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Products of composition and iterated differentation operators from fractional Cauchy transforms to weighted Bloch-type spaces

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Abstract

We consider products of composition and iterated differentiation operators from the space of fractional Cauchy transforms to weighted Bloch-type spaces and little weighted Bloch-type spaces. Upper and lower bounds for norm of these operators are computed and compactness is completely characterized.

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1 Introduction and Preliminaries

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$, $\partial\mathbb{D}$ its boundary, $dA(z)$ the normalized area measure on $\mathbb{D}$ (i.e. $A(\mathbb{D}) = 1$) and H^∞ the space of all bounded holomorphic functions on $\mathbb{D}$ with the norm $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$. Let $H(\mathbb{D})$ the class of all holomorphic functions on $\mathbb{D}$. $H(\mathbb{D})$ is a locally convex linear topological space with respect to the topology given by uniform convergence on compact subsets of $\mathbb{D}$. We denote by $\mathfrak{M}$ the space of all complex Borel measures on $\partial\mathbb{D}$ and let $\mathfrak{M}^*$ be the subset of $\mathfrak{M}$ consisting of probability measures. Let $\alpha > 0$ be a real number. The family $\mathcal{F}_\alpha$ of fractional Cauchy transforms is the collection of functions $f \in H(\mathbb{D})$ which admits a representation of the form

$$f(z) = \int_{\partial\mathbb{D}} \frac{1}{(1-\overline{\zeta}z)^\alpha} d\mu(\zeta) \quad (z \in \mathbb{D}) \tag{1.1}$$

for some $\mu \in \mathfrak{M}$. The principal branch is used in the power function in (1.1) and throughout the rest of the paper. The space $\mathcal{F}_\alpha$ is a Banach space with respect to the norm

$$\|f\|_{\mathcal{F}_\alpha} = \inf_{\mu\in\mathfrak{M}} \left\{ \|\mu\| \ : \ f(z) = \int_{\partial\mathbb{D}} \frac{1}{(1-\overline{\zeta}z)^\alpha} d\mu(\zeta) \right\},$$

where $\|\mu\|$ denotes the total variation of measure μ. According to the Lebesgue decomposition theorem $\mathfrak{M} = \mathfrak{M}_a + \mathfrak{M}_s$, where $\mathfrak{M}_a = \{\mu_a \in \mathfrak{M} : \mu_a << m\}$, where m is the normalized Lebesgue measure on the unit circle $\partial\mathbb{D}$, and $\mathfrak{M}_s = \{\mu_s \in \mathfrak{M} : \mu_s \perp m\}$. Thus any μ can be written as $\mu = \mu_a + \mu_s$, where $\mu_a \in \mathfrak{M}_a$, $\mu_s \in \mathfrak{M}_s$ and $\|\mu\| = \|\mu_a\| + \|\mu_s\|$. Consequently, the space $\mathcal{F}_\alpha$ may also be written as $\mathcal{F}_\alpha = (\mathcal{F}_\alpha)_a + (\mathcal{F}_\alpha)_s$, where $(\mathcal{F}_\alpha)_a$ is isometrically isomorphic to $\mathfrak{M}/\overline{H_0^1}$, the closed subspace of $\mathfrak{M}$ of absolutely continuous measures and $(\mathcal{F}_\alpha)_s$ is isomorphic to $\mathfrak{M}_s$ the closed subspace of $\mathfrak{M}$ of singular measures. If $f \in (\mathcal{F}_\alpha)_a$, then the singular part is null and the measure μ for which the integral in (1.1) holds reduces to $d\mu(e^{it}) = g(e^{it})dt$, where $g(e^{it}) \in L^1$ and dt is the Lebesgue measure on $\partial\mathbb{D}$. For more about the space $\mathcal{F}_\alpha$, we refer [1], [2] [3], [4], [8], [9] and [10]. Let

$$\eta_a(z) = \frac{a - z}{1 - \bar{a}z}, \quad a, z \in \mathbb{D},$$

that is, the involutive automorphism of $\mathbb{D}$ interchanging points a and 0. Also we need the following well known identity

$$(1 - |z|^2)|\eta_a'(z)| = 1 - |\eta_a(z)|^2 = \frac{(1 - |a|^2)(1 - |z|^2)}{|1 - \bar{a}z|^2} \tag{1.2}$$

The Bloch-type space $\mathcal{B}_\nu(\mathbb{D}) = \mathcal{B}_\nu$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathcal{B}_\nu} := |f(0)| + b_\nu(f) = |f(0)| + \sup_{z\in\mathbb{D}} \nu(z)|f'(z)| < \infty,$$

where ν is a positive continuous function on $\mathbb{D}$ (*weight*). A weight ν is called *typical* if it is radial, i.e. $\nu(z) = \nu(|z|)$, $z \in \mathbb{D}$ and $\nu(|z|)$ decreasingly converges to 0 as $|z| \to 1$. A positive continuous function ν on the interval $[0, 1)$ is called normal if there are $\delta \in [0, 1)$ and τ and t, $0 < \tau < t$ such that

$$\frac{\nu(r)}{(1 - r)^\tau} \text{ is decreasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1 - r)^\tau} = 0;$$

$$\frac{\nu(r)}{(1 - r)^t} \text{ is increasing on } [\delta, 1) \text{ and } \lim_{r\to 1} \frac{\nu(r)}{(1 - r)^t} = \infty.$$

If we say that a function $\nu : \mathbb{D} \to [0, \infty)$ is normal we also assume that it is radial. The little Bloch-type space $\mathcal{B}_{\nu,0}(\mathbb{D}) = \mathcal{B}_{\nu,0}$ consists of all $f \in H(\mathbb{D})$ such that

$$\lim_{|z|\to 1} \nu(z)|f'(z)| = 0.$$

With the norm $\|\cdot\|_{\mathcal{B}_\nu}$ the Bloch-type space $\mathcal{B}_\nu$ is a Banach space and the little Bloch-type space $\mathcal{B}_{\nu,0}$ is a closed subspace of the Bloch-type space $\mathcal{B}_\nu$.
Let φ be a holomorphic self-map of $\mathbb{D}$. For a non-negative integer n, we define a linear operator D_φ^n as follows:

$$D_\varphi^n f = f^{(n)} \circ \varphi, \quad f \in H(\mathbb{D}).$$

If $n = 0$, then we have $D_\varphi^n = C_\varphi$, the composition operator induced by φ, defined as $C_\varphi f = f \circ \varphi$, $f \in H(\mathbb{D})$. We recall that an operator T from a Banach space X to a Banach space Y is bounded if there exists a positive constant C such that $\|Tf\|_Y \le C\|f\|_X$. A bounded operator $T : X \to Y$ is compact if the image of every bounded set in X is relatively compact in Y. Equivalently, $T : X \to Y$ is compact if for every bounded sequence $\{f_m\}$ in X, $\{Tf_m\}$ has a convergent sequence in Y. In [8], Hibschweiler and MacGregor proved that if $\alpha \geq 1$, then every holomorphic self-map φ of $\mathbb{D}$ induces a bounded composition operator on $\mathcal{F}_\alpha$. In fact, Bourdon and Cima [1] proved that

$$\|C_\varphi\|_{\mathcal{F}_1 \to \mathcal{F}_1} \leq \frac{2 + 2\sqrt{2}}{1 - |\varphi(0)|}$$

which was improved to

$$\|C_\varphi\|_{\mathcal{F}_1 \to \mathcal{F}_1} \leq \frac{1 + 2|\varphi(0)|}{1 - |\varphi(0)|}$$

by Cima and Matheson [3]. Moreover, equality is attained for certain linear fractional maps.

In contrast with the situation when $\alpha \geq 1$, a self-map φ of $\mathbb{D}$ need not induce a bounded composition operator on $\mathcal{F}_\alpha$ when $0 < \alpha < 1$. In fact, the condition $\varphi \in \mathcal{F}_\alpha$ is necessary for C_φ to be bounded on $\mathcal{F}_\alpha$. Hibschweiler and MacGregor [8], constructed a self-map φ of $\mathbb{D}$ with $\varphi \notin \mathcal{F}_\alpha (0 < \alpha < 1)$. For some recent results in this area, see [2],[6],[7], [11], [13] and the references therein. In this paper, we characterize boundedness and compactness of products of composition and iterated differentation from fractional Cauchy transforms to weighted Bloch-type spaces. Throughout the paper constants are denoted by C, they are positive and not necessarily the same at each occurrence. The notation $A \asymp B$ means that there is a positive constant C such that $A/C \leq B \leq CA$.

2 Boundedness and Compactness of $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$

In this section, we characterize the boundedness and compactness of D_φ^n from the space of fractional Cauchy transforms to weighted Bloch-type spaces.

The following lemma can be found in [7], and is used throughout the rest of the paper.

Lemma 1. *Let $\alpha > 0$ and $f \in H(\mathbb{D})$.*

(1) If $f \in \mathcal{F}_\alpha$ and $z \in \mathbb{D}$, then $|f(z)| \leq \|f\|_{\mathcal{F}_\alpha}/(1 - |z|)^\alpha$.

(2) If $f \in \mathcal{F}_\alpha$, then $f' \in \mathcal{F}_{\alpha+1}$ and $\|f'\|_{\mathcal{F}_{\alpha+1}} \leq \alpha \|f\|_{\mathcal{F}_\alpha}$.

Theorem 2. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$ and φ a holomorphic self-map of $\mathbb{D}$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded if and only if*

$$M_1 := \sup_{\zeta \in \partial\mathbb{D}} \sup_{z \in \mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1 - \overline{\zeta}\varphi(z)|^{n+\alpha+1}} < \infty. \tag{2.1}$$

Moreover, if $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded, then

$$\begin{aligned} \alpha(\alpha+1)\cdots(\alpha+n)M_1 &\leq \|D_\varphi^n\|_{\mathcal{F}_\alpha \to \mathcal{B}_\nu} \\ &\leq \alpha(\alpha+1)\cdots(\alpha+n-1)\left\{(\alpha+n)M_1 + \frac{1}{(1 - |\varphi(0)|)^{n+\alpha}}\right\}. \end{aligned} \tag{2.2}$$

Proof. First, suppose that (2.1) holds. Let $f \in \mathcal{F}_\alpha$. Then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{F}_\alpha}$ and

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{(1 - \overline{\zeta}z)^\alpha}.$$

Thus, we have

$$f^{(n+1)}(z) = \alpha(\alpha+1)\cdots(\alpha+n) \int_{\partial\mathbb{D}} \frac{(\overline{\zeta})^{n+1}}{(1 - \overline{\zeta}z)^{n+\alpha+1}} d\mu(\zeta). \tag{2.3}$$

Replacing z in (2.3) by $\varphi(z)$, using a known inequality and multiplying such obtained inequality by $\nu(z)|\varphi'(z)|$, we obtain

$$\begin{aligned}\nu(z)|\varphi'(z)||f^{(n+1)}(\varphi(z))| &\le \alpha(\alpha+1)\cdots(\alpha+n)\int_{\partial\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}d|\mu|(\zeta) \\ &\le \alpha(\alpha+1)\cdots(\alpha+n)\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}\int_{\partial\mathbb{D}}d|\mu|(\zeta) \\ &= \alpha(\alpha+1)\cdots(\alpha+n)\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}\|\mu\|\end{aligned}\tag{2.4}$$

from which it follows that

$$\nu(z)|(D_\varphi^n f)'(z)| \le \alpha(\alpha+1)\cdots(\alpha+n)\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}\|f\|_{\mathcal{F}_\alpha}.$$

Taking the supremum over $z\in\mathbb{D}$, we get

$$\sup_{z\in\mathbb{D}}\nu(z)|(D_\varphi^n f)'(z)| \le \alpha(\alpha+1)\cdots(\alpha+n)M_1\|f\|_{\mathcal{F}_\alpha}.\tag{2.5}$$

By Lemma 1, we have

$$|(D_\varphi^n f)(0)| = |f^{(n)}(\varphi(0))| \le \frac{\|f^n\|_{\mathcal{F}_{n+\alpha}}}{(1-|\varphi(0)|)^{n+\alpha}} \le \alpha(\alpha+1)\cdots(\alpha+n-1)\frac{\|f\|_{\mathcal{F}_\alpha}}{(1-|\varphi(0)|)^{n+\alpha}}.\tag{2.6}$$

Thus from (2.5) and (2.6), we have

$$\|D_\varphi^n f\|_{\mathcal{B}_\nu} \le \alpha(\alpha+1)\cdots(\alpha+n-1)\left\{(\alpha+n)M_1+\frac{1}{(1-|\varphi(0)|)^{n+\alpha}}\right\}\|f\|_{\mathcal{F}_\alpha}.$$

Hence $D_\varphi^n:\mathcal{F}_\alpha\to\mathcal{B}_\nu$ is bounded and

$$\|D_\varphi^n\|_{\mathcal{F}_\alpha\to\mathcal{B}_\nu} \le \alpha(\alpha+1)\cdots(\alpha+n-1)\left\{(\alpha+n)M_1+\frac{1}{(1-|\varphi(0)|)^{n+\alpha}}\right\}.\tag{2.7}$$

Next suppose that $D_\varphi^n:\mathcal{F}_\alpha\to\mathcal{B}_\nu$ is bounded. Let

$$f_\zeta(z)=\frac{1}{(1-\overline{\zeta}z)^\alpha},\quad \zeta\in\partial\mathbb{D}.\tag{2.8}$$

Then $\|f_\zeta\|_{\mathcal{F}_\alpha}=1$ and

$$f_\zeta^{(n+1)}(z)=\alpha(\alpha+1)\cdots(\alpha+n)\frac{(\overline{\zeta})^{n+1}}{(1-\overline{\zeta}z)^{n+\alpha+1}}.$$

From this and the boundedness of the operator $D_\varphi^n:\mathcal{F}_\alpha\to\mathcal{B}_\nu$, we have that $\|D_\varphi^n f_\zeta\|_{\mathcal{B}_\nu}\le\|D_\varphi^n\|_{\mathcal{F}_\alpha\to\mathcal{B}_\nu}$, for every $\zeta\in\partial\mathbb{D}$ and so

$$\alpha(\alpha+1)\cdots(\alpha+n)\sup_{\zeta\in\partial\mathbb{D}}\sup_{z\in\mathbb{D}}\frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} \le \|D_\varphi^n\|_{\mathcal{F}_\alpha\to\mathcal{B}_\nu}.\tag{2.9}$$

If $D_\varphi^n:\mathcal{F}_\alpha\to\mathcal{B}_\nu$ is bounded, then from (2.7) and (2.9), inequality in (2.2) follows.

□

Theorem 3. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$, φ a holomorphic self-map of $\mathbb{D}$ and $d\lambda(z) = dA(z)/(1-|z|^2)^2$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded if and only if*

$$L_1 := \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{|\varphi'(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty. \tag{2.10}$$

Moreover, if $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded, then asymptotic relation $L_1 \asymp M_1^2$ holds.

Proof. First assume that (2.10) holds. Since ν is normal, $\nu(a) \asymp \nu(z)$ when $z \in D(a,(1-|a|)/2) = \{|z-a| < (1-|a|)/2\}$. Also it is known that $|1-\bar{a}z| \asymp 1-|a|^2$, for $z \in D(a,(1-|a|)/2)$. Using these two facts, (1.2) and the subharmonicity of the function

$$g(z) = \frac{|\varphi'(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}}$$

we obtain

$$\begin{aligned} L_1 &\geq \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|\varphi'(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) \\ &= \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{D(a,(1-|a|)/2)} \frac{|\varphi'(z)|^2}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} \nu^2(z) \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) \\ &\geq \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \frac{\nu^2(a)|\varphi'(a)|^2}{|1-\overline{\zeta}\varphi(a)|^{2(n+\alpha+1)}} = M_1^2. \end{aligned} \tag{2.11}$$

Thus by Theorem 1, the operator $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded.
Next assume that the operator $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded. By Theorem 1, we have that (2.1) holds. From this, we have

$$L_1 \leq M_1^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \frac{(1-|a|^2)^2}{|1-\bar{a}z|^4} dA(z) = M_1^2 C < \infty. \tag{2.12}$$

The asymptotic relation $L_1 \asymp M_1^2$ follows from (2.11) and (2.12). □

Proceeding as in the proof of Theorem 2, we can easily prove the following lemma. We omit the proof.

Lemma 4. *Let $\nu : \mathbb{D} \to [0,\infty)$ be a normal weight function and $d\lambda(z) = dA(z)/(1-|z|^2)^2$. Then $f \in \mathcal{B}_\nu$ if and only if*

$$I := |f(0)|^2 + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 d\lambda(z) < \infty.$$

Moreover, the following asymptotic relationship holds

$$\|f\|_{\mathcal{B}_\nu}^2 \asymp I.$$

By Lemma 1, the unit ball $B_{\mathcal{F}_\alpha}$ of $\mathcal{F}_\alpha$ is a normal family, a standard argument from Proposition 3.11 in [5] yields the proof of the next lemma.

Lemma 5. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$ and φ a holomorphic self-map of $\mathbb{D}$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is compact if and only if for any bounded sequence $\{f_m\}_{m\in\mathbb{N}}$ in $\mathcal{F}_\alpha$ converging to zero on compact subsets of $\mathbb{D}$, we have that $\lim_{m\to\infty} \|D_\varphi^n f_m\|_{\mathcal{B}_\nu} = 0$.*

Theorem 6. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N} \cup \{0\}$, φ a holomorphic self-map of $\mathbb{D}$, $d\lambda(z) = dA(z)/(1-|z|^2)^2$ and $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded. Then the following statements are equivalent:*

1. *$D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is compact.*
2. $M_3 := \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \infty$
 and

$$\lim_{r\to1} \sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}}(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) = 0. \quad (2.13)$$

Proof. (1) $\Rightarrow$ (2). Since $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded, for $f(z) = z^n/n! \in \mathcal{F}_\alpha$, we get

$$M_3 = \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} \nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \infty.$$

Let $f_m(z) = z^m$, $m \in \mathbb{N}$. It is a norm bounded sequence in $\mathcal{F}_\alpha$ converging to zero uniformly on compact subsets of $\mathbb{D}$. Hence by Lemma 2, it follows that $\|D^n_\varphi f_m\|_{\mathcal{B}_\nu} \to 0$ as $m \to \infty$. Thus for every $\epsilon > 0$, there is an $m_0 \in \mathbb{N}$ such that for $m \geq m_0$, we have

$$\Big(\prod_{j=0}^{n}(m-j)\Big)^2 \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |\varphi(z)|^{2(m-n-1)}\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.14)$$

From (2.14), we have that for each $r \in (0,1)$

$$r^{2(m-n-1)}\Big(\prod_{j=0}^{n}(m-j)\Big)^2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.15)$$

Hence for $r \in \big[\prod_{j=0}^{n}(m-j)^{-\frac{1}{m-n-1}}, 1\big)$, we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.16)$$

Let $f \in B_{\mathcal{F}_\alpha}$ and $f_t(z) = f(tz)$, $0 < t < 1$. Then $\sup_{0<t<1}\|f_t\|_{\mathcal{F}_\alpha} \leq \|f\|_{\mathcal{F}_\alpha}$, $f_t \in \mathcal{F}_\alpha$, $t \in (0,1)$ and $f_t \to f$ uniformly on compact subsets of $\mathbb{D}$ as $t \to 1$. The compactness of $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ implies that $\lim_{t\to1}\|D^n_\varphi f_t - D^n_\varphi f\|_{\mathcal{B}_\nu} = 0$. Hence for every $\epsilon > 0$, there is a $t \in (0,1)$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t^{(n+1)}(\varphi(z)) - f^{(n+1)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) < \epsilon. \quad (2.17)$$

By inequalities (2.16) and (2.17), we have

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n+1)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) \\
&\leq 2\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f_t^{(n+1)}(\varphi(z)) - f^{(n+1)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) \\
&\quad + 2\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_t^{(n+1)}(\varphi(z))|^2\nu^2(z)(1-|\eta_a(z)|^2)^2|\varphi'(z)|^2 d\lambda(z) \\
&\leq 2\varepsilon(1 + \|f_t^{(n+1)}\|^2_\infty).
\end{aligned}$$

Hence for every $f \in B_{\mathcal{F}_\alpha}$, there is a $\delta_0 \in (0,1)$, $\delta_0 = \delta_0(f,\epsilon)$, such that for $r \in (\delta_0, 1)$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \epsilon.$$

From the compactness of $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_\nu$, we have that for every $\epsilon > 0$ there is a finite collection of functions $f_1, f_2, \dots, f_k \in B_{\mathcal{F}_\alpha}$ such that for each $f \in B_{\mathcal{F}_\alpha}$, there is a $j \in \{1,2,\dots,k\}$ such that

$$\sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f^{(n+1)}(\varphi(z)) - f_j^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \varepsilon. \tag{2.18}$$

On the other hand, from (2.18) it follows that if $\delta := \max_{1\le j\le k} \delta_j(f_j, \varepsilon)$, then for $r \in (\delta, 1)$ and all $j \in \{1,2,\dots,k\}$ we have

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_j^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \varepsilon. \tag{2.19}$$

From (2.18) and (2.19), we have that for $r \in (\delta, 1)$ and every $f \in B_{\mathcal{F}_\alpha}$

$$\sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f^{(n+1)}(\varphi(z))|^2 \nu^2(z)(1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < 4\varepsilon. \tag{2.20}$$

Applying (2.20) to the functions $f_\zeta(z) = 1/(1-\overline{\zeta}z)^\alpha$, $\zeta \in \partial\mathbb{D}$, we obtain

$$\sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z)$$
$$< 4\varepsilon/(\alpha(\alpha+1)\cdots(\alpha+n))^2$$

from which (2.13) follows.
(2) $\Rightarrow$ (1). Assume that $\{f_m\}_{m\in\mathbb{N}}$ is a bounded sequence in $\mathcal{F}_\alpha$, say by L, converging to 0 uniformly on compacts of $\mathbb{D}$ as $m \to \infty$. Then by the Weierstrass theorem, $f_m^{(k)}$ also converges to 0 uniformly on compacts of $\mathbb{D}$, for each $k \in \mathbb{N}$. We need to show that $\|D^n_\varphi f_m\|_{\mathcal{B}_\nu} \to 0$ as $m \to \infty$. For each $m \in \mathbb{N}$, we can find a $\mu_m \in \mathfrak{M}$ with $\|\mu_m\| = \|f_m\|_{\mathcal{F}_\alpha}$ such that

$$f_m(z) = \int_{\partial\mathbb{D}} \frac{d\mu_m(\zeta)}{(1-\overline{\zeta}z)^\alpha}. \tag{2.21}$$

Differentiating (2.21) $n+1$ times, composing such obtained equation by φ, applying Jensen's inequality, as well as the boundedness of sequence $\{f_m\}_{m\in\mathbb{N}}$, we obtain

$$|f_m^{(n+1)}(\varphi(w))|^2 \le L(\alpha(\alpha+1)\cdots(\alpha+n))^2 \int_{\partial\mathbb{D}} \frac{d|\mu_m|(\zeta)}{|1-\overline{\zeta}\varphi(w)|^{2(n+\alpha+1)}}. \tag{2.22}$$

By the second condition in (2), we have that for every $\varepsilon > 0$, there is an $r_1 \in (0,1)$ such that for $r \in (r_1, 1)$, we have

$$\sup_{\zeta\in\partial\mathbb{D}} \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(z)|^{2(n+\alpha+1)}} (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) < \varepsilon. \tag{2.23}$$

By Lemma 2, we have

$$\|D^n_\varphi f_m\|^2_{\mathcal{B}_\nu} \asymp |f^n_m(\varphi(0))|^2 + \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\le r} |f_m^{(n+1)}(\varphi(z))|^2 (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(z) d\lambda(z)$$
$$+ \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|>r} |f_m^{(n+1)}(\varphi(z))|^2 (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(z) d\lambda(z).$$

Using first condition in (2), (2.23), Fubini's theorem and the fact that

$$|f_m^{(n)}(\varphi(0))|^2 < \varepsilon \text{ and } \sup_{|w|\le r} |f_m^{(n+1)}(w)|^2 < \varepsilon,$$

for sufficiently large m, say $m \ge m_0$, we have that

$$\begin{aligned}
\|D_\varphi^n f_m\|_{\mathcal{B}_\nu}^2 &\le |f_m^{(n)}(\varphi(0))|^2 \\
&+ \sup_{|\varphi(z)|\le r} |f_m^{(n+1)}(\varphi(z))|^2 \sup_{a\in\mathbb{D}} \int_{|\varphi(z)|\le r} (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 \nu^2(z) d\lambda(z) \\
&+ \sup_{a\in\mathbb{D}} \int_{\partial\mathbb{D}} \int_{|\varphi(z)|>r} \frac{\nu^2(z)}{|1-\overline{\zeta}\varphi(w)|^{2(n+\alpha+1)}} (1-|\eta_a(z)|^2)^2 |\varphi'(z)|^2 d\lambda(z) d|\mu_m|(\zeta) \\
&\le \left(1 + M_3 + \int_{\partial\mathbb{D}} d|\mu_m|(\zeta)\right)\varepsilon \\
&\le (1 + M_3 + L)\varepsilon.
\end{aligned}$$

Since ε is an arbitrary, the result follows by Lemma 3. □

Theorem 7. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N}\cup\{0\}$ and φ a holomorphic self-map of $\mathbb{D}$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is bounded if and only if following conditions hold*

$$M_1 := \sup_{\zeta\in\partial\mathbb{D}} \sup_{z\in\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} < \infty. \tag{2.24}$$

$$\lim_{|z|\to 1} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} = 0 \tag{2.25}$$

for every $\zeta \in \partial\mathbb{D}$.

Proof. First suppose that (2.24) and (2.25) hold. By (2.25), the integrand in (2.4) tends to zero for every $\zeta \in \partial\mathbb{D}$, as $|z| \to 1$, and is dominated by the function $f(z) = M_1$. Thus by the Lebesgue convergence theorem, the integral in (2.4) tends to zero as $|z| \to 1$, implying

$$\lim_{|z|\to 1} \nu(z)|(D_\varphi^n f)'(z)| = 0.$$

Hence, for every $f \in \mathcal{F}_\alpha$ we have that $D_\varphi^n f \in \mathcal{B}_{\nu,0}$, from which the boundedness of $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ follows. Conversely, suppose that $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is bounded. Then $D_\varphi^n f_\zeta \in \mathcal{B}_{\nu,0}$ for every function f_ζ, $\zeta \in \partial\mathbb{D}$, defined in (2.8), that is

$$\lim_{|z|\to 1} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} = 0$$

for every $\zeta \in \partial\mathbb{D}$. Since $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is bounded, then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_\nu$ is bounded too. Thus by Theorem 1, (2.24) follows, as claimed. □

Theorem 8. *Let ν be a normal weight, $\alpha > 0$, $n \in \mathbb{N}\cup\{0\}$ and φ a holomorphic self-map of $\mathbb{D}$. Then $D_\varphi^n : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is compact if and only if*

$$\lim_{|z|\to 1} \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} = 0. \tag{2.26}$$

Proof. By a know result (see, e.g. Lemma 1 in [12], a closed set E in $\mathcal{B}_{\nu,0}$ is compact if and only if it is bounded and satisfies

$$\lim_{|z|\to 1} \sup_{f\in E} \nu(z)|f'(z)| = 0.$$

Thus the set $\{D^n_\varphi f : f \in \mathcal{F}_\alpha, \|f\|_{\mathcal{F}_\alpha} \leq 1\}$ has compact closure in $\mathcal{B}_{\nu,0}$ if and only if

$$\lim_{|z|\to 1} \sup\{\nu(z)|(D^n_\varphi f)'(z)| : f \in \mathcal{F}_\alpha, \|f\|_{\mathcal{F}_\alpha} \leq 1\} = 0. \tag{2.27}$$

Let $f \in B_{\mathcal{F}_\alpha}$, then there is a $\mu \in \mathfrak{M}$ such that $\|\mu\| = \|f\|_{\mathcal{F}_\alpha}$ and

$$f(z) = \int_{\partial\mathbb{D}} \frac{d\mu(\zeta)}{(1-\overline{\zeta}z)^\alpha}.$$

Thus we easily get that for each $f \in B_{\mathcal{F}_\alpha}$

$$\begin{aligned} \nu(z)|(D^n_\varphi f)'(z)| &\leq \alpha(\alpha+1)\cdots(\alpha+n)\|\mu\| \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}} \\ &\leq \alpha(\alpha+1)\cdots(\alpha+n) \sup_{\zeta\in\partial\mathbb{D}} \frac{\nu(z)|\varphi'(z)|}{|1-\overline{\zeta}\varphi(z)|^{n+\alpha+1}}. \end{aligned} \tag{2.28}$$

Using (2.26) in (2.28), we get (2.27). Hence $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is compact. Conversely, suppose that $D^n_\varphi : \mathcal{F}_\alpha \to \mathcal{B}_{\nu,0}$ is compact. Taking the test functions in (2.8), we can easily obtain that (2.26) follows from (2.27). □

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