

# Boundedness of the solution set for a fourth-order nonlinear differential equation with multiple deviating arguments

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## Abstract

This paper deals with a fourth-order non-linear differential equation with multiple deviating arguments. Some sufficient conditions are set up for all solutions and their derivatives to be bounded. Our results are new and complement to previously known results.

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## 1 Introduction

We consider the nonlinear differential equation of fourth order with multiple deviating arguments

$$\begin{aligned} x^{(4)}(t) + f_1(t, x(t))x^{(3)}(t) + f_2(t, x(t))x^{(2)}(t) + f_3(t, x(t))x^{(1)}(t) \\ + g_0(t, x(t)) + \sum_{i=1}^n g_i(t, x(t - \tau_i(t))) = p(t) \end{aligned} \quad (1.1)$$

where  $f_1, f_2, f_3$  and  $g_i$  ( $i = 0, 1, 2, \dots, n$ ) are continuous functions on  $R^+ \times R$ ,  $\tau_i(t) \geq 0$  ( $i = 1, 2, \dots, n$ ) and  $p(t)$  are bounded continuous functions on  $R^+ = [0, +\infty)$ .

Define  $y(t) = \frac{dx(t)}{dt} + d_1x(t)$ ,  $z(t) = \frac{dy(t)}{dt} + d_2y(t)$  and  $w(t) = \frac{dz(t)}{dt} + d_3z(t)$  where  $d_1, d_2$  and  $d_3$  are some constants. Then, we can transform Eq. (1.1) into the system, as

follows,

$$\begin{aligned}
\frac{dx(t)}{dt} &= -d_1x(t) + y(t) \\
\frac{dy(t)}{dt} &= -d_2y(t) + z(t) \\
\frac{dz(t)}{dt} &= -d_3z(t) + w(t) \\
\frac{dw(t)}{dt} &= -(f_1(t, x(t)) - d_1 - d_2 - d_3)w(t) \\
&\quad + (-(d_1 + d_2 - f_1(t, x(t)))(d_1 + d_2 + d_3) + (d_1d_2 - f_2(t, x(t))) - d_3^2)z(t) \\
&\quad + ((d_1 + d_2 - f_1(t, x(t)))(d_1^2 + d_1d_2 + d_2^2) - (d_1d_2 - f_2(t, x(t)))(d_1 + d_2) \\
&\quad - f_3(t, x(t)))y(t) \\
&\quad + ((f_1(t, x(t)) - d_1 - d_2)d_1^3 + (d_1d_2 - f_2(t, x(t)))d_1^2 + f_3(t, x(t))d_1)x(t) \\
&\quad - g_0(t, x(t)) - \sum_{i=1}^n g_i(t, x(t - \tau_i(t))) + p(t). \tag{1.2}
\end{aligned}$$

In applied science, some practical problems are associated with higher-order nonlinear differential equations, such as nonlinear oscillations ([1]–[4]), electronic theory [5], biological model and other models ([6], [7]). Just as above, in the past few decades, the study of qualitative behaviors for higher order differential equations has been paid attention to by many scholars. And, many results relative to the stability and boundedness of solutions of higher order differential equations with delays or without delays have been obtained in view of various methods, especially, Liapunov’s method (see [8]–[23] and references therein). On the other hand, some researchers have obtained their results for higher order differential equations with several deviating arguments without using Liapunov’s method and Liapunov functional (see [24]–[27]). However, to the best of our knowledge, no authors have considered the boundedness of solutions of fourth order differential equations with multiple deviating arguments in non-Liapunov sense. By the way, we interpret that forming the forthcoming conditions in non-Liapunov sense for our results is easier and more useful than determining a Liapunov functional for higher order differential equations with delays. Thus, it is worthwhile to continue to investigate the boundedness of solutions of Eq. (1.1) in this case.

The main objective of this paper is to study the uniformly boundedness of solutions of (1.2). We will establish some sufficient conditions satisfying the solutions of (1.2) to be uniformly bounded. Our results are new and complement to previously known results.

## 2 Definition and Assumptions

We assume that  $h = \max_{1 \leq i \leq n} \left\{ \sup_{t \in R} \tau_i(t) \right\} \geq 0$ . Let  $C([-h, 0], R)$  denote the space of continuous functions  $\phi: [-h, 0] \rightarrow R$  with the supremum norm. It is known from ([28]–[30]) that for  $g_i (i = 0, 1, 2, \dots, n)$ ,  $\phi$ ,  $f_1$ ,  $f_2$ ,  $f_3$ ,  $p$  and  $\tau_i(t) (i = 1, 2, \dots, n)$  continuous, given a continuous initial function  $\phi \in C([-h, 0], R)$  and a vector  $(y_0, z_0, w_0) \in R^3$ , there exists a solution of (1.2) on an interval  $[0, T)$  satisfying the initial condition and satisfying (1.2) on  $[0, T)$ . If the solution remains bounded, then  $T = +\infty$ . We denote such a solution by  $(x(t), y(t), z(t), w(t)) = (x(t, \phi, y_0, z_0, w_0), y(t, \phi, y_0, z_0, w_0), z(t, \phi, y_0, z_0, w_0), w(t, \phi, y_0, z_0, w_0))$  where  $y(s) = y_0$ ,  $z(s) = z_0$  and  $w(s) = w_0$  for all  $s \in [-h, 0]$ . Then, it follows that  $(x(t), y(t), z(t), w(t))$  can be defined on  $[-h, +\infty)$ .

**Definition.** Solutions of (1.2) are called uniformly bounded (UB) if for each  $B_1 > 0$  there is a  $B_2 > 0$  such that  $(\phi, y_0, z_0, w_0) \in C([-h, 0], R) \times R^3$  and  $\|\phi\| + \|y_0\| + \|z_0\| + \|w_0\| \leq B_1$  imply that  $|x(t, \phi, y_0, z_0, w_0)| + |y(t, \phi, y_0, z_0, w_0)| + |z(t, \phi, y_0, z_0, w_0)| + |w(t, \phi, y_0, z_0, w_0)| \leq B_2$  for all  $t \in R^+$ .

In this work, we also assume that the following conditions hold:

There exist constants  $K > 0$ ,  $d_1 > 1$ ,  $d_2 > 1$ ,  $d_3 > 1$ ,  $d_4 > 0$  and nonnegative constants  $L_i$  and  $q_i$  ( $i = 0, 1, 2, \dots, n$ ) such that

i)  $|(f_1(t, x(t)) - d_1 - d_2)d_1^3 + (d_1d_2 - f_2(t, x(t)))d_1^2 + f_3(t, x(t))d_1)u - g_0(t, u)| \leq L_0|u|$ , for all  $u \in R$  and  $t \geq K$ ,

ii)  $|g_1(t, u)| \leq L_1|u| + q_1$ ,  $|g_2(t, u)| \leq L_2|u| + q_2, \dots, |g_n(t, u)| \leq L_n|u| + q_n$  for all  $u \in R$  and  $t \geq K$ ,

iii)  $d_4 = \inf_{t \geq K} (f_1(t, x(t)) - d_1 - d_2 - d_3)$

$-(\sup_{t \geq K} |(d_1 + d_2 - f_1(t, x(t)))(d_1^2 + d_1d_2 + d_2^2) - (d_1d_2 - f_2(t, x(t)))(d_1 + d_2) - f_3(t, x(t))|)$

$+ \sup_{t \geq K} |-(d_1 + d_2 - f_1(t, x(t)))(d_1 + d_2 + d_3) + (d_1d_2 - f_2(t, x(t))) - d_3^2|) > \sum_{i=0}^n L_i.$

### 3 Main Results

**Theorem 1.** Suppose (i)-(iii) hold. Then solutions of (1.2) are uniformly bounded.

*Proof.* Let  $(x(t), y(t), z(t), w(t))$  be a solution of system (1.2) with initial conditions  $x(s) = \phi(s)$ ,  $y(0) = y_0$ ,  $z(0) = z_0$  and  $w(0) = w_0$  for all  $s \in [-h, 0]$  where  $\phi \in C([-h, 0], R)$  and  $(y_0, z_0, w_0) \in R^3$ .

Calculating the upper right derivatives of  $|x(s)|$ ,  $|y(s)|$ ,  $|z(s)|$  and  $|w(s)|$  along (1.2), in view of (i)-(iii), we have

$$\begin{aligned} D^+(|x(s)|)_{s=t} &= \operatorname{sgn}(x(t))\{-d_1x(t) + y(t)\} \\ &\leq -d_1|x(t)| + |y(t)|, \end{aligned} \quad (3.1)$$

$$\begin{aligned} D^+(|y(s)|)_{s=t} &= \operatorname{sgn}(y(t))\{-d_2y(t) + z(t)\} \\ &\leq -d_2|y(t)| + |z(t)|, \end{aligned} \quad (3.2)$$

$$\begin{aligned} D^+(|z(s)|)_{s=t} &= \operatorname{sgn}(z(t))\{-d_3z(t) + w(t)\} \\ &\leq -d_3|z(t)| + |w(t)|, \end{aligned} \quad (3.3)$$

and

$$\begin{aligned}
& D^+(|w(s)|)_{s=t} \\
&= \operatorname{sgn}(w(t))\{-f_1(t, x(t)) - d_1 - d_2 - d_3\}w(t) \\
&+ (-(d_1 + d_2 - f_1(t, x(t)))(d_1 + d_2 + d_3) + (d_1d_2 - f_2(t, x(t))) - d_3^2)z(t) \\
&+ ((d_1 + d_2 - f_1(t, x(t)))(d_1^2 + d_1d_2 + d_2^2) - (d_1d_2 - f_2(t, x(t)))(d_1 + d_2) \\
&- f_3(t, x(t)))y(t) \\
&+ ((f_1(t, x(t)) - d_1 - d_2)d_1^3 + (d_1d_2 - f_2(t, x(t)))d_1^2 + f_3(t, x(t))d_1)x(t) \\
&- g_0(t, x(t)) - \sum_{i=1}^n g_i(t, x(t - \tau_i(t)) + p(t)\} \\
&\leq \{(-\inf_{t \geq K} (f_1(t, x(t)) - d_1 - d_2 - d_3)) |w(t)| \\
&+ \sup_{t \geq K} |-(d_1 + d_2 - f_1(t, x(t)))(d_1 + d_2 + d_3) + (d_1d_2 - f_2(t, x(t))) - d_3^2| |z(t)| \\
&+ \sup_{t \geq K} |(d_1 + d_2 - f_1(t, x(t)))(d_1^2 + d_1d_2 + d_2^2) \\
&- (d_1d_2 - f_2(t, x(t)))(d_1 + d_2) - f_3(t, x(t))| |y(t)| \\
&+ L_0 |x(t)| + \sum_{i=1}^n L_i |x(t - \tau_i(t))|\} + \sum_{i=0}^n q_i + |p(t)|. \tag{3.4}
\end{aligned}$$

Let

$$M(t) = \max_{-h \leq s \leq t} \{\max\{|x(s)|, |y(s)|, |z(s)|, |w(s)|\}\}.$$

It is clear that  $\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} \leq M(t)$  and  $M(t)$  is non-decreasing. Now, we consider the following two cases:

Case I): If there exists a sufficiently large constant  $K_1 > K$  such that

$$M(t) > \max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} \tag{3.5}$$

for all  $t \geq K_1$ , then we claim that

$$M(t) \equiv M(K_1) \tag{3.6}$$

is a constant for all  $t \geq K_1$ .

By contrapositive, assume (3.6) does not hold, then, there exists  $t_1 \geq K_1$  such that  $M(t_1) > M(K_1)$ .

Here  $\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} \leq M(K_1)$  for all  $-h \leq t \leq K_1$  and there exists  $\beta \in (K_1, t_1)$  such that  $\max\{|x(\beta)|, |y(\beta)|, |z(\beta)|, |w(\beta)|\} = M(t_1) \geq M(\beta)$  which contradicts (3.5). This implies that (3.6) holds. It follows that there exists  $t_2 \geq K_1$  such that  $\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} < M(t) = M(K_1)$  for all  $t \geq t_2$ .

Case II): There is a point  $t_0 \geq K_1$  such that

$$M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\}.$$

Let  $\eta = \min\left\{d_1 - 1, d_2 - 1, d_3 - 1, d_4 - \sum_{i=0}^n L_i\right\} > 0$  and  $\theta = \sum_{i=0}^n q_i + \sup_{t \in R^+} |p(t)| + 1$  where  $t \geq K_1$ . Then, if  $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |x(t_0)|$ , then, in

view of (3.1), we obtain

$$\begin{aligned} 0 &\leq D^+(|x(s)|)_{s=t_0} \leq -d_1|x(t)| + |y(t)| \\ &\leq (1-d_1)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \quad (3.7)$$

if  $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |y(t_0)|$ , then, in view of (3.2), we have

$$\begin{aligned} 0 &\leq D^+(|y(s)|)_{s=t_0} \leq -d_2|y(t)| + |z(t)| \\ &\leq (1-d_2)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \quad (3.8)$$

if  $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |z(t_0)|$ , then, in view of (3.3), we get

$$\begin{aligned} 0 &\leq D^+(|z(s)|)_{s=t_0} \leq -d_3|z(t)| + |w(t)| \\ &\leq (1-d_3)M(t_0) \\ &< -\eta M(t_0) + \theta, \end{aligned} \quad (3.9)$$

if  $M(t_0) = \max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} = |w(t_0)|$ , then, in view of (3.4), we get

$$\begin{aligned} 0 &\leq D^+(|w(s)|)_{s=t_0} \\ &\leq \{(-\inf_{t \geq K} (f_1(t) - d_1 - d_2 - d_3))|w(t)| \\ &\quad + \sup_{t \geq K} |-(d_1 + d_2 - f_1(t, x(t)))(d_1 + d_2 + d_3) + (d_1 d_2 - f_2(t, x(t))) - d_3^2||z(t)| \\ &\quad + \sup_{t \geq K} |(d_1 + d_2 - f_1(t, x(t)))(d_1^2 + d_1 d_2 + d_2^2) \\ &\quad - (d_1 d_2 - f_2(t, x(t)))(d_1 + d_2) - f_3(t, x(t))||y(t_0)| \\ &\quad + L_0|x(t_0)| + \sum_{i=1}^n L_i|x(t_0 - \tau_i(t_0))|\} + \sum_{i=0}^n q_i + |p(t)| \\ &\leq (\sum_{i=0}^n L_i - d_4)M(t_0) + \sum_{i=0}^n q_i + |p(t)| \\ &< -\eta M(t_0) + \theta. \end{aligned} \quad (3.10)$$

In addition, if  $M(t_0) \geq \frac{\theta}{\eta}$ , (3.7), (3.8), (3.9) and (3.10) imply that  $M(t)$  is strictly decreasing in a small neighborhood  $(t_0, t_0 + \delta_0)$ . This contradicts that  $M(t)$  is non-decreasing. Therefore,  $M(t_0) < \frac{\theta}{\eta}$  and

$$\max\{|x(t_0)|, |y(t_0)|, |z(t_0)|, |w(t_0)|\} < \frac{\theta}{\eta}. \quad (3.11)$$

For  $\forall t > t_0$ , by the same approach used in the proof of (3.11), we have

$$\max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} < \frac{\theta}{\eta}, \text{ if } M(t) = \max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\}.$$

On the other hand, if  $M(t) > \max\{|x(t)|, |y(t)|, |z(t)|, |w(t)|\}$ ,  $t > t_0$ , then, we can choose  $t_0 \leq t_3 < t$  such that  $M(t_3) = \max\{|x(t_3)|, |y(t_3)|, |z(t_3)|, |w(t_3)|\} < \frac{\theta}{\eta}$

and  $M(s) > \max \{|x(s)|, |y(s)|, |z(s)|, |w(s)|\}$  for all  $s \in (t_3, t]$ . Using a similar argument as in the proof of Case (I), we can show that  $M(s) \equiv M(t_3)$  is a constant, for all  $s \in (t_3, t]$ , which implies  $\max \{|x(t)|, |y(t)|, |z(t)|, |w(t)|\} < M(t) = M(t_3) = \max \{|x(t_3)|, |y(t_3)|, |z(t_3)|, |w(t_3)|\} < \frac{\theta}{\eta}$ .

To sum up, the solutions of (1.2) are uniformly bounded. The proof is complete.  $\square$

#### 4 An Example

Consider the following fourth-order non-linear differential equation

$$\begin{aligned} & x^{(4)}(t) + \left(18 - \frac{1}{1+t+x^2(t)}\right)x^{(3)}(t) + \left(78 - \frac{4}{1+t+x^2(t)}\right)x^{(2)}(t) \\ & + \left(127 - \frac{3}{1+t+x^2(t)}\right)x^{(1)}(t) + \left(69 + \frac{3}{1+t+x^2(t)}\right)x(t) + \sin x(t - |\sin t|) \\ & + \cos x(t - 2|\sin t|) + \sin t \sin x(t - e^{|\sin t|}) + \cos t \cos x(t - e^{2|\sin t|}) = \frac{1}{1+t^2} \end{aligned} \quad (4.1)$$

Setting  $y(t) = \frac{dx(t)}{dt} + 2x(t)$ ,  $z(t) = \frac{dy(t)}{dt} + 2y(t)$  and  $w(t) = \frac{dz(t)}{dt} + 2z(t)$  we can transform (4.1) into the following system

$$\begin{aligned} \frac{dx(t)}{dt} &= -2x(t) + y(t) \\ \frac{dy(t)}{dt} &= -2y(t) + z(t) \\ \frac{dz(t)}{dt} &= -2z(t) + w(t) \end{aligned}$$

$$\begin{aligned} \frac{dw(t)}{dt} &= -\left(10 - \frac{1}{1+t+x^2(t)}\right)w(t) + \left(2 - \frac{2}{1+t+x^2(t)}\right)z(t) + \left(1 - \frac{1}{1+t+x^2(t)}\right)y(t) \\ & + \left(1 - \frac{1}{1+t+x^2(t)}\right)x(t) - \sin x(t - |\sin t|) - \cos x(t - 2|\sin t|) \\ & - \sin t \sin x(t - e^{|\sin t|}) - \cos t \cos x(t - e^{2|\sin t|}) + \frac{1}{1+t^2}. \end{aligned} \quad (4.2)$$

Then we can satisfy the assumptions ( $i - iii$ ):

- (i)  $\left|1 - \frac{1}{1+t+u^2}\right| \leq L_0 |u| + q_0$  for all  $t, u \in R$ ,
- (ii)  $|g_1(t, u)| = |\sin u| \leq L_1 |u| + q_1$ ,  $|g_2(t, u)| = |\cos u| \leq L_2 |u| + q_2$ ,  $|g_3(t, u)| = |\sin t \sin u| \leq L_3 |u| + q_3$ ,  $|g_4(t, u)| = |\cos t \cos u| \leq L_4 |u| + q_4$  for all  $t, u \in R$ ,
- (iii)  $d_4 = \inf_{t \geq K} \left(10 - \frac{1}{1+t+x^2(t)}\right) - \left(\sup_{t \geq K} \left|2 - \frac{2}{1+t+x^2(t)}\right| + \sup_{t \geq K} \left|1 - \frac{1}{1+t+x^2(t)}\right|\right) > \sum_{i=0}^4 L_i$  by taking suitable  $L_i$  and  $q_i$  such as  $L_0 = L_1 = L_2 = L_3 = L_4 = 1$  for appropriate  $q_i$  ( $i = 0, 1, 2, 3, 4$ ). Hence, all solutions of the system (4.2) are uniformly bounded.

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## References

- [1] A.U. Afuwape, P. Omari, and F. Zanolin, Nonlinear perturbations of differential operators with nontrivial kernel and applications to third-order periodic boundary problems, *J. Math. Anal. Appl.* **143** (1989), 35–56.
- [2] J. Andres, Periodic boundary value problem for certain nonlinear differential equation of the third order, *Math. Slovaca* **35** (1985), 305–309.
- [3] K.O. Fridedrichs, *On nonlinear vibrations of the third-order*, in: “Studies in Nonlinear Vibrations Theory”, Inst. Math. Mech., New York University, 1949.
- [4] P. Amster and M.C. Mariani, Oscillating solutions of a nonlinear fourth order ordinary differential equation, *J. Math. Anal. Appl.* **325** (2007), 1133–1141.
- [5] L.L. Rauch, Oscillations of a third order nonlinear autonomous system, in: Contributions to the Theory of Nonlinear Oscillations, *Ann. Math. Stud.* **20** (1950), 39–88.
- [6] J. Cronin, Some mathematics of biological oscillations, *SIAM Rev.* **19** (1977), 100–137.
- [7] K. Gopalsamy, “Stability and Oscillations in Delay Differential Equations of Population Dynamic”, Mathematics and Its Applications, vol. 74, Kluwer Academic, Dordrecht, 1992.
- [8] R. Reissig, G. Sansone and R. Conti, “Non-linear Differential Equations of Higher Order”, Translated from the German. Noordhoff International Publishing, Leyden, 1974.
- [9] H.O. Tejumola, A note on the boundedness and the stability of solutions of certain third-order differential equations, *Ann. Mat. Pura Appl.* (4) **92** (1972), 65–75.
- [10] J.O.C. Ezeilo and H.O. Tejumola, Boundedness theorems for certain third order differential equations, *Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur.* (8) **55** (1973), 194–201.
- [11] T. Hara, On the uniform ultimate boundedness of the solutions of certain third order differential equations, *J. Math. Anal. Appl.* **80**(2) (1981), 533–544.
- [12] S. Murakami, Asymptotic behavior of solutions of some differential equations, *J. Math. Anal. Appl.* **109** (1985), 534–545.
- [13] B. Mehri and D. Shadman, Boundedness of solutions of certain third order differential equation. *Math. Inequal. Appl.* **2**(4) (1999), 545–549.
- [14] M.O. Omeike, Stability and boundedness of solutions of some non-autonomous delay differential equation of the third order, *An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.)* **55**(1) (2009), 49–58.
- [15] T.A. Ademola, M.O. Ogundiran, P.O. Arawomo, and O.A. Adesina, Boundedness results for a certain third order nonlinear differential equation, *Appl. Math. Comput.* **216**(10) (2010), 3044–3049.
- [16] C. Tunç, On the stability and boundedness of solutions to third order nonlinear differential equations with retarded argument, *Nonlinear Dynam.* **57**(1-2) (2009), 97–106.
- [17] C. Tunç, Some remarks on the stability and boundedness of solutions of certain differential equations of fourth-order, *Comput. Appl. Math.* **26**(1) (2007), 1–17.
- [18] C. Tunç and H. Ergören, On boundedness of a certain non-linear differential equation of third order, *J. Comput. Anal. Appl.* **12**(3) (2010), 687–94.
- [19] C. Tunç, A boundedness criterion for fourth order nonlinear ordinary differential equations with delay, *Int. J. Nonlinear Sci.* **6**(3) (2008), 195–201.
- [20] C. Tunç, The boundedness to nonlinear differential equations of fourth order with delay, *Nonlinear Stud.* **17**(1) (2010), 47–56.
- [21] C. Tunç, Boundedness of solutions to third-order nonlinear differential equations with bounded delay, *J. Franklin Inst.* **347**(2) (2010), 415–425.

- [22] H. Ergören and C. Tunç, Anti-Periodic Solutions for a Class of Fourth-Order Nonlinear Differential Equations with Multiple Deviating Arguments, *Azerbaijan Journal of Mathematics* **3**(1) (2013), 111–126.
- [23] W. Xionga, G. Yue, Almost periodic solutions for a class of fourth-order nonlinear differential equations with a deviating argument, *Computers & Mathematics with Applications* **60**(5) (2010), 1184–1190.
- [24] A.U. Afuwape and J.E. Castellanos, Asymptotic and exponential stability of certain third-order non-linear delayed differential equations: frequency domain method, *Appl. Math. Comput.* **216**(3) (2010), 940–950.
- [25] H. Gao and B. Liu, Almost periodic solutions for a class of Lienard-type systems with multiple varying time delays, *Appl. Math. Model.* **34** (2010), 72–79.
- [26] Y. Yu and C. Zhao, Boundedness of solutions for a Lienard equation with multiple deviating arguments, *Elec. J. Differential Equations.* **2009**(14) (2009), 1–5.
- [27] H. Ergören and C. Tunç, Uniformly boundedness for a class of non-linear differential equations of third order with multiple deviating arguments. *Cubo. A Mathematical Journal* **14**(3) (2012), 63–69.
- [28] T.A. Burton, “Stability and Periodic Solutions of Ordinary and Functional Differential Equations”, Academic Press, Orland, FL, 1985.
- [29] J.K. Hale, “Theory of Functional Differential Equations”, Springer-Verlag, New York, 1977.
- [30] Y. Kuang, “Delay Differential Equations with Applications in Population Dynamics”, Academic Press, New York, 1993.