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Coincidence point theorems for a family of multivalued mappings in partially ordered metric spaces

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Abstract

In this paper we establish certain multivalued coincidence point results of a family of multivalued mappings with a singlevalued mapping under the assumptions of certain almost contractive type inequalities. Our results are derived in metric spaces with a partial ordering. The corresponding singled valued cases are shown to extend a number of existing results. We have given one illustrative example. The methodology applied here is a blending of order theoretic and analytic methodologies.

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1 Introduction

In the fixed point theory of setvalued maps two types of distances are generally used. One is the Hausdorff distance. Nadler [22] had proved a multivalued version of the Banach's contraction mapping principle by using the Hausdorff metric. There are many other fixed point results using this Hausdorff metric, some instances of these works are in [9, 17, 29, 30, 31]. The another distance is the δ - distance. This is not metric like the Hausdorff distance, but shares most of the properties of a metric. It has been used in many problem on fixed point theory like those in [1, 2, 19, 33].

In recent times, fixed point theory has developed rapidly in partially ordered metric spaces; that is, metric spaces endowed with a partial ordering. References [10, 15, 23, 25, 27] are some recent instances of these works. A speciality of these problems is that they use both analytic and order theoretic methods. It is also one of the main reasons why they are considered interesting.

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Khan et al. [21] initiated the use of a control function in metric fixed point theory which they called Altering distance function. Several works on fixed point theory like those noted in [12, 16, 26, 28] have utilized this control function.

The concept of almost contractions were introduced by Berinde [5, 6]. It was shown in [5] that any strict contraction, the Kannan [20] and Zamfirescu [34] mappings, as well as a large class of quasi-contractions, are all almost contractions. Almost contractions and its generalizations were further considered in several works like [7, 11, 24].

The purpose of this paper is to establish some coincidence point results of a family of multivalued mappings with a single valued mapping under the assumptions of certain almost contractive type inequalities in partially ordered metric spaces. We have also utilized δ-compatible pairs in our theorems. In another theorem we have replaced the continuities of the functions with an order condition. We also give here the corresponding singlevalued versions of the theorems which generalize a number of existing works. An illustrative example for the multivalued case is given.

2 Mathematical Preliminaries

In the following we give some technical definitions which are used in our theorems.
Let $(X,\ d)$ be a metric space. We denote the class of nonempty and bounded subsets of X by $B(X)$. For $A,\ B \in B(X)$, functions $D(A,\ B)$ and $\delta(A,\ B)$ are defined as follows:

$$D(A,\ B) = \ \inf\{d(a,\ b) : a \in A,\ b \in B\}$$

and

$$\delta(A,\ B) = \ \sup\{d(a,\ b) : a \in A,\ b \in B\}.$$

If $A = \{a\}$, then we write $D(A,\ B) = D(a,\ B)$ and $\delta(A,\ B) = \delta(a,\ B)$. Also, in addition, if $B = \{b\}$, then $D(A,\ B) = d(a,\ b)$ and $\delta(A,\ B) = d(a,\ b)$. Obviously, $D(A,\ B) \leq \delta(A,\ B)$. For all $A,\ B,\ C \in B(X)$, the definition of $\delta(A,\ B)$ yields the following:

$$\delta(A,\ B) = \delta(B,\ A),$$

$$\delta(A,\ B) \leq \delta(A,\ C) + \delta(C,\ B),$$

$$\delta(A,\ B) = 0 \text{ iff } A = B = \{a\},$$

$$\delta(A,\ A) = \ \text{diam } A.$$ [13]

There are several works which have utilized δ - distance [2, 4, 13, 14, 19, 33].

Definition 1. ([13]) A sequence $\{A_n\}$ of subsets of metric space $(X,\ d)$ is said to be convergent to subset A of X if

(i) given $a \in A$, there is a sequence $\{a_n\}$ in X such that $a_n \in A_n$, for $n = 1, 2, 3, ...$, and $\{a_n\}$ converges to a.

(ii) given $\epsilon > 0$, there exists a positive integer N such that $A_n \subseteq A_\epsilon$, for all $n > N$, where A_ϵ is the union of all open sphere with centers in A and radius ϵ.

Lemma 2. *([13, 14]) If $\{A_n\}$ and $\{B_n\}$ are sequences in $B(X)$, where (X,d) is a complete metric space and $\{A_n\} \to A$ and $\{B_n\} \to B$ where $A, B \in B(X)$ then*

$$\delta(A_n, B_n) \to \delta(A,\ B) \quad \textit{as} \quad n \to \infty.$$

Lemma 3. *([14]) If $\{A_n\}$ is a sequence of bounded subsets of a complete metric space (X, d) and if $\lim\limits_{n\to\infty} \delta(A_n, \{y\}) = 0$, for some $y \in X$, then $\{A_n\} \to \{y\}$ as $n \longrightarrow \infty$.*

Definition 4. ([14]) A set-valued mapping $T : X \longrightarrow B(X)$, where $(X,\ d)$ is a metric space, is continuous at a point $x \in X$ if $\{x_n\}$ is a sequence in X converging to x, then the sequence $\{Tx_n\}$ in $B(X)$ converges to Tx. T is said to be continuous in X if it is continuous at each point $x \in X$.

Lemma 5. *([14]) If $\{A_n\}$ is a sequence of nonempty subsets of X and $z \in X$ such that*

$$\lim_{n\to\infty} a_n = z,$$

where z is independent of the particular choice of each $a_n \in A_n$. If a self map g of X is continuous, $\{gz\}$ is the limit of the sequence $\{gA_n\}$.

Definition 6. ([18]) Two self maps g and T of a metric space (X, d) are said to be compatible mappings if $\lim\limits_{n\to\infty} d(gTx_n, Tgx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $\lim\limits_{n\to\infty} gx_n = \lim\limits_{n\to\infty} Tx_n = t$, for some $t \in X$.

Definition 7. ([19]) The mappings $g : X \longrightarrow X$ and $T : X \longrightarrow B(X)$, where $(X,\ d)$ is a metric space, are δ- compatible if $\lim\limits_{n\to\infty} \delta(Tgx_n, gTx_n) = 0$ whenever $\{x_n\}$ is a sequence in X such that $gTx_n \in B(X)$ and $Tx_n \to \{t\}$, $gx_n \to t$, for some t in X.

Definition 8. Let $(X,\ d)$ be a metric space and $g : X \longrightarrow X$ and $T : X \longrightarrow B(X)$. Then $u \in X$ is called a coincidence point of g and T if $\{gu\} = Tu$.

Definition 9. ([4]) Let A and B be two nonempty subsets of a partially ordered set $(X,\ \preceq)$. The relation between A and B is denoted and defined as follows:
$A \prec_1 B$, if for every $a \in A$ there exists $b \in B$ such that $a \preceq b$.

Definition 10. ([21]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:

(i) ψ is monotone increasing and continuous,

(ii) $\psi(t) = 0$ if and only if $t = 0$.

3 Main Results

Lemma 11. *Let $(X,\ d)$ be a metric space and let $\{x_n\}$ be a sequence in X such that*

$$\lim_{n\to\infty} d(x_{n+1},\ x_n) = 0. \tag{3.1}$$

If $\{x_n\}$ is not a Cauchy sequence in $(X,\ d)$, then there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that $n(k) > m(k) > k$ and the following four sequences tend to ϵ when $k \longrightarrow \infty$:

$$d(x_{m(k)},\ x_{n(k)}),\ d(x_{m(k)},\ x_{n(k)+1}),\ d(x_{n(k)},\ x_{m(k)+1}),\ d(x_{m(k)+1},\ x_{n(k)+1}). \tag{3.2}$$

Proof. Suppose that $\{x_n\}$ is a sequence in $(X,\ d)$ satisfying (3.1) which is not Cauchy. Then there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for all positive integers k,

$$n(k) > m(k) > k,\ d(x_{m(k)},\ x_{n(k)-1}) < \epsilon,\ d(x_{m(k)},\ x_{n(k)}) \geq \epsilon.$$

Now,

$$\epsilon \leq d(x_{m(k)},\ x_{n(k)}) \leq d(x_{n(k)},\ x_{n(k)-1}) + d(x_{n(k)-1},\ x_{m(k)}) < d(x_{n(k)},\ x_{n(k)-1}) + \epsilon.$$

Letting $k \longrightarrow \infty$ in the above inequality and using (3.1), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)}) = \epsilon. \tag{3.3}$$

Again,

$$d(x_{m(k)},\ x_{n(k)}) \leq d(x_{m(k)},\ x_{n(k)+1}) + d(x_{n(k)+1},\ x_{n(k)})$$

and

$$d(x_{m(k)},\ x_{n(k)+1}) \leq d(x_{m(k)},\ x_{n(k)}) + d(x_{n(k)},\ x_{n(k)+1}).$$

Letting $k \longrightarrow \infty$ in the above inequalities and using (3.1) and (3.3), we have

$$\lim_{k\to\infty} d(x_{m(k)},\ x_{n(k)+1}) = \epsilon. \tag{3.4}$$

That the remaining two sequences in (3.2) tend to ϵ can be proved in a similar way. $\square$

Theorem 12. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Let $\{T_\alpha : X \longrightarrow B(X) : \alpha \in \Lambda\}$ be a family of multivalued mappings. Let $g : X \longrightarrow X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that*

(i) T_{α_0} and g are continuous,

(ii) $T_{\alpha_0}x \subseteq g(X)$ and $gT_{\alpha_0}x \in B(X)$, for every $x \in X$,

(iii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$,

(iv) for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \prec_1 T_{\alpha_0}y$,

(v) the pair $(g,\ T_{\alpha_0})$ is δ - compatible,

(vi) $\psi(\delta(T_{\alpha_0}x, T_\alpha y))$

$$\leq \theta(d(gx, gy))\ \mathit{max}\ \{\psi(d(gx, gy)),\ \psi(D(gx, T_{\alpha_0}x)),\ \psi(D(gy, T_\alpha y)),$$
$$\sqrt{\psi(D(gx, T_\alpha y))\ .\ \psi(D(gy, T_{\alpha_0}x))}\ \}$$
$$+ L\ \mathit{min}\, \{\psi(D(gx,\ T_{\alpha_0}x)),\ \psi(D(gy,\ T_\alpha y)),\ \psi(D(gx,\ T_\alpha y)),\ \psi(D(gy,\ T_{\alpha_0}x))\},$$

where $x,\ y \in X$ such that gx and gy are comparable and $L \geq 0$.

Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.

Proof. First we establish that any coincidence point of g and T_{α_0} is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$ and conversely. Suppose that $z \in X$ be a coincidence point of g

and T_{α_0}. Then $\{gz\} = T_{\alpha_0}z$. From (vi) and using the properties of ψ, we have

$$\begin{aligned}
\psi(\delta(gz,\ T_\alpha z)) &= \psi(\delta(T_{\alpha_0}z,\ T_\alpha z)) \\
&\le \theta(d(gz,gz)) \max\{\psi(d(gz,gz)),\ \psi(D(gz,T_{\alpha_0}z)),\ \psi(D(gz,T_\alpha z)), \\
&\qquad \sqrt{\psi(D(gz,T_\alpha z))\ .\ \psi(D(gz,T_{\alpha_0}z))}\ \} \\
&\quad + L \min\{\psi(D(gz,\ T_{\alpha_0}z)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gz,\ T_{\alpha_0}z))\} \\
&= \theta(d(gz,gz)) \max\{\psi(d(gz,gz)),\ \psi(d(gz,gz)),\ \psi(D(gz,T_\alpha z)), \\
&\qquad \sqrt{\psi(D(gz,T_\alpha z))\ .\ \psi(d(gz,gz))}\ \} \\
&\quad + L \min\{\psi(d(gz,\ gz)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(d(gz,\ gz))\} \\
&= \theta(d(gz,gz))\psi(D(gz,T_\alpha z)) \\
&< \psi(D(gz,T_\alpha z)), (\text{since } \theta(t) < 1, \text{ for all } t \in [0,\ \infty)).
\end{aligned}$$

Again using the monotone property of ψ, we have

$$\delta(gz,\ T_\alpha z) < D(gz,\ T_\alpha z) \le \delta(gz,\ T_\alpha z),$$

which implies that $\delta(gz,\ T_\alpha z) = 0$, that is, $\{gz\} = T_\alpha z$, for all $\alpha \in \Lambda$. Hence z is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. Converse part is trivial.

Now it is sufficient to prove that g and T_{α_0} have a coincidence point. Let $x_0 \in X$ be such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$. Then there exists $u \in T_{\alpha_0}x_0$ such that $gx_0 \preceq u$. Since $T_{\alpha_0}x_0 \subseteq g(X)$ and $u \in T_{\alpha_0}x_0$, there exists $x_1 \in X$ such that $gx_1 = u$. So $gx_0 \preceq gx_1$. Then by the assumption (iv), $T_{\alpha_0}x_0 \prec_1 T_{\alpha_0}x_1$. Since $u = gx_1 \in T_{\alpha_0}x_0$, there exists $v \in T_{\alpha_0}x_1$ such that $gx_1 \preceq v$. As $T_{\alpha_0}x_1 \subseteq g(X)$ and $v \in T_{\alpha_0}x_1$, there exists $x_2 \in X$ such that $gx_2 = v$. So $gx_1 \preceq gx_2$. Continuing this process we construct a sequence $\{x_n\}$ in X such that

$$gx_{n+1} \in T_{\alpha_0}x_n, \quad \text{for all} \quad n \ge 0, \tag{3.5}$$

and

$$gx_0 \preceq gx_1 \preceq gx_2 \preceq ..... \preceq gx_n \preceq gx_{n+1}.... \tag{3.6}$$

Let $\tau_n = d(gx_n, gx_{n+1})$.
Since $gx_n \preceq gx_{n+1}$, putting $\alpha = \alpha_0$, $x = x_n$ and $y = x_{n+1}$ in (vi) and using the properties of ψ, we have

$$\begin{aligned}
\psi(\tau_{n+1}) &\le \psi(\delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n+1})) \\
&\le \theta(\tau_n) \max\{\psi(\tau_n),\ \psi(D(gx_n,T_{\alpha_0}x_n)), \psi(D(gx_{n+1},T_{\alpha_0}x_{n+1})), \\
&\qquad \sqrt{\psi(D(gx_n,T_{\alpha_0}x_{n+1})).\ \psi(D(gx_{n+1},T_{\alpha_0}x_n))}\ \} \\
&\quad + L \min\{\psi(D(gx_n,T_{\alpha_0}x_n)), \psi(D(gx_{n+1},T_{\alpha_0}x_{n+1})), \\
&\qquad \psi(D(gx_n,T_{\alpha_0}x_{n+1})), \psi(D(gx_{n+1},T_{\alpha_0}x_n))\} \\
&\le \theta(\tau_n) \max\{\psi(\tau_n),\ \psi(d(gx_n,gx_{n+1})),\ \psi(d(gx_{n+1},gx_{n+2})), \\
&\qquad \sqrt{\psi(d(gx_n,gx_{n+2})).\ \psi(d(gx_{n+1},gx_{n+1}))}\ \} \\
&\quad + L \min\{\psi(d(gx_n,\ gx_{n+1})),\ \psi(d(gx_{n+1},\ gx_{n+2})), \\
&\qquad \psi(d(gx_n,\ gx_{n+2})),\ \psi(d(gx_{n+1},\ gx_{n+1}))\} \\
&= \theta(\tau_n) \max\{\psi(\tau_n),\ \psi(\tau_{n+1})\}.
\end{aligned} \tag{3.7}$$

Suppose that, $\max\{\psi(\tau_n),\ \psi(\tau_{n+1})\} = \psi(\tau_{n+1})$. Then from (3.7), it follows that

$$\psi(\tau_{n+1}) \le \theta(\tau_n)\ \psi(\tau_{n+1}) < \psi(\tau_{n+1}),\ (\text{since } \theta(\tau_n) < 1),$$

which is a contradiction. Hence

$$\psi(\tau_{n+1}) \leq \theta(\tau_n)\ \psi(\tau_n) < \psi(\tau_n),\ \text{(since } \theta(\tau_n) < 1).$$

By the monotone property of ψ, it follows that

$$\tau_{n+1} < \tau_n, \quad \text{for all} \ \ n \geq 0,$$

that is, $\{\tau_n\}$ is a monotone decreasing sequence of nonnegative real numbers. Hence there exists a $\tau \geq 0$ such that

$$\tau_n \longrightarrow \tau \quad \text{as} \quad n \longrightarrow \infty.$$

Taking $n \longrightarrow \infty$ in (3.7), using the continuities of θ and ψ, we have

$$\psi(\tau) \leq \theta(\tau)\ \psi(\tau) < \psi(\tau), \quad \text{(since} \ \ \theta(\tau) < 1),$$

which is a contradiction unless $\tau = 0$. Thus we have

$$\lim_{n\to\infty} \tau_n = \lim_{n\to\infty} d(gx_n,\ gx_{n+1}) = 0. \tag{3.8}$$

Next we show that $\{gx_n\}$ is a Cauchy sequence. If $\{gx_n\}$ is not a Cauchy sequence, then following Lemma 11, there exists $\epsilon > 0$ and two sequences $\{m(k)\}$ and $\{n(k)\}$ of positive integers such that for all positive integers k, $n(k) > m(k) > k$ and

$$\lim_{k\to\infty} d(gx_{m(k)},\ gx_{n(k)}) = \epsilon, \tag{3.9}$$

$$\lim_{k\to\infty} d(gx_{m(k)},\ gx_{n(k)+1}) = \epsilon, \tag{3.10}$$

$$\lim_{k\to\infty} d(gx_{n(k)},\ gx_{m(k)+1}) = \epsilon, \tag{3.11}$$

and

$$\lim_{k\to\infty} d(gx_{m(k)+1},\ gx_{n(k)+1}) = \epsilon. \tag{3.12}$$

For each positive integer k, $gx_{m(k)}$ and $gx_{n(k)}$ are comparable. Then putting $\alpha = \alpha_0$, $x = x_{m(k)}$ and $y = x_{n(k)}$ in (vi) and using the monotone property of ψ, we have

$$\begin{aligned}
\psi(d(gx_{m(k)+1}, gx_{n(k)+1})) &\leq \psi(\delta(T_{\alpha_0}x_{m(k)},\ T_{\alpha_0}x_{n(k)})) \\
&\leq \theta(d(gx_{m(k)}, gx_{n(k)}))\ \max\ \{\psi(d(gx_{m(k)}, gx_{n(k)})), \psi(D(gx_{m(k)}, T_{\alpha_0}x_{m(k)})), \\
&\qquad \psi(D(gx_{n(k)}, T_{\alpha_0}x_{n(k)})), \\
&\qquad \sqrt{\psi(D(gx_{m(k)}, T_{\alpha_0}x_{n(k)})).\ \psi(D(gx_{n(k)}, T_{\alpha_0}x_{m(k)}))}\ \} \\
&\quad + L\ \min\ \{\psi(D(gx_{m(k)}, T_{\alpha_0}x_{m(k)})), \psi(D(gx_{n(k)}, T_{\alpha_0}x_{n(k)})), \\
&\qquad \psi(D(gx_{m(k)}, T_{\alpha_0}x_{n(k)})), \psi(D(gx_{n(k)}, T_{\alpha_0}x_{m(k)}))\} \\
&\leq \theta(d(gx_{m(k)}, gx_{n(k)}))\ \max\ \{\psi(d(gx_{m(k)}, gx_{n(k)})), \psi(d(gx_{m(k)}, gx_{m(k)+1})), \\
&\qquad \psi(d(gx_{n(k)}, gx_{n(k)+1})), \\
&\qquad \sqrt{\psi(d(gx_{m(k)}, gx_{n(k)+1})).\ \psi(d(gx_{n(k)}, gx_{m(k)+1}))}\ \} \\
&\quad + L\ \min\ \{\psi(d(gx_{m(k)},\ gx_{m(k)+1})),\ \psi(d(gx_{n(k)},\ gx_{n(k)+1})), \\
&\qquad \psi(d(gx_{m(k)},\ gx_{n(k)+1})),\ \psi(d(gx_{n(k)},\ gx_{m(k)+1}))\}.
\end{aligned}$$

Letting $k \longrightarrow \infty$ in the above inequality, using (3.8), (3.9), (3.10), (3.11) and (3.12) and using the properties of θ and ψ, we have

$$\psi(\epsilon) \leq \theta(\epsilon)\ \psi(\epsilon) < \psi(\epsilon),\ \ (\text{since}\ \ \theta(\epsilon) < 1),$$

which is a contradiction. Hence $\{gx_n\}$ is a Cauchy sequence in $g(X)$. Since X is complete and $g(X)$ is closed in X, there exists $u \in g(X)$ such that

$$gx_n \longrightarrow u \quad \text{as} \quad n \longrightarrow \infty.$$

Since $u \in g(X)$, there exists $z \in X$ such that $u = gz$. Then

$$gx_n \longrightarrow u = gz \quad \text{as} \quad n \longrightarrow \infty. \tag{3.13}$$

Since $\{\tau_n\}$ is monotone decreasing, from (3.7), we have

$$\psi(\tau_{n+1}) \leq \psi(\delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n+1})) \leq \theta(\tau_n)\psi(\tau_n).$$

As $\theta(\tau_n) < 1$, it follows that

$$\psi(\tau_{n+1}) \leq \psi(\delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n+1})) < \psi(\tau_n),$$

which, by the monotone property of ψ, implies that

$$\tau_{n+1} \leq \delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n+1}) < \tau_n.$$

Taking $n \to \infty$ in the above inequality, and using (3.8), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}x_{n+1},\ T_{\alpha_0}x_n) = 0. \tag{3.14}$$

Now,

$$\delta(T_{\alpha_0}x_n,\ \{u\}) \leq \delta(T_{\alpha_0}x_n,\ gx_n) + \delta(gx_n,\ \{u\}) \leq \delta(T_{\alpha_0}x_n,\ T_{\alpha_0}x_{n-1}) + d(gx_n,\ u).$$

Taking $n \to \infty$ in the above inequality, and using (3.13) and (3.14), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}x_n,\ \{u\}) = 0,$$

which, by Lemma 3, implies that

$$T_{\alpha_0}x_n \longrightarrow \{u\} \quad \text{as} \quad n \longrightarrow \infty. \tag{3.15}$$

Since the pair (g, T_{α_0}) is δ - compatible, from (3.13) and (3.15), we have

$$\lim_{n\to\infty} \delta(T_{\alpha_0}gx_n,\ gT_{\alpha_0}x_n) = 0.$$

As g and T_{α_0} are continuous, it follows by Lemma 5 that $\delta(T_{\alpha_0}u,\ gu) = 0$, that is, $T_{\alpha_0}u = \{gu\}$. Hence $u \in g(X) \subseteq X$ is a coincidence point of g and T_{α_0}. By what we have already proved, u is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. □

In our next theorem we relax the continuity assumption on T_{α_0} and g by imposing an order condition. We also relax the δ - compatibility assumption of the pairs $(g,\ T_{\alpha_0})$ and the condition that $gT_{\alpha_0}x \in B(X)$, for every $x \in X$.

Theorem 13. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that if $x_n \longrightarrow x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $\{T_\alpha : X \longrightarrow B(X) : \alpha \in \Lambda\}$ be a family of multivalued mappings. Let $g : X \longrightarrow X$ be a mapping such that $g(X)$ is closed in X. Suppose that there exists $\alpha_0 \in \Lambda$ such that*

(i) $T_{\alpha_0}x \subseteq g(X)$, for every $x \in X$,

(ii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 T_{\alpha_0}x_0$,

(iii) for $x,\ y \in X$, $gx \preceq gy$ implies $T_{\alpha_0}x \prec_1 T_{\alpha_0}y$,

(iv)
$$\psi(\delta(T_{\alpha_0}x,\ T_\alpha y)) \le \theta(d(gx,gy))\ \max\Big\{\psi(d(gx,gy)),\psi(D(gx,T_{\alpha_0}x)),\psi(D(gy,T_\alpha y)),$$
$$\sqrt{\psi(D(gx,T_\alpha y))\ .\ \psi(D(gy,T_{\alpha_0}x))}\ \Big\}$$
$$+L\ \min\{\psi(D(gx,\ T_{\alpha_0}x)),\ \psi(D(gy,\ T_\alpha y)),\ \psi(D(gx,\ T_\alpha y)),\ \psi(D(gy,\ T_{\alpha_0}x))\},$$

where $x,\ y \in X$ such that gx and gy are comparable and $L \ge 0$.

Then g and $\{T_\alpha : \alpha \in \Lambda\}$ have a coincidence point.

Proof. We take the same sequence $\{gx_n\}$ as in the proof of Theorem 12. Then we have $gx_{n+1} \in T_{\alpha_0}x_n$, for all $n \ge 0$, $\{gx_n\}$ is monotonic nondecreasing and $gx_n \longrightarrow gz$ as $n \longrightarrow \infty$. Then by the order condition of the metric space, we have $gx_n \preceq gz$, for all n. Using the monotone property of ψ and the condition (iv), we have

$$\begin{aligned}\psi(\delta(gx_{n+1},\ T_\alpha z)) &\le \psi(\delta(T_{\alpha_0}x_n,\ T_\alpha z))\\ &\le \theta(d(gx_n,gz))\ \max\Big\{\psi(d(gx_n,gz)),\psi(D(gx_n,T_{\alpha_0}x_n)),\psi(D(gz,T_\alpha z)),\\ &\qquad \sqrt{\psi(D(gx_n,T_\alpha z)).\ \psi(D(gz,T_{\alpha_0}x_n))}\ \Big\}\\ &+ L\ \min\{\psi(D(gx_n,\ T_{\alpha_0}x_n)),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gx_n,\ T_\alpha z)),\ \psi(D(gz,\ T_{\alpha_0}x_n))\}\\ &\le \theta(d(gx_n,gz))\ \max\Big\{\psi(d(gx_n,gz)),\psi(d(gx_n,gx_{n+1})),\psi(D(gz,T_\alpha z)),\\ &\qquad \sqrt{\psi(D(gx_n,T_\alpha z)).\ \psi(d(gz,gx_{n+1}))}\ \Big\}\\ &+ L\ \min\{\psi(d(gx_n,\ gx_{n+1})),\ \psi(D(gz,\ T_\alpha z)),\ \psi(D(gx_n,\ T_\alpha z)),\ \psi(d(gz,\ gx_{n+1}))\}.\end{aligned}$$

Letting $n \longrightarrow \infty$ in the above inequality and using the properties of θ and ψ, we have

$$\psi(\delta(gz,T_\alpha z)) \le \theta(0)\psi(D(gz,T_\alpha z)) \le \theta(0)\psi(\delta(gz,T_\alpha z)) < \psi(\delta(gz,T_\alpha z))\ (\text{since } \theta(0) < 1),$$

which implies that $\delta(gz,\ T_\alpha z) = 0$, that is, $\{gz\} = T_\alpha z$, for all $\alpha \in \Lambda$. Hence z is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$. □

Considering $\{T_\alpha : X \longrightarrow B(X) : \alpha \in \Lambda\} = \{T\}$ in Theorem 12, we have the following corollary.

Corollary 14. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a* *metric d on X such that $(X,\ d)$ is a complete metric space. Let $T : X \longrightarrow B(X)$ be a multivalued mapping and $g : X \longrightarrow X$ a mapping such that*

(i) T and g are continuous,

(ii) $Tx \subseteq g(X)$ and $gTx \in B(X)$, for every $x \in X$, and $g(X)$ is closed in X,

(iii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 Tx_0$,

(iv) for $x,\ y \in X$, $gx \preceq gy$ implies $Tx \prec_1 Ty$,

(v) the pair $(g,\ T)$ is δ - compatible,

(vi)
$$\begin{aligned}&\psi(\delta(Tx,Ty))\\ &\leq \theta(d(gx,gy))\ max\ \{\psi(d(gx,gy)),\psi(D(gx,Tx)),\psi(D(gy,Ty)),\\ &\qquad\qquad\qquad \sqrt{\psi(D(gx,Ty))\ .\ \psi(D(gy,Tx))}\ \}\\ &\qquad + L\ min\ \{\psi(D(gx,\ Tx)),\ \psi(D(gy,\ Ty)),\ \psi(D(gx,\ Ty)),\ \psi(D(gy,\ Tx))\},\end{aligned}$$

where $x,\ y \in X$ such that gx and gy are comparable and $L \geq 0$.

Then g and T have a coincidence point.

Considering $\{T_\alpha : X \longrightarrow B(X) : \alpha \in \Lambda\} = \{T\}$ in Theorem 13, we have the following corollary.

Corollary 15. *Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be a continuous function and ψ be an altering distance function. Let $(X,\ \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that $(X,\ d)$ is a complete metric space. Assume that if $x_n \longrightarrow x$ is a nondecreasing sequence in X, then $x_n \preceq x$, for all n. Let $T : X \longrightarrow B(X)$ be a multivalued mapping and $g : X \longrightarrow X$ a mapping such that*

(i) $Tx \subseteq g(X)$, for every $x \in X$, and $g(X)$ is closed in X,

(ii) there exists $x_0 \in X$ such that $\{gx_0\} \prec_1 Tx_0$,

(iii) for $x,\ y \in X$, $gx \preceq gy$ implies $Tx \prec_1 Ty$,

(iv)
$$\begin{aligned}&\psi(\delta(Tx,Ty))\\ &\leq \theta(d(gx,gy))\ max\ \{\psi(d(gx,gy)),\psi(D(gx,Tx)),\psi(D(gy,Ty)),\\ &\qquad\qquad\qquad \sqrt{\psi(D(gx,Ty))\ .\ \psi(D(gy,Tx))}\ \}\\ &\qquad + L\ min\ \{\psi(D(gx,\ Tx)),\ \psi(D(gy,\ Ty)),\ \psi(D(gx,\ Ty)),\ \psi(D(gy,\ Tx))\},\end{aligned}$$

where $x,\ y \in X$ such that gx and gy are comparable and $L \geq 0$.

Then g and T have a coincidence point.

The following theorems are single valued cases of the Theorems 12 and 13 respectively. Here we treat T as a multivalued mapping in which case Tx is a singleton set for every $x \in X$.

Theorem 16. *Let* $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ *be a continuous function and* ψ *be an altering distance function. Let* $(X,\ \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* $(X,\ d)$ *is a complete metric space. Let* $\{T_\alpha : X \longrightarrow X : \alpha \in \Lambda\}$ *be a family of mappings. Let* $g : X \longrightarrow X$ *be a mapping such that* $g(X)$ *is closed in* X*. Suppose that there exists* $\alpha_0 \in \Lambda$ *such that*

(i) T_{α_0} *and* g *are continuous,*

(ii) $T_{\alpha_0}(X) \subseteq g(X)$*,*

(iii) there exists $x_0 \in X$ *such that* $gx_0 \preceq T_{\alpha_0}x_0$*,*

(iv) for $x,\ y \in X$*,* $gx \preceq gy$ *implies* $T_{\alpha_0}x \preceq T_{\alpha_0}y$*,*

(v) the pair $(g,\ T_{\alpha_0})$ *is compatible,*

(vi) $\psi(d(T_{\alpha_0}x, T_\alpha y))$

$$\leq \theta(d(gx,gy))\ max\ \{\psi(d(gx,gy)), \psi(d(gx,T_{\alpha_0}x)), \psi(d(gy,T_\alpha y)),$$

$$\sqrt{\psi(d(gx,T_\alpha y))\ .\ \psi(d(gy,T_{\alpha_0}x))}\ \}$$

$$+\ L\ min\ \{\psi(d(gx,\ T_{\alpha_0}x)),\ \psi(d(gy,\ T_\alpha y)),\ \psi(d(gx,\ T_\alpha y)),\ \psi(d(gy,\ T_{\alpha_0}x))\},$$

where $x,\ y \in X$ *such that* gx *and* gy *are comparable and* $L \geq 0$*.*

Then g *and* $\{T_\alpha : \alpha \in \Lambda\}$ *have a coincidence point.*

Theorem 17. *Let* $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ *be a continuous function and* ψ *be an altering distance function. Let* $(X,\ \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* $(X,\ d)$ *is a complete metric space. Assume that if* $x_n \longrightarrow x$ *is a nondecreasing sequence in* X*, then* $x_n \preceq x$*, for all* n*. Let* $\{T_\alpha : X \longrightarrow X : \alpha \in \Lambda\}$ *be a family of mappings. Let* $g : X \longrightarrow X$ *be a mapping such that* $g(X)$ *is closed in* X*. Suppose that there exists* $\alpha_0 \in \Lambda$ *such that*

(i) $T_{\alpha_0}(X) \subseteq g(X)$*,*

(ii) there exists $x_0 \in X$ *such that* $gx_0 \preceq T_{\alpha_0}x_0$*,*

(iii) for $x,\ y \in X$*,* $gx \preceq gy$ *implies* $T_{\alpha_0}x \preceq T_{\alpha_0}y$*,*

(iv) $\psi(d(T_{\alpha_0}x, T_\alpha y))$

$$\leq \theta(d(gx,gy))\ max\ \{\psi(d(gx,gy)), \psi(d(gx,T_{\alpha_0}x)), \psi(d(gy,T_\alpha y)),$$

$$\sqrt{\psi(d(gx,T_\alpha y))\ .\ \psi(d(gy,T_{\alpha_0}x))}\ \}$$

$$+ L\ min\ \{\psi(d(gx,\ T_{\alpha_0}x)),\ \psi(d(gy,\ T_{\alpha}y)),\ \psi(d(gx,\ T_{\alpha}y)),\ \psi(d(gy,\ T_{\alpha_0}x))\},$$

where $x,\ y \in X$ *such that* gx *and* gy *are comparable and* $L \geq 0$.

Then g *and* $\{T_{\alpha} : \alpha \in \Lambda\}$ *have a coincidence point.*

Corollary 18. *Let* $p,\ q,\ r,\ s$ *be four continuous functions from* $[0,\ \infty)$ *into* $[0,\ 1)$ *which satisfy the property* $p(t) + q(t) + r(t) + s(t) < 1$, *for all* $t \in [0,\ \infty)$ *and* ψ *be an altering distance function. Let* $(X,\ \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* $(X,\ d)$ *is a complete metric space. Let* $T : X \longrightarrow X$ *and* $g : X \longrightarrow X$ *be two mappings such that*

(i) T *and* g *are continuous,*

(ii) $T(X) \subseteq g(X)$ *and* $g(X)$ *is closed in* X,

(iii) there exists $x_0 \in X$ *such that* $gx_0 \preceq Tx_0$,

(iv) for $x,\ y \in X$, $gx \preceq gy$ *implies* $Tx \preceq Ty$,

(v) the pair $(g,\ T)$ *is compatible,*

(vi) $\psi(d(Tx, Ty))$

$$\leq p(d(gx, gy))\psi(d(gx, gy)) + q(d(gx, gy))\psi(d(gx, Tx)) + r(d(gx, gy))\psi(d(gy, Ty))$$
$$+s(d(gx, gy))\sqrt{\psi(d(gx, Ty))\ .\ \psi(d(gy, Tx))},$$

where $x,\ y \in X$ *such that* gx *and* gy *are comparable.*

Then g *and* T *have a coincidence point.*

Proof. Starting with the inequality (vi), we have

$$\begin{aligned}\psi(d(Tx, Ty)) &\leq p(d(gx, gy))\psi(d(gx, gy)) + q(d(gx, gy))\psi(d(gx, Tx))\\ &\quad + r(d(gx, gy))\psi(d(gy, Ty)) + s(d(gx, gy))\sqrt{\psi(d(gx, Ty))\ .\ \psi(d(gy, Tx))},\\ &\leq \theta(d(gx, gy))\ \max\{\psi(d(gx, gy)), \psi(d(gx, Tx)), \psi(d(gy, Ty))\\ &\qquad \sqrt{\psi(d(gx, Ty))\ .\ \psi(d(gy, Tx))}\ \},\end{aligned}$$

where $\theta(d(gx, gy)) = p(d(gx, gy)) + q(d(gx, gy)) + r(d(gx, gy)) + s(d(gx, gy))$, which is a special case of the inequality (vi) of Theorem 16 obtained by considering $\{T_{\alpha} : X \longrightarrow X : \alpha \in \Lambda\} = \{T\}$ and $L = 0$. □

Corollary 19. *Let* $p,\ q,\ r,\ s$ *be four continuous functions from* $[0,\ \infty)$ *into* $[0,\ 1)$ *which satisfy the property* $p(t) + q(t) + r(t) + s(t) < 1$, *for all* $t \in [0,\ \infty)$ *and* ψ *be an altering distance function. Let* $(X,\ \preceq)$ *be a partially ordered set and suppose that there exists a metric* d *on* X *such that* $(X,\ d)$ *is a complete metric space. Assume that if* $x_n \longrightarrow x$ *is a nondecreasing sequence in* X, *then* $x_n \preceq x$, *for all* n. *Let* $T : X \longrightarrow X$ *and* $g : X \longrightarrow X$ *be two mappings such that* *(i)* $T(X) \subseteq g(X)$ *and* $g(X)$ *is closed in* X*,*

(ii) there exists $x_0 \in X$ *such that* $gx_0 \preceq Tx_0$*,*

(iii) for $x,\ y \in X$*,* $gx \preceq gy$ *implies* $Tx \preceq Ty$*,*

(iv) $\psi(d(Tx, Ty))$

$$\leq p(d(gx, gy))\psi(d(gx, gy)) + q(d(gx, gy))\psi(d(gx, Tx)) + r(d(gx, gy))\psi(d(gy, Ty))$$

$$+s(d(gx, gy))\sqrt{\psi(d(gx, Ty))\ .\ \psi(d(gy, Tx))},$$

where $x,\ y \in X$ *such that* gx *and* gy *are comparable.*

Then g *and* T *have a coincidence point.*

Proof. Like the proof of the Corollary 18, we can show that the inequality (iv) is a special case of the inequality (iv) of Theorem 17 obtained by considering $\{T_\alpha : X \longrightarrow X : \alpha \in \Lambda\} = \{T\}$ and $L = 0$. □

Example 20. Let $X = [1,\ \infty)$ with usual order $\preceq$ be a partially ordered set. Let $d : X \times X \longrightarrow \mathbb{R}$ be given as

$$d(x,\ y) = |x - y|,\ \text{for}\ \ x,\ y \in X.$$

Then $(X,\ d)$ is a complete metric space with the required properties mentioned in Theorems 12 and 13.
Let $g : X \to X$ be defined as follows:

$$gx = x^2,\ \text{for}\ \ x \in X.$$

Then g has the properties mentioned in Theorems 12 and 13.
Let $\Lambda = \{1,\ 2,\ 3,\ ...\}$. Let the family of mappings $\{T_\alpha : X \to B(X) : \alpha \in \Lambda\}$ be defined as follows:

$$T_1x = \{1\},\quad \text{for}\ \ x \in X \quad \text{and for}\ \alpha \geq 2,\ \ T_\alpha x = \begin{cases} \{1\}, & \text{if}\ 1 \leq x \leq 4, \\ \{1, \dfrac{2\ \alpha}{\alpha + 1}\}, & \text{if}\ x > 4. \end{cases}$$

For any sequence $\{x_n\}$ in X, $T_1x_n \to \{t\}$, $gx_n \to t$, for some t in X implies $t = 1$. Then clearly, the pair $(g,\ T_1)$ is δ - compatible. Also, g and T_1 satisfy required conditions mentioned in Theorems 12 and 13.
Let $\psi : [0,\ \infty) \longrightarrow [0,\ \infty)$ be defined as follows:

$$\psi(t) = t^2,\ \text{for}\ \ t \in [0,\ \infty).$$

Then ψ has the properties mentioned in Theorems 12 and 13.
Let $\theta : [0,\ \infty) \longrightarrow [0,\ 1)$ be defined as follows:

$$\theta(t) = \frac{1}{2},\ \text{for all}\ \ t \in [0,\ \infty).$$

Then θ satisfies the required properties mentioned in Theorems 12 and 13.
The condition (vi) of Theorem 12 and the condition (iv) of Theorem 13 are satisfied for any $L \geq 0$. Hence all the condition of Theorems 12 and 13 are satisfied and it is seen that 1 is a coincidence point of g and $\{T_\alpha : \alpha \in \Lambda\}$.

Note In the above example if one takes $g : X \to X$ to be function as follows:

$$gx = \begin{cases} \dfrac{x}{2}, & \text{if } 1 \leq x \leq 4, \\ 200, & \text{if } x > 4. \end{cases}$$

Then the above example is still applicable to Theorem 13 but not applicable to Theorem 12 because g is not continuous and hence does not satisfy required properties mentioned in Theorem 12.

Remark 21. Theorems 16 and 17 are generalizations of ordered versions of theorem 3.1 in [8] which generalizes the Banach contraction principle [3], theorem 2 of Khan et al [21], the theorem of Skof [32], and the theorem of Kannan [20]. Also, Theorems 16 and 17 generalize the ordered versions of the main result of Berinde [5].

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