nu + 2}{4\nu^2 + 10\nu + 2} \sum_{i=1}^n \alpha_i &\leq 1 + \frac{\delta + 1}{2\delta(\delta - 1)} \\ \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i &\leq \frac{2\delta^2 - 2\delta + \delta + 1}{2\delta(\delta - 1)} \\ &= \frac{2\delta^2 - \delta + 1}{2\delta(\delta - 1)} \end{aligned}$$

Hence from Lemma 3,  $F_{\nu_i, \alpha_i}(z) \in S^* \left( \frac{\delta + 1}{2\delta} \right)$  for some  $\delta \leq -1$ .

Thus the proof of Theorem 5 is established. □

**Theorem 6.** Let  $n$  be a natural number such that  $\nu_1, \nu_2, \dots, \nu_n > \left( \frac{-5 + \sqrt{5}}{4} \right)$ . Consider the function  $g_{\nu_i} : \mathbb{U} \rightarrow \mathbb{C}$  defined by

$$g_{\nu_i}(z) = 2^{\nu_i} \Gamma(\nu_i + 1) z^{1 - \frac{\nu_i}{2}} J_{\nu_i}(z^{\frac{1}{2}}).$$

Let  $\nu = \min \{ \nu_1, \nu_2, \dots, \nu_n \}$  and suppose that the inequality

$$\frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i \leq \frac{5\delta - 1}{2(\delta + 1)} \tag{2.4}$$

is satisfied. Then the function  $F_{\nu_i, \alpha_i}(z)$  defined by (1.5) is in the class  $S^*$  for some  $1 \leq \delta \leq 2$ .

*Proof.* First we observe that, since for all  $i \in \{1, 2, \dots, n\}$ . We have  $g_{\nu_i}(z) \in \mathcal{A}$  i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$\begin{aligned} F'_{\nu_i, \alpha_i}(z) &= \prod_{i=1}^n \left( \frac{g_{\nu_i}(z)}{z} \right)^{\alpha_i} \\ \left( \frac{F''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right) &= \sum_{i=1}^n \alpha_i \left( \frac{g'_{\nu_i}(z)}{g_{\nu_i}(z)} - \frac{1}{z} \right) \\ \Re \left\{ 1 + \frac{z F''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right\} &= 1 + \sum_{i=1}^n \alpha_i \Re \left( \frac{z g'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right) \\ &\leq 1 + \sum_{i=1}^n \alpha_i \left| \frac{z g'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right| \\ &\leq 1 + \sum_{i=1}^n \alpha_i \left( \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \right) \\ &\leq 1 + \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i. \end{aligned}$$

For all  $z \in \mathbb{U}$  and  $\nu_1, \nu_2, \dots, \nu_n > \frac{-5 + \sqrt{5}}{4}$ .

Since the function,

$\phi : \left( \frac{-5 + \sqrt{5}}{4}, \infty \right) \rightarrow \mathbb{R}$  defined by

$$\phi(x) = \frac{x + 2}{4x^2 + 10x + 5}$$

is decreasing and consequently for all  $i \in \{1, 2, \dots, n\}$ , we have

$$\frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \leq \frac{\nu + 2}{4\nu^2 + 10\nu + 5}$$

Since,

$$\begin{aligned} 1 + \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i &\leq \frac{5\delta - 1}{2(\delta + 1)} \\ \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i &\leq \frac{5\delta - 1}{2(\delta + 1)} - 1 \\ &= \frac{5\delta - 1 - 2(\delta + 1)}{2\delta + 1} \\ &= \frac{5\delta - 1 - 2\delta - 2}{2(\delta + 1)} \\ &= \frac{3(\delta - 1)}{2(\delta + 1)}. \end{aligned}$$

Hence from Lemma 2,  $F_{\nu_i, \alpha_i}(z) \in S^*$  for some  $1 \leq \delta \leq 2$

Thus the proof of Theorem 6 is established.  $\square$

**Theorem 7.** Let  $n$  be a natural number and  $\nu_1, \nu_2, \dots, \nu_n > \left(\frac{-5+\sqrt{5}}{4}\right)$ . Consider the function  $g_{\nu_i} : \mathbb{U} \rightarrow \mathbb{C}$  defined by

$$g_{\nu_i}(z) = 2^{\nu_i} \Gamma(\nu_i + 1) z^{1-\frac{\nu_i}{2}} J_{\nu_i}(z^{\frac{1}{2}}).$$

Let  $\nu = \max\{\nu_1, \nu_2, \dots, \nu_n\}$  be positive real numbers and suppose that the inequality

$$\frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i \leq \frac{\delta + 1 - 2\delta^2}{2\delta(\delta + 1)}$$

is satisfied then  $F_{\nu_i, \alpha_i}(z)$  defined by (1.5) is in the class  $S^*\left(\frac{\delta+1}{2\delta}\right)$  for some  $\delta > 1$ .

*Proof.* First we observe that since for all  $i \in \{1, 2, \dots, n\}$ . We have  $g_{\nu_i} \in \mathcal{A}$  i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i, \alpha_i}(z) = \prod_{i=1}^n \left( \frac{g_{\nu_i}(z)}{z} \right)^{\alpha_i}.$$

Taking logarithmic differentiation

$$\begin{aligned} \Re \left\{ 1 + \frac{z F''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right\} &= \sum_{i=1}^n \alpha_i \Re \left\{ \frac{z g'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right\} + 1 \\ \left| \frac{z g'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1 \right| &\leq \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \\ \Re \left\{ 1 - \frac{z g'_{\nu_i}(z)}{g_{\nu_i}(z)} \right\} &\leq \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \\ \Re \left\{ \frac{z g'_{\nu_i}(z)}{g_{\nu_i}(z)} \right\} &\geq 1 - \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \\ \Re \left\{ 1 + \frac{z F''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right\} &\geq \sum_{i=1}^n \alpha_i \left( 1 - \frac{\nu_i + 2}{4\nu_i^2 + 10\nu_i + 5} \right) + 1 - \sum_{i=1}^n \alpha_i \\ \Re \left\{ 1 + \frac{z F''_{\nu_i, \alpha_i}(z)}{F'_{\nu_i, \alpha_i}(z)} \right\} &\geq \left( 1 - \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \right) \sum_{i=1}^n \alpha_i + 1 - \sum_{i=1}^n \alpha_i. \end{aligned}$$

Here  $\phi(x) = \frac{x+2}{4x^2+10x+5}$  is decreasing. So for all  $i \in \{1, 2, \dots, n\}$ .

$$\begin{aligned} \left( 1 - \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \right) \sum_{i=1}^n \alpha_i + 1 - \sum_{i=1}^n \alpha_i &\geq \frac{3\delta + 1}{2\delta(\delta + 1)} \\ - \frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^n \alpha_i &\geq \frac{3\delta + 1}{2\delta(\delta + 1)} - 1 \\ &\geq \frac{3\delta + 1 - 2\delta(\delta + 1)}{2\delta(\delta + 1)} \end{aligned}$$

$$\begin{aligned}
&= \frac{3\delta + 1 - 2\delta^2 - 2\delta}{2\delta(\delta + 1)} \\
&= \frac{\delta + 1 - 2\delta^2}{2\delta(\delta + 1)}.
\end{aligned}$$

Hence from Lemma (2),  $F_{\nu_i, \alpha_i} \in S^*\left(\frac{\delta+1}{2\delta}\right)$  for some  $\delta > 1$ .

Thus the proof of Theorem 7 is established.  $\square$

**Remark 8.** 1. One may also obtain the analogues of these results for various subclasses of analytic functions e.g. Convex functions, Strongly convex functions and Strongly starlike functions, Spiral-like functions etc.

2. The results of this paper can also be extended by using the integral operator studied by Breaz *et al.* [4], Merkes and Wright [7], Miller *et al.* [8], Pesker [10], Porwal and Kumar [13] and Porwal and Singh [14].
3. Recently Baricz [1] introduced generalized Bessel functions of first kind which is a natural generalization of Bessel function, Modified Bessel function, Spherical Bessel function and Modified spherical Bessel function and give wide applications in Geometric Function Theory. For detailed study one may refer [2], (see also [11], [12]).
4. It is interesting to find the analogues of these results for harmonic starlike and convex functions.

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