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On Stability and Boundedness of Solutions of Certain Non Autonomous Fourth-Order Delay Differential Equations

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Abstract
This paper establishes sufficient conditions to ensure the boundedness and asymptotic stability of solutions of a certain fourth order nonlinear non-autonomous differential equation by constructing Lyapunov functionals.

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1 Introduction

In applied science, some practical problems are associated with higher-order nonlinear differential equations, such as , electronic theory [22], biological model and other models [8] and [18] . Many results concerning the stability and boundedness of solutions of fourth order differential equations without delay have been obtained in view of various methods, especially, Lyapunov's method, see, the book of Reissig et al. [26] as a survey and the papers of Abou-El-Ela and Sadek [1], Adesina and Ogundare [3], Cartwright [9], Chukwu [10], Ezeilo [12], [14] Ezeilo and Tejumola [15], Harrow [16], Hu [17], Tejumola [30], Tunc [36], [37], [38], [39], Wu and Xiong [44] and the references cited therein. Besides, it should be noted that there are only a few results on the same problem for nonlinear differential equations of fourth order with delay, it have been discussed by a few authors, see, Bereketoglu [4], Abou-El-Ela et al. [2], Kang and Si [20], Sadek [27], Sinha [28], Tejumola [31], and Tunc [40], [41], [42]. The most efficient tool for the study of the stability and boundedness of solutions of a given nonlinear differential equation is provided by Lyapunov theory. But the construction of such functions which are positive definite with negative definite derivatives for higher-order differential equations with delay, is is in general a difficult task.

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In [7], Chin has tried to use a new technique (the intrinsic method) to construct new Lyapunov functions for the following fourth-order differential equations

$$x'''' + a_1x''' + a_2x'' + a_3x' + f(x) = 0, \tag{1.1}$$

$$x'''' + a_1x''' + \psi(x')x'' + a_3x' + a_4x = 0, \tag{1.2}$$

$$x'''' + a_1x''' + f(x, x')x'' + a_3x' + a_4x = 0, \tag{1.3}$$

in which a_1, a_2, a_3 and a_4 are constants. In [44], Wu and Xiong also investigated the asymptotic stability of the zero solution of the differential equations (1.1) and (1.3). Later, in 2004, Sadek [27] considered the fourth-order nonlinear delay differential equations of the form

$$x'''' + a_1x''' + a_2x'' + a_3x' + f(x(t-r)) = 0,$$

and he derived sufficient conditions for the asymptotic stability of the zero-solution of these equations by constructing new Lyapunov functional.
In [33], Tunc investigated the asymptotic stability of zero solution of the fourth order non-linear differential equations with delay as follows

$$x'''' + \varphi(x'')x''' + a_2x'' + a_3x' + f(x(t-r)) = 0.$$

In this article, we establish the uniform asymptotic stability of the differential equation of the form

$$\begin{aligned} \Big(g\big(x(t)\big)x''(t)\Big)'' + a\,(t)\,\Big(p\big(x(t)\big)x''(t)\Big)' \quad &+ b\,(t)\,\Big(q\big(x(t)\big)x'(t)\Big)' + c\,(t)\,f\big(x(t)\big)x'(t) \\ &+ d\,(t)\,h\big(x(t-r)\big) = 0, \end{aligned} \tag{1.4}$$

where $g(x) > 0$ and r is a positive constant to be determined later; the primes in (1.4) denote differentiation with respect to t; the functions a, b, c, d, are continuously differentiable functions. The functions f, g, h, p, q, are continuous. It is also supposed that the derivatives, $g'(x)$, $p'(x)$, $q'(x)$, $f'(x)$ and $h'(x)$ exist and are continuous.
Equation (1.4) is equivalent to the system

$$\begin{cases} x' = y, \\ y' = \dfrac{1}{g\,(x)}z, \\ z' = w, \\ w' = -a(t)\dfrac{p\,(x)}{g\,(x)}w + \left(a\,(t)\,p(x)\theta_1(t) - b(t)\dfrac{q\,(x)}{g\,(x)} - a(t)g(x)\theta_2(t)\right)z \\ \qquad -\Big(b(t)g^2(x)\theta_3(t) + c\,(t)\,f\,(x)\Big)y - d\,(t)\,h\,(x) + d\,(t)\displaystyle\int_{t-r}^{t} y\,(s)\,h'\,(x\,(s))\,ds, \end{cases} \tag{1.5}$$

where
$$\theta_1\,(t) = \frac{g'\,(x\,(t))}{g^2\,(x\,(t))}x'\,(t)\,,\quad \theta_2\,(t) = \frac{p'\,(x\,(t))}{g^2\,(x\,(t))}x'\,(t)\,,\ \text{ and } \theta_3\,(t) = \frac{q'\,(x\,(t))}{g^2\,(x\,(t))}x'\,(t)\,.$$
The continuity of the functions $a, b, c, d, f, g, g', h, p, p', q$, and q' guarantees the existence of the solutions of (1.4) (see [11], pp.15). It is assumed that the right hand side of the system (1.5) satisfies a Lipschitz condition in $x(t), y(t), z(t), w(t)$ and $x(t-r)$. This assumption guarantees the uniqueness of solutions of (1.4) ([11], pp.15). The motivation for the present paper comes from the results mentioned above. Our purpose is to

extend and improve the result established by Tunc [33], and Sadek [27] to the equation (1.4). Clearly the equation discussed in [27] and is a special case of equation (1.1) when $g(x) = p(x) = q(x) = 1$, and $a(t) = a,\ b(t) = b,\ c(t) = c$. Our approach is based on Lyapunov's second (direct) method. We shall use appropriate Lyapunov function and impose suitable conditions on the functions $g(x)$, $p(x)$, and $q(x)$.

2 Preliminaries

In this section, we shall state and prove certain results useful in the proof of our main result. Consider the functional differential equation

$$x' = f(t, x_t), \quad x_t(\theta) = x(t+\theta)\ , \quad -r \le \theta \le 0, \quad t \ge 0, \tag{2.1}$$

where $f : I \times C_H \to \mathbb{R}^n$ is a continuous mapping, $f(t, 0) = 0$, $C_H := \{\phi \in (C[-r, 0], \mathbb{R}^n) : \|\phi\| \le H\}$, and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(t, \phi)| < L(H_1)$ when $\|\phi\| < H_1$.

Definition 1. [6] An element $\psi \in C$ is in the $\omega - limit$ set of ϕ, say $\Omega(\phi)$, if $x(t, 0, \phi)$ is defined on $[0, +\infty)$ and there is a sequence $\{t_n\}, t_n \to \infty$, as $n \to \infty$, with $\|x_{t_n}(\phi) - \psi\| \to 0$ as $n \to \infty$ where $x_{t_n}(\phi) = x(t_n + \theta, 0, \phi)\ for\ \ -r \le \theta \le 0$.

Definition 2. [6] A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solution of (2.1), $x(t, 0, \phi)$, is defined on $[0, \infty)$ and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.

Lemma 3. *[5] If $\phi \in C_H$ is such that the solution $x_t(\phi)$ of (2,1) with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\|x_t(\phi)\| \le H_1 < H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a non-empty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0 \quad as \quad t \to \infty.$$

Lemma 4. *[5] Let $V(t, \phi) : I \times C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition, $V(t, 0) = 0$, and wedges W_i such that:*
(i) $W_1(\|\phi\|) \le V(t, \phi) \le W_2(\|\phi\|)$.
(ii) $V'_{(2,1)}(t, \phi) \le -W_3(\|\phi\|)$.
Then, the zero solution of (2.1) is uniformly asymptotically stable.

3 Assumptions and main results

First, we state some assumptions on the functions that appeared in (1.4), and suppose that there are positive constants $a_0, b_0, c_0, d_0, f_0, g_0, p_0, q_0, a_1, b_1, c_1, d_1, f_1, g_1, p_1, q_1, h_0, m$, $M, \delta, \delta_0, \eta_1$ and η_2 such that the following conditions are satisfied

i) $0 < a_0 \le a\,(t) \le a_1;\ 0 < b_0 \le b\,(t) \le b_1;\ 0 < c_0 \le c\,(t) \le c_1;$
$0 < d_0 \le d\,(t) \le d_1$ for $t \ge 0$.

ii) $0 < f_0 \le f\,(x) \le f_1;\ g_0 \le g\,(x) \le g_1;\ 0 < p_0 \le p\,(x) \le p_1;\quad 0 < q_0 \le q\,(x) \le q_1$
for $x \in \mathbb{R}$ and $0 < m < \min\left\{f_0, p_0, g_0, 1\right\}$, $M > \max\left\{f_1, , g_1, p_1, 1\right\}$.

iii) $\dfrac{h(x)}{x} \ge \delta > 0 \quad (\text{ for } x \ne 0)\,;\ h\,(0) = 0.$

iv) $\dfrac{h_0}{m} - \dfrac{a_0 m \delta_0}{d_1} \le h'\,(x) \le \dfrac{h_0}{2M} \quad$ for $x \in \mathbb{R}$,

v) $b_0q_0 > \max(\kappa_1, \kappa_2)$ where

$$\begin{cases} \kappa_1 = \dfrac{a_1h_0d_1M^2}{c_0m^3} + \dfrac{M^3(c_1+\delta_0)}{a_0m^2} + a_0a_1m(M-1), \\ \kappa_2 = \dfrac{2d_1h_0a_0}{c_0(M-1)}\left(\dfrac{1}{m}-\dfrac{1}{M}\right)^2 + 2\dfrac{c_0M}{a_0} + 2a_1\dfrac{d_1h_0M}{c_0m^3} + \dfrac{c_0c_1(M^2+2)mM}{d_1h_0}. \end{cases}$$

vi) $\displaystyle\int_0^{+\infty} (|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|)\, dt < \eta_1 < +\infty.$

vii) $\displaystyle\int_{-\infty}^{+\infty} (|g'(s)| + |p'(s)| + |q'(s)| + |f'(s)|)\, ds < \eta_2 < \infty.$

Now we dispose of the following lemma which will be required in the proof of next theorem.

Lemma 5. *[19] Let* $h(0) = 0$, $xh(x) > 0$ $(x \neq 0)$ *and* $\delta(t) - h'(x) \geq 0$ $(\delta(t) > 0)$, *then*

$$2\delta(t)H(x) \geq h^2(x) \qquad \text{where} \quad H(x) = \int_0^x h(s)ds.$$

The main result of this paper is the following theorem.

Theorem 6. *Suppose that assumptions i) ∼ vii) hold. Then, every solution* $x(t)$ *of (1.4) and their derivatives* $x'(t), x''(t)$ *and* $x'''(t)$ *are uniformly asymptotically stable, provided that*

$$r < \frac{1}{\lambda}\min\left\{\epsilon c_0 m,\ \epsilon\frac{a_0m}{M},\ \frac{m^2(b_0q_0-\kappa_1) - \epsilon M^2(a_1+c_1mM)}{Mm^2}\right\}, \tag{3.1}$$

where

$$\lambda = d_1\lambda_0(\alpha+\beta+1),\ \lambda_0 = \max\left\{\frac{h_0}{2M}, \left|\frac{h_0}{m} - \frac{a_0m\delta_0}{d_1}\right|\right\}, \quad \text{and} \tag{3.2}$$

$$\epsilon < \min\left\{\frac{M}{a_0m}, \frac{d_1h_0}{c_0m}, \frac{m^2(b_0q_0-\kappa_1)}{M^2(a_1+mMc_1)}\right\}. \tag{3.3}$$

Proof. The proof depend on some fundamental properties of a continuously differentiable Lyapunov functional $W = W(t, x_t, y_t, z_t, w_t)$ defined by

$$W = e^{-\frac{1}{\eta}\int_0^t \gamma(s)\,ds} V, \tag{3.4}$$

where

$$\gamma(t) = |a'(t)| + |b'(t)| + |c'(t)| + |d'(t)| + |\theta_1(t)| + |\theta_2(t)| + |\theta_3(t)| + |\theta_4(t)|,$$

and

$$V = V_0(t, x_t, y_t, z_t, w_t) + \lambda\int_{-r}^0\int_{t+s}^t y^2(\theta)\, d\theta ds,$$

such that

$$\begin{aligned}
\theta_4(t) &= \frac{f'(x(t))}{g^2(x(t))}x'(t),\\
2V_0 &= 2\beta d(t)H(x) + c(t)g(x)f(x)y^2 + \alpha b(t)\frac{q(x)}{g(x)}z^2 + a(t)\frac{p(x)}{g(x)}z^2\\
&\quad +2\beta a(t)\frac{p(x)}{g(x)}yz + [\beta b(t)q(x) - \alpha h_0 d(t)]y^2 - \beta\frac{1}{g(x)}z^2 + \alpha w^2\\
&\quad +2d(t)g(x)h(x)y + 2\alpha d(t)h(x)z + 2\alpha c(t)f(x)yz + 2\beta yw + 2zw,
\end{aligned}$$

with $H(x) = \int_0^x h(s)ds$, $\alpha = \dfrac{M}{a_0 m} + \epsilon$, $\beta = \dfrac{d_1 h_0}{c_0 m} + \epsilon$ and η is positive constant to be determined later in the proof. $2V_0$ can be rearranged as the following

$$\begin{aligned}
2V_0 &= a(t)p(x)\Big(\frac{w}{a(t)p(x)} + z + \beta\frac{1}{g(x)}y\Big)^2 + c(t)f(x)\left(\frac{d(t)h(x)}{c(t)f(x)} + y + \alpha z\right)^2\\
&\quad +c(t)f(x)\left[\Big(g(x)-1\Big)y + \frac{d(t)h(x)}{c(t)f(x)}\right]^2 + 2\epsilon d(t)H(x) + V_1 + V_2 + V_3,
\end{aligned}$$

where

$$\begin{aligned}
V_1 &= 2d(t)\int_0^x h(s)\left(\frac{d_1 h_0}{c_0 m} - 2\frac{d(t)}{c(t)f(x)}h'(s)\right)ds,\\
V_2 &= \left(\alpha b(t)\frac{q(x)}{g(x)} - \beta\frac{1}{g(x)} - \alpha^2 c(t)f(x) + a(t)p(x)\Big(\frac{1}{g(x)} - 1\Big)\right)z^2,
\end{aligned}$$

and

$$\begin{aligned}
V_3 &= \left(\beta b(t)q(x) - \alpha h_0 d(t) - \beta^2 a(t)\frac{p(x)}{g^2(x)} - c(t)f(x)\left(g^2(x) - 3g(x) + 2\right)\right)y^2\\
&\quad + \left(\alpha - \frac{1}{a(t)p(x)}\right)w^2 + 2\beta\Big(1 - \frac{1}{g(x)}\Big)yw.
\end{aligned}$$

To prove that V is positive definite it suffice to show that V_1, V_2 and V_3 are positive. Using conditions i) $\sim$ v), and inequality (3.3) we obtain the following

$$\begin{aligned}
V_1 &\geq 2d(t)\int_0^x 2h(s)\frac{d_1}{c_0 m}\left(\frac{h_0}{2} - h'(s)\right)ds\\
&\geq 4d_0\frac{d_1}{c_0 m}\int_0^x h(s)\left(\frac{h_0}{2M} - h'(s)\right)ds \geq 0.
\end{aligned}$$

Since (3.3) we get

$$\frac{M}{a_0 m} < \alpha < 2\frac{M}{a_0 m}, \quad \frac{d_1 h_0}{c_0 m} < \beta < 2\frac{d_1 h_0}{c_0 m}. \tag{3.5}$$

From (3.5) and rearrange V_2 we obtain

$$\begin{aligned}
V_2 &= \alpha\Big(b(t)\frac{q(x)}{g(x)} - \beta\frac{a(t)}{g(x)} - \alpha c(t) f(x) - \frac{a(t)p(x)}{\alpha}\Big(1-\frac{1}{g(x)}\Big)\Big)z^2 \\
&\quad + \beta\Big(\alpha\frac{a(t)}{g(x)} - \frac{1}{g(x)}\Big)z^2 \\
&\geq \alpha\Big(\frac{b_0q_0}{M} - (\frac{d_1h_0}{c_0m}+\epsilon)\frac{a_1}{m} - (\frac{M}{a_0m}+\epsilon)c_1M - a_1\frac{a_0m}{M}(M-1)\Big)z^2 \\
&\quad + \beta\Big(\alpha\frac{a_0}{M} - \frac{1}{m}\Big)z^2 \\
&\geq \alpha\Big(\frac{b_0q_0}{M} - \frac{d_1h_0a_1}{c_0m^2} - \frac{c_1M^2}{a_0m} - a_1\frac{a_0m}{M}(M-1) - \frac{\epsilon}{m}(a_1+c_1mM)\Big)z^2 \\
&\geq \frac{\alpha}{Mm}\Big(m(b_0q_0-\kappa_1) - \epsilon M\left(a_1+c_1mM\right)\Big)z^2 \geq 0,
\end{aligned}$$

and

$$\begin{aligned}
V_3 &\geq \beta\Big(b_0q_0 - \frac{\alpha}{\beta}h_0d_1 - a_1\beta\frac{M}{g^2(x)} - \frac{c_1M(M^2+2)}{\beta}\Big)y^2 + \Big(\frac{M-1}{a_0m}\Big)w^2 \\
&\quad +2\beta\Big(1-\frac{1}{g(x)}\Big)yw \\
&\geq \beta\Big(b_0q_0 - 2\frac{M}{a_0}c_0 - 2a_1\frac{d_1h_0M}{c_0m^3} - \frac{c_0c_1(M^2+2)mM}{d_1h_0}\Big)y^2 + \Big(\frac{M-1}{a_0m}\Big)w^2 \\
&\quad +2\beta\Big(1-\frac{1}{g(x)}\Big)yw \\
&\geq \psi(y,\omega),
\end{aligned}$$

such that

$$\psi(y,\omega) = \beta\frac{2d_1h_0a_0}{c_0(M-1)}\left(\frac{1}{m}-\frac{1}{M}\right)^2 y^2 + \Big(\frac{M-1}{a_0m}\Big)w^2 + 2\beta\Big(1-\frac{1}{g(x)}\Big)yw.$$

Observe that $\psi(y,\omega)$ is positive definite. Indeed by calculating the discriminant

$$\triangle = \beta^2\Big(1-\frac{1}{g(x)}\Big)^2 - \beta\frac{2d_1h_0}{c_0m}\left(\frac{1}{m}-\frac{1}{M}\right)^2,$$

since

$$\frac{1}{M} < \frac{1}{g(x)} < \frac{1}{m}, \textit{and} \ \frac{1}{M} < 1 < \frac{1}{m},$$

we get

$$\left|1-\frac{1}{g(x)}\right| < \frac{1}{m} - \frac{1}{M}.$$

Using (3.5) we obtain

$$\triangle \leq \beta\left[\frac{2d_1h_0}{c_0m}\left(\frac{1}{M}-\frac{1}{m}\right)^2 - \frac{2d_1h_0}{c_0m}\left(\frac{1}{M}-\frac{1}{m}\right)^2\right] = 0.$$

Using the fact that the integral $\int_{-r}^{0}\int_{t+s}^{t} y^2(\theta)d\theta ds$ is positive, we deduce that there exists positive number D_0 such that

$$2V \geq D_0\left(y^2+z^2+w^2+H(x)\right). \tag{3.6}$$

By lemma 5 and conditions iii) and iv) we conclude that there exists a positive number D_1 such that

$$2V \geq D_1\left(x^2+y^2+z^2+w^2\right), \tag{3.7}$$

thus V is positive definite which implies that W is also positive definite.
Then, we can find positive definite functions $U_1(\|X\|)$ and $U_2(\|X\|)$ such that $U_1(\|X\|) \leq V \leq U_2(\|X\|)$. By (ii) and (vii), we get

$$\begin{aligned}\int_0^t \Big(\sum_{i=1}^{4}|\theta_i(s)|\Big)ds &= \int_{\alpha_1(t)}^{\alpha_2(t)} \frac{|g'(u)|+|p'(u)|+|q'(u)|+|f'(u)|}{g^2(u)}du \\ &\leq \frac{1}{m^2}\int_{-\infty}^{+\infty}\big(|g'(u)|+|p'(u)|+|q'(u)|+|f'(u)|\big)du<\infty, \end{aligned}\tag{3.8}$$

where $\alpha_1(t)=\min\{x(0),x(t)\}$, and $\alpha_2(t)=\max\{x(0),x(t)\}$. By inequalities (3.4), (3.7), and (3.8), we have

$$W \geq D_2(x^2+y^2+z^2+w^2), \text{ where } D_2=\frac{D_1}{2}e^{-\frac{1}{\eta}(\eta_1+\frac{\eta_2}{m^2})}. \tag{3.9}$$

Therefore we can find positive definite functions $W_1(\|X\|)$ and $W_2(\|X\|)$ such that

$$W_1(\|X\|) \leq W \leq W_2(\|X\|).$$

Now we prove that $\dot{W}$ is negative definite functional.
The derivative of V along any solution $(x(t),y(t),z(t),w(t))$ of system (1.5), we have

$$\begin{aligned}2\dot{V}_{(1.5)} &= -2\epsilon c(t)f(x)y^2+V_4+V_5+V_6+V_7+2\frac{\partial V_0}{\partial t}+\lambda r y^2(t)-\lambda\int_{t-r}^{t}y^2(u)\,du \\ &\quad +2\alpha w d(t)\int_{t-r}^{t}y(s)h'(x(s))\,ds+2\beta y d(t)\int_{t-r}^{t}y(s)h'(x(s))\,ds \\ &\quad +2zd(t)\int_{t-r}^{t}y(s)h'(x(s))\,ds,\end{aligned}$$

where

$$\begin{aligned}V_4 &= -2\left(\frac{d_1h_0}{c_0m}c(t)f(x)-d(t)g(x)h'(x)\right)y^2-2\alpha d(t)\left(\frac{h_0}{g(x)}-h'(x)\right)yz,\\ V_5 &= -2\left(\frac{b(t)q(x)}{g(x)}-\alpha c(t)\frac{f(x)}{g(x)}-\beta a(t)\frac{p(x)}{g^2(x)}\right)z^2,\\ V_6 &= -2\left(\alpha\frac{a(t)p(x)}{g(x)}-1\right)w^2\end{aligned}$$

and

$$\begin{aligned}V_7 &= \theta_1\Big(a(t)p(x)z^2-\alpha b(t)q(x)z^2+c(t)f(x)g^2(x)y^2+\beta z^2+2d(t)g^2(x)h(x)y\\ &\quad +2\alpha a(t)p(x)zw\Big)-b(t)\theta_3g(x)\Big(\alpha z^2+2\alpha g(x)zw+\beta g(x)y^2+2g(x)yz\Big)\\ &\quad -a(t)\theta_2g(x)\Big(z^2+2\alpha zw\Big)+\theta_4\Big(c(t)g^3(x)y^2+2\alpha c(t)g^2(x)yz\Big).\end{aligned}$$

By conditions i), ii), iv), v) and inequalities (3.2), (3.3) and (3.5) we get

$$
\begin{aligned}
V_4 &\leq -2\Big(d(t)\,h_0 - d(t)\,g(x)\,h'(x)\Big)y^2 - 2\alpha d(t)\left(\frac{h_0}{g(x)} - h'(x)\right)yz \\
&\leq -2d(t)\,g(x)\left(\frac{h_0}{g(x)} - h'(x)\right)y^2 - 2\alpha d(t)\left(\frac{h_0}{g(x)} - h'(x)\right)yz \\
&\leq -2d(t)\,m\Big(\frac{h_0}{g(x)} - h'(x)\Big)\left[\Big(y + \frac{\alpha}{2m}z\Big)^2 - \Big(\frac{\alpha}{2m}z\Big)^2\right] \\
&\leq -2d(t)\,m\Big(\frac{h_0}{M} - h'(x)\Big)\Big(y + \frac{\alpha}{2m}z\Big)^2 + 2d(t)m\Big(\frac{h_0}{m} - h'(x)\Big)\left(\frac{\alpha}{2m}z\right)^2 \\
&\leq \frac{\alpha^2}{2m}d(t)\left(\frac{h_0}{m} - h'(x)\right)z^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
V_4 + V_5 &\leq -2\left[b(t)\frac{q(x)}{g(x)} - \alpha c(t)\frac{f(x)}{g(x)} - \beta a(t)\frac{p(x)}{g^2(x)} - \frac{\alpha^2}{4m}d(t)\left(\frac{h_0}{m} - h'(x)\right)\right]z^2 \\
&\leq -2\left[\frac{b_0 q_0}{M} - (\frac{M}{a_0 m} + \epsilon)\frac{c_1 M}{m} - (\frac{d_1 h_0}{c_0 m} + \epsilon)\frac{a_1 M}{m^2} - \frac{\alpha^2}{4m}(a_0\delta_0)\right]z^2 \\
&\leq -2\left[\frac{b_0 q_0}{M} - \frac{M^2}{a_0 m^2}c_1 - \frac{d_1 h_0 a_1 M}{c_0 m^3} - \frac{M^2\delta_0}{a_0 m^2} - \epsilon\frac{M}{m}\left(\frac{a_1}{m} + c_1\right)\right]z^2 \\
&\leq -\frac{2}{Mm^2}\Big(m^2(b_0 q_0 - \kappa_1) - \epsilon M^2\,(a_1 + c_1 m)\Big)z^2 \leq 0.
\end{aligned}
$$

We have also,

$$
V_6 \leq -2\left(\alpha\frac{a_0 m}{M} - 1\right)w^2 = -2\epsilon\frac{a_0 m}{M}w^2 \leq 0.
$$

Putting $\lambda_1 = \min\left\{\epsilon c_0 m,\ \epsilon\frac{a_0 m}{M},\ \frac{1}{Mm^2}\Big(m^2(b_0 q_0 - \kappa_1) - \epsilon M^2\,(a_1 + c_1 mM)\Big)\right\}$ and using Cauchy Schwartz inequality we have

$$
\begin{aligned}
&-2\epsilon c(t)\,f(x)y^2 + V_4 + V_5 + V_6 + 2\alpha w d(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds - \lambda\int_{t-r}^{t} y^2(s)\,ds \\
&+ 2\beta y d(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds + 2z d(t)\int_{t-r}^{t} y(s)\,h'(x(s))\,ds + \lambda r y^2(t) \\
&\leq -2\lambda_1\left(y^2 + z^2 + w^2\right) + \alpha d_1\lambda_0\left(w^2 r + \int_{t-r}^{t} y^2(s)\,ds\right) + \beta d_1\lambda_0\left(y^2 r + \int_{t-r}^{t} y^2(s)\,ds\right) \\
&\quad + d_1\lambda_0\left(z^2 r + \int_{t-r}^{t} y^2(s)\,ds\right) + \lambda r y^2(t) - \lambda\int_{t-r}^{t} y^2(s)\,ds \\
&\leq -2\lambda_1\left(y^2 + z^2 + w^2\right) + d_1\lambda_0 r\left(\beta y^2 + z^2 + \alpha w^2\right) + \left(d_1\lambda_0(\alpha + \beta + 1) - \lambda\right)\int_{t-r}^{t} y^2(s)\,ds \\
&\leq -2\lambda_1\left(y^2 + z^2 + w^2\right) + d_1\lambda_0(\alpha + \beta + 1)r\left(y^2 + z^2 + w^2\right) \\
&\leq -2D_3\left(y^2 + z^2 + w^2\right). \qquad (3.10)
\end{aligned}
$$

Where $D_3 = \lambda_1 - \lambda r$. It can be seen that if $r < \dfrac{\lambda_1}{\lambda}$, then $D_2 > 0$.

Now by the inequalities (3.6), (3.10), the lemma 5 and the Cauchy Schwartz inequality we get the following.

$$\begin{aligned} V_7 \leq{}& |\theta_1|\Big(a(t)\,p(x)z^2 + \alpha b(t)\,q(x)z^2 + c(t)\,f(x)\,g^2(x)\,y^2 + \beta z^2 + d(t)\,g^2(x)\,(h^2(x)+y^2) \\ &+ \alpha a(t)\,p(x)(z^2+w^2)\Big) + |\theta_4|\Big(c(t)\,g^3(x)\,y^2 + \alpha c(t)\,g^2(x)\,(y^2+z^2)\Big) \\ &+ b(t)|\theta_3|g(x)\Big(\alpha z^2 + \alpha g(x)(z^2+w^2) + \beta g(x)y^2 + g(x)(y^2+z^2)\Big) \\ &+ a(t)|\theta_2|g(x)\Big(z^2 + \alpha(z^2+w^2)\Big) \\ \leq{}& \lambda_2\big(|\theta_1|+|\theta_2|+|\theta_3|+|\theta_4|\big)\big(y^2+z^2+w^2+H(x)\big) \\ \leq{}& \frac{2\lambda_2}{D_0}\big(|\theta_1|+|\theta_2|+|\theta_3|+|\theta_4|\big)V. \end{aligned}$$

such that,

$$\begin{aligned} \lambda_2 = \max\big\{&d_1h_0M, \alpha M(a_1+Mb_1), c_1M^3 + (d_1+\alpha c_1+\beta b_1+b_1)M^2, \\ &\beta + a_1M(\alpha+1) + b_1M(2\alpha+1) + \alpha c_1M^2\big\}. \end{aligned}$$

Similarly we have

$$\begin{aligned} 2\frac{\partial V_0}{\partial t} ={}& d'(t)\left[2\beta H(x) - \alpha h_0 y^2 + 2g(x)\,h(x)\,y + 2\alpha h(x)\,z\right] \\ &+c'(t)\left[g(x)\,f(x)y^2 + 2\alpha f(x)yz\right] + b'(t)\left[\alpha\frac{q(x)}{g(x)}z^2 + \beta q(x)y^2\right] \\ &+a'(t)\left[\frac{p(x)}{g(x)}z^2 + 2\beta\frac{p(x)}{g(x)}yz\right]. \end{aligned}$$

There exist positive constant λ_3 such that

$$\begin{aligned} 2\left|\frac{\partial V_0}{\partial t}\right| \leq{}& |d'(t)|\Big(2\beta H(x) + \alpha h_0y^2 + g(x)\big(h^2(x)+y^2\big) + \alpha\big(h^2(x)+z^2\big)\Big) \\ &+ |c'(t)|f(x)\Big(g(x)\,y^2 + \alpha\big(y^2+z^2\big)\Big) + |b'(t)|q(x)\Big(\alpha\frac{1}{g(x)}z^2 + \beta y^2\Big) \\ &+ |a'(t)|\frac{p(x)}{g(x)}\Big(z^2 + \beta\big(y^2+z^2\big)\Big) \\ \leq{}& \lambda_3\Big(|a'(t)|+|b'(t)|+|c'(t)|+|d'(t)|\Big)\big(y^2+z^2+w^2+H(x)\big) \\ \leq{}& 2\frac{\lambda_3}{D_0}\Big(|a'(t)|+|b'(t)|+|c'(t)|+|d'(t)|\Big)V, \end{aligned}$$

where

$$\lambda_3 = \max\big\{2\beta + h_0(\alpha+1), M^2 + \beta\frac{M}{m} + \alpha(h_0+M), \frac{M}{m}(\alpha+\beta+1)\big\}.$$

Thus for $\dfrac{1}{\eta} = \dfrac{1}{D_0}\max\{\lambda_2, \lambda_3\}$ we have

$$\begin{aligned} \dot{V}_{(1.5)} \leq{}& -D_3(y^2+z^2+w^2) + \frac{1}{\eta}\Big(|a'(t)|+|b'(t)|+|c'(t)|+|d'(t)| \\ &+|\theta_1|+|\theta_2|+|\theta_3|+|\theta_4|\Big)V. \end{aligned} \tag{3.11}$$

By conditions vi), vii) and inequalities (3.8), (3.11) we have

$$\begin{aligned} \dot{W}_{(1.5)} &= \left(\dot{V}_{(1.5)} - \frac{1}{\eta}\gamma(t) V\right) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\, ds} \\ &\leq -D_3\left(y^2+z^2+w^2\right) e^{-\frac{1}{\eta}\int_0^t \gamma(s)\, ds} \\ &\leq -D_4\left(y^2+z^2+w^2\right), \end{aligned}$$

where $D_4 = D_3 e^{-\frac{\eta_1+\eta_2}{\eta}}$. From (1.5, $W_3(\|X\|) = D_4(y^2+z^2+w^2)$ is positive definite function. Thus, we conclude that the solutions of system (1.5) are uniformly asymptotically stable. Now, it is evident from (1.5) that

$$\begin{cases} |x'(t)| = |y(t)|, \\ |x''(t)| = |\dfrac{z(t)}{g(x)}| \leq \dfrac{|z(t)|}{m}, \\ |x'''(t)| = |\dfrac{w(t)}{g(x)} - \dfrac{g'(x)x'(t)}{g^2(x)}| \leq \dfrac{|w(t)|}{m} + \dfrac{|g'(x)||x'(t)|}{m^2}. \end{cases}$$

Clearly, from the above discussion

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} x'(t) = 0, \quad \lim_{t\to\infty} x''(t) = 0, \ \textit{and} \ \lim_{t\to\infty} x'''(t) = 0.$$

This fact completes the proof of the Theorem. □

4 Example

We consider the following fourth order non-autonomous delay differential equation

$$\begin{aligned} &\left(\left(\frac{x^2(t)\sin x(t) + 5x^4(t) + 5}{5\left(1+x^4(t)\right)}\right) x''(t)\right)'' \\ &+ \left(e^{-t}\sin t + 2\right)\left(\left(\frac{x(t) + 4e^{x(t)} + 4e^{-x(t)}}{4\left(e^{x(t)} + e^{-x(t)}\right)}\right) x''(t)\right)' \\ &+ \left(\frac{\cos t + 7t^2 + 7}{1+t^2}\right)\left(\left(\frac{\sin x(t) + 6e^{x(t)} + 6e^{-x(t)}}{e^{x(t)} + e^{-x(t)}}\right) x'(t)\right)' \\ &+ \left(e^{-2t}\sin^3 t + 2\right)\left(\frac{x(t)\cos x(t) + 5x^4(t) + 5}{5\left(1+x^4(t)\right)}\right) x'(t) \\ &+ \left(\frac{\cos^2 t + t^2 + 1}{10\left(1+t^2\right)}\right)\left(\frac{x(t-r)}{x^2(t-r)+1}\right) = 0. \end{aligned} \tag{4.1}$$

By taking $g(x) = \dfrac{x^2\sin x + 5x^4 + 5}{5(1+x^4)}$, $p(x) = \dfrac{x + 4e^x + 4e^{-x}}{4(e^x + e^{-x})}$, $q(x) = \dfrac{\sin x + 6e^x + 6e^{-x}}{e^x + e^{-x}}$, $f(x) = \dfrac{x\cos x + 5x^4 + 5}{5(1+x^4)}$, $h(x) = \dfrac{x}{x^2+1}$, $a(t) = e^{-t}\sin t + 2$, $b(t) = \dfrac{\cos t + 7t^2 + 7}{1+t^2}$, $c(t) = e^{-2t}\sin^3 t + 2$ and $d(t) = \dfrac{\cos^2 t + t^2 + 1}{10(1+t^2)}$.

We have

$m=\dfrac{9}{10}\ ,\ M=\dfrac{11}{10}\ ,\ q_0=\dfrac{11}{2}\ ,\ q_1=\dfrac{13}{2}\ ,\ h_0=\dfrac{5}{2}\ ,\ \delta_0=\dfrac{5}{3}\ ,\ a_0=1\ ,\ a_1=3\ ,$
$b_0=6\ ,\ b_1=8\ ,\ c_0=1\ ,\ c_1=3\ ,\ d_0=\dfrac{1}{10}\ ,\ d_1=\dfrac{1}{5}\ ,$ and

$$\begin{aligned}
\frac{h_0}{m}-\frac{a_0 m\delta_0}{d_1} &\le -4.55\le h'(x)\le 1.1\le \frac{h_0}{2M},\\
\kappa_1 &= \frac{a_1h_0d_1M^2}{c_0m^3}+\frac{M^3(c_1+\delta_0)}{a_0m^2}+a_0a_1m\,(M-1)\le 11,\\
\kappa_2 &= \frac{2d_1h_0a_0}{c_0\,(M-1)}\left(\frac{1}{m}-\frac{1}{M}\right)^2+2\frac{c_0M}{a_0}+2a_1\frac{d_1h_0M}{c_0m^3}+\frac{c_0c_1(M^2+2)mM}{d_1h_0}\\
&\le 27,\\
\epsilon &< \min\left\{\frac{M}{a_0m}\ ,\ \frac{d_1h_0}{c_0m}\ ,\ \frac{m^2(b_0q_0-\kappa_1)}{M^2(a_1+mMc_1)}\right\}=\frac{5}{9}.
\end{aligned}$$

By choosing $\epsilon=\dfrac{1}{4}$ we get $\alpha=\dfrac{M}{a_0m}+\epsilon=\dfrac{53}{36},\quad \beta=\dfrac{d_1h_0}{c_0m}+\epsilon=\dfrac{29}{36},$

$$\begin{aligned}
\lambda_0 &= \max\left\{\frac{h_0}{2M},\ \left|\frac{h_0}{m}-\frac{a_0m\delta_0}{d_1}\right|\right\}=\frac{85}{18},\\
\lambda &= d_1\lambda_0(\alpha+\beta+1)=\frac{1003}{324},\\
r &< \frac{1}{\lambda}\min\left\{\epsilon c_0m,\ \epsilon\frac{a_0m}{M},\ \frac{m^2(b_0q_0-\kappa_1)-\epsilon M^2(a_1+c_1mM)}{Mm^2}\right\}=\frac{729}{11033}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{-\infty}^{+\infty}|g'(x)|\,dx &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{-4x^5\sin x+\left(2x\sin x+x^2\cos x\right)\left(x^4+1\right)}{\left(x^4+1\right)^2}\right|dx\le\frac{3}{5}\sqrt{2}\pi,\\
\int_{-\infty}^{+\infty}|p'(x)|\,dx &= \frac{1}{4}\int_{-\infty}^{+\infty}\left|\frac{1}{e^x+e^{-x}}+x\frac{e^{-x}-e^x}{\left(e^x+e^{-x}\right)^2}\right|dx\le\frac{\pi}{4},
\end{aligned}$$

$$\begin{aligned}
\int_{-\infty}^{+\infty}|q'(x)|\,dx &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{\left(e^x+e^{-x}\right)\cos x-\left(e^x-e^{-x}\right)\sin x}{\left(e^x+e^{-x}\right)^2}\right|dx\le\frac{\pi}{5},\\
\int_{-\infty}^{+\infty}|f'(x)|\,dx &= \frac{1}{5}\int_{-\infty}^{+\infty}\left|\frac{\left(\cos x-x\sin x\right)\left(x^4+1\right)-4x^4\cos x}{\left(x^4+1\right)^2}\right|dx\\
&\le \frac{1}{5}\int_{-\infty}^{+\infty}\frac{5+x^2}{x^4+1}dx=\frac{9}{10}\sqrt{2}\pi,
\end{aligned}$$

then,

$$\int_{-\infty}^{+\infty}\left(|g'(s)|+|p'(s)|+|q'(s)|+|f'(s)|\right)ds<\infty.$$

We have also

$$\begin{aligned}
\int_0^{+\infty} |a'(t)|\, dt &= \int_0^{+\infty} \left| (\cos t)\, e^{-t} - (\sin t)\, e^{-t} \right| dt \leq \int_0^{+\infty} 2e^{-t} dt = 2, \\
\int_0^{+\infty} |b'(t)|\, dt &= \int_0^{+\infty} \left| -\frac{\sin t}{t^2+1} - 2t\frac{\cos t}{(t^2+1)^2} \right| dt \leq \int_0^{+\infty} \left(\frac{1}{t^2+1} + \frac{2t}{(t^2+1)^2} \right) dt \\
&\leq \int_0^{+\infty} \frac{2}{t^2+1} dt = \pi, \\
\int_0^{+\infty} |c'(t)|\, dt &= \int_0^{+\infty} \left| 3\left(\cos t \sin^2 t\right) e^{-2t} - 2\left(\sin^3 t\right) e^{-2t} \right| dt \leq \int_0^{+\infty} 5e^{-2t} dt = \frac{5}{2}, \\
\int_0^{+\infty} |d'(t)|\, dt &= \int_0^{+\infty} \left| -2(\cos t)\frac{\sin t}{t^2+1} - 2t\frac{\cos^2 t}{(t^2+1)^2} \right| dt \leq \int_0^{+\infty} \frac{3}{t^2+1} dt = \frac{3\pi}{2}.
\end{aligned}$$

Hence

$$\int_0^{+\infty} \left(|a'(t)| + |b'(t)| + |c'(t)| + |d'(t)|\right) dt < +\infty.$$

Thus all the assumptions of Theorem (6) hold, this shows that every solution $x(t)$ of (4.1) and their derivatives $x'(t), x''(t)$ and $x'''(t)$ are uniformly asymptotically stable.

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