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Numerical solution of a class of nonlinear Volterra integral equations using Bernoulli operational matrix of integration

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Abstract

In this paper, a simple efficient method for the numerical solution of a class of nonlinear Volterra integral equations (VIEs) is presented. The approach starts by expanding the existing functions in terms of Bernoulli polynomials. Subsequently, using the new introduced Bernoulli operational matrices of integration and the product along with the so-called collocation method, the considered problem is reduced into a set of nonlinear algebraic equations with unknown Bernoulli coefficients. The error analysis and rate of convergence are also given. Finally, some tests of other authors are included and a comparison has been done between the results.

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1 Introduction

As is noted in [32], Volterra integral equations arise in many physical problems, e.g., heat conduction problem [5], concrete problem of mechanics or physics [44], on the unsteady Poiseuille flow in a pipe [16], diffusion problems [4], electroelastic [14], contact problems [23], etc. Due to this fact that analytical solutions of integral equations either do not exist or are hard to find, many different methods have been proposed to approximate solutions of these equations [1, 7, 8, 13, 21, 25, 27].

Recently, in [2] Aziz and Islam used Haar wavelets and in [34] Maleknejad and Rahimi used ϵ modified block pulse functions (ϵMBPFs) to solve these kinds of equations. A method based on Bernstein polynomials is also presented by Maleknejad, Basirat and Hashemizadeh in [31].

In the present paper, we consider the nonlinear Volterra integral equations of the form

$$u(x) = f(x) + \int_0^x k(x,t)N\big(u(t)\big)dt, \quad x \in \Omega := [0,1], \tag{1.1}$$

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where $u(x)$ is an unknown real valued function and $f(x)$ and $k(x,t)$ are given continuous functions defined, respectively on Ω and $\Omega \times \Omega$, and $N\big(u(x)\big)$ is a polynomial of $u(x)$ with constant coefficients. It follows from the classical theory of Volterra equations (see, for example, [8], [9]) that (1.1) has a unique continuous solution $u^*(x)$ on Ω. Moreover, if functions f and k are r times continuously differentiable on Ω and $S := \{(x,t) : 0 \leq t \leq x \leq 1\}$, respectively, then u^* is r times continuously differentiable on Ω.

The method of this paper consists of reducing (1.1) into a set of nonlinear algebraic equations. The underlying idea employed is the following integral property

$$\int_0^x \Psi(t)dt \simeq P\Psi(x), \tag{1.2}$$

where $\Psi(x) = [\psi_0(x), \psi_1(x), \dots, \psi_{n-1}(x)]^T$ is the basis vector and P is a square constant matrix called the *operational matrix of integration*. Up to now, the operational matrix of integration P has been derived for several types of basis functions such as Walsh [12], block-pulse [41], Legendre wavelets [40], Haar wavelets [18], Laguerre [20], Chebyshev [19, 37], Legendre [11], Bernstein [45], Bessel [38], Fourier [39] and Jacobi [28]. We are interested here in the use of the Bernoulli polynomials. Some interesting properties of the Bernoulli polynomials are:

- Comparing the structure of the Bernoulli operational matrix of integration P given in (2.17) with the corresponding matrices of other basis functions, we may observe that the setting up of P is simpler.
- The Bernoulli operational matrix of integration P appears to be computationally very attractive because, compared with other types of basis functions, it has more zero elements. Indeed, the nonzero entries of the Bernoulli operational matrix of integration are located only on the superdiagonal and its first column, while the corresponding matrices of the Bessel and the Bernstein polynomials are full and it is an upper triangular matrix for the block-pulse functions and a tridiagonal matrix for the Legendre wavelet basis. The nonzero elements of the shifted Chebyshev and shifted Jacobi operational matrices of integration are located on the subdiagonals, diagonals, superdiagonals and their first columns which are more than the case of Bernoulli polynomials. Also, the shifted Legendre, Laguerre and Hermite operational matrices of integration have the same number of nonzero elements with the Bernoulli operational matrix of integration. A same argument can be made for the operational matrix of derivatives.
- The Bernoulli polynomials have less terms than the shifted Chebyshev, shifted Legendre and shifted Jacobi polynomials which makes them attractive from the computational point of view. For example $B_6(x)$ (the 6th Bernoulli polynomial), has five terms while $T_6(x)$ (the 6th shifted Chebyshev polynomial) and $L_6(t)$ (the 6th shifted legendre polynomial), have seven terms, and this difference will increase by increasing the degree. Hence for approximating an arbitrary function we use less CPU time by applying Bernoulli polynomials as compared to any classical orthogonal polynomials; this issue is claimed in [35] for shifted Legendre polynomials.
- The coefficient of individual terms in Bernoulli polynomials $B_k(t)$, are smaller than the coefficient of individual terms in the shifted Legendre and shifted Chebyshev polynomials $L_k(t)$ and $T_k(t)$, respectively (it can be easily checked by the *Mathematica* software). Since the computational errors in the product are related to the coefficients of individual terms, the computational errors are less by using Bernoulli polynomials [35].

For convenience, we assume that

$$N(u(x)) = u^m(x), \tag{1.3}$$

where m is a positive integer, but the method can be easily extended and applied to any nonlinear VIE of the form (1.1), where $N\big(u(x)\big)$ is a polynomial of $u(x)$ with constant coefficients.

The reminder of the paper is organized as follows. We give a brief review of Bernoulli polynomials and their properties in Sections 2.1 and 2.2. New Bernoulli operational matrices of integration and the product are derived in Section 2.3. In Section 3, how the new introduced Bernoulli operational matrices can be used to reduce the problem (1.1)-(1.3) into a set of nonlinear algebraic equations is explained. The error analysis and rate of convergence are also given in this section. In Section 4, we show that the Bernoulli polynomial coefficients vector of $u^m(x)$ can be computed in terms of the Bernoulli polynomial coefficients vector of $u(x)$. Some numerical examples are presented in Section 5, which show the efficiency and accuracy of the proposed method. Conclusions of the work are given in Section 6.

2 Some properties of Bernoulli polynomials

To facilitate the presentation of the material that follows, we present in this section some background on the Bernoulli polynomials.

2.1 Definition

The generalized Bernoulli polynomials $B_k^{(a)}(x)$ of degree k can be defined by the generating formula [29, Section 2.8]

$$\frac{t^a e^{xt}}{(e^t-1)^a} = \sum_{k=0}^{\infty} \frac{t^k}{k!} B_k^{(a)}(x), \quad |t| \le 2\pi.$$

If $a = 1$, we have the Bernoulli polynomials $B_k^{(1)}(x) \equiv B_k(x)$, and if, further, $x = 0$, we have the Bernoulli numbers $B_k(0) = B_k$.
The Bernoulli polynomials satisfy the familiar expansion [15, Section 1.13]

$$\sum_{r=0}^{k-1} \binom{k}{r} B_r(x) = kx^{k-1}, \quad k = 1, 2, \ldots. \tag{2.1}$$

The first five Bernoulli polynomials are as follows

$$\begin{aligned}
B_0(x) &= 1,\\
B_1(x) &= x - \tfrac{1}{2},\\
B_2(x) &= x^2 - x + \tfrac{1}{6},\\
B_3(x) &= x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x,\\
B_4(x) &= x^4 - 2x^3 + x^2 - \tfrac{1}{30}.
\end{aligned}$$

Also, the Bernoulli polynomials satisfy the following relations ([15], Section 1.13)

$$\begin{aligned} &B_k'(x) = kB_{k-1}(x), \quad k \ge 1, \\ &\textstyle\int_0^1 B_k(x)dx = 0, \quad k \ge 1, \\ &B_k(x+1) - B_k(x) = kx^{k-1}, \quad k \ge 1, \\ &B_k(x) = \sum_{r=0}^{k} \binom{k}{r} B_r x^{k-r}, \quad k \ge 1. \end{aligned} \tag{2.2}$$

With the aid of equation (2.1), the Bernoulli polynomial vector

$$B(x) = [B_0(x), B_1(x), \ldots, B_N(x)]^T, \tag{2.3}$$

can be written of the form

$$B(x) = D^{-1} T_N(x), \tag{2.4}$$

where

$$T_N(x) = [1, x, x^2, \ldots, x^N]^T, \tag{2.5}$$

and D is a lower triangular matrix defined by

$$D = [d_{ij}]_{i,j=0}^{N}, \quad d_{ij} = \begin{cases} \frac{1}{i+1}\binom{i+1}{j}, & 0 \le j \le i, \\ 0, & i < j \le N. \end{cases}$$

On the other hand, if in the third part of equation (2.2), k varies from 0 to N we have

$$B(x) = \widehat{D} T_N(x), \tag{2.6}$$

where $\widehat{D}$ is a lower triangular matrix as

$$\widehat{D} = [\hat{d}_{ij}]_{i,j=0}^{N}, \quad \hat{d}_{ij} = \begin{cases} \binom{i}{i-j} B_{i-j}, & 0 \le j \le i, \\ 0, & i < j \le N, \end{cases} \tag{2.7}$$

and $T_N(x)$ is the vector defined by equation (2.5). So, from equations (2.4) and (2.6) we obtain $\widehat{D} = D^{-1}$. The dual matrix of $B(x)$ is defined by

$$\begin{aligned} Q = \int_0^1 B(x)B^T(x)dx &= \int_0^1 \left(\widehat{D}T_N(x)\right)\left(\widehat{D}T_N(x)\right)^T dx \\ &= \widehat{D}\left(\int_0^1 T_N(x)T_N^T(x)dx\right)\widehat{D}^T = \widehat{D}H\widehat{D}^T, \end{aligned} \tag{2.8}$$

where $\widehat{D}$ is the matrix defined in (2.7) and H is the Hilbert matrix

$$H = \int_0^1 T_N(x)T_N^T(x)dx = \left[\frac{1}{i+j+1}\right]_{i,j=0}^{N}.$$

2.2 Function approximation and error analysis

Let $\mathcal{H} = \mathfrak{L}^2([0,1])$ be the space of square integrable functions with respect to Lebesgue measure on the closed interval $[0,1]$. The inner product in this space is defined by

$$\langle f, g\rangle = \int_0^1 f(x)g(x)dx, \tag{2.9}$$

and the norm is as follows

$$\|f\|_2 = \langle f, f\rangle^{\frac{1}{2}} = \Big(\int_0^1 f^2(x)dx\Big)^{\frac{1}{2}}. \tag{2.10}$$

Let

$$\mathcal{H}_N = \operatorname{span}\{B_0(x), B_1(x), \ldots, B_N(x)\}. \tag{2.11}$$

Since $\mathcal{H}_N$ is a finite dimensional subspace of $\mathcal{H}$, then it is closed [24, Theorem 2.4-3] and for every given $g \in \mathcal{H}$ there exists a unique best approximation $\bar{g} \in \mathcal{H}_N$ [24, Theorem 6.2-5] such that

$$\|g - \bar{g}\|_2 \le \|g - f\|_2, \quad \forall f \in \mathcal{H}_N. \tag{2.12}$$

Since $\bar{g} \in \mathcal{H}_N$, there exist unique coefficients g_0, g_1, ..., g_N such that

$$g(x) \simeq \bar{g}(x) = \sum_{k=0}^{N} g_k B_k(x) = B^T(x)G, \tag{2.13}$$

where $B(x)$ is the Bernoulli polynomial vector defined in equation (2.3) and G is the Bernoulli polynomial coefficients vector of $g(x)$ defined as

$$G = [g_0, g_1, \ldots, g_N]^T. \tag{2.14}$$

Also, for a positive integer m, $g^m(x)$ may be approximated as

$$g^m(x) \simeq B^T(x)G^{(m)},$$

where $G^{(m)}$ is a column vector whose elements are nonlinear functions of the elements of G. The form of these functions will be explained later in Section 4.

Let us denote by $\mathcal{C}^m(\Omega)$ the space of functions $f : \Omega \to \mathbb{R}$ with continuous derivatives

$$f^{(i)}(x) = \frac{d^i}{dx^i} f(x), \quad x \in \Omega,$$

for all i such that $0 \le i \le m$ and by $\mathcal{C}^{m,n}(\Omega \times \Omega)$ the space of functions $f : \Omega \times \Omega \to \mathbb{R}$ with continuous partial derivatives

$$f^{(i,j)}(x,t) = \frac{\partial^{i+j}}{\partial x^m \partial t^n} f(x,t), \quad (x,t) \in \Omega \times \Omega,$$

for all (i, j) such that $0 \le i \le m, 0 \le j \le n$. The following results are satisfied.

Corollary 1. *[42] Suppose that $g(x) \in \mathcal{C}^N(\Omega)$ is approximated by the truncated Bernoulli series $P_N[g](x) = \sum_{k=0}^{N} g_k B_k(x)$. Then the coefficients g_n can be calculated from the following relation*

$$g_n = \frac{1}{n!} \int_0^1 g^{(n)}(x)dx, \quad n = 0, 1, \ldots, N.$$

It follows from the next corollary that Bernoulli coefficients will decay rapidly with increasing n.

Corollary 2. *[42] Assume that the function $g(x) \in \mathcal{C}^N(\Omega)$ is approximated by Bernoulli polynomials as argued in Corollary 1. Then the coefficients g_n decay as follows*

$$g_n \leq \frac{\bar{G}_n}{n!}, \quad n = 0, 1, \dots, N,$$

where $\bar{G}_n$ denotes the maximum of $g^{(n)}(x)$ in the interval Ω.

The following theorem provides an error term for the approximation presented in Corollary 1.

Theorem 3. *[42] Suppose that $g(x) \in \mathcal{C}^N(\Omega)$ and $P_N[g](x)$ is its approximation in terms of Bernoulli polynomials and $R_N[g](x)$ is the remainder term. Then, the associated formulas are stated as follows*

$$g(x) = P_N[g](x) + R_N[g](x), \quad x \in \Omega,$$

$$P_N[g](x) = \int_0^1 g(x)dx + \sum_{j=1}^{N} \frac{B_j(x)}{j!}(g^{(j-1)}(1) - g^{(j-1)}(0)),$$

$$R_N[g](x) = -\frac{1}{N!}\int_0^1 B_N^*(x-t)g^{(N)}(t)dt,$$

where $B_N^*(x) = B_N(x - [x])$ and $[x]$ denotes the largest integer not greater than x.

Theorem 4. *[43] Suppose $g(x) \in \mathcal{C}^N(\Omega)$ and $P_N[g](x)$ is its approximation in terms of Bernoulli polynomials. Then the error bound would be obtained as follows*

$$E(g) = \|g(x) - P_N[g](x)\|_\infty \leq C\hat{G}(2\pi)^{-N}, \quad x \in \Omega,$$

where C is a positive constant independent of N and $\hat{G}$ is such that

$$\|g^{(i)}(x)\|_\infty \leq \hat{G}, \quad i = 0, 1, \dots, N.$$

The above results can be extended to the case of functions of two (or more) variables. Let $k(x,t) \in H \times H$, then it can be approximated in terms of truncated Bernoulli series as

$$k(x,t) \simeq \sum_{i=0}^{N}\sum_{j=0}^{N} k_{ij}B_i(x)B_j(t) = B^T(x)KB(t), \tag{2.15}$$

where $K = [k_{ij}]_{i,j=0}^N$ is an $(N+1) \times (N+1)$ matrix.

Corollary 5. *[6] Assume that the function $k(x,t) \in \mathcal{C}^{N,N}(\Omega \times \Omega)$ is approximated by the two variable truncated Bernoulli series $P_N[k](x,t) = \sum_{i=0}^{N}\sum_{j=0}^{N} k_{ij}B_i(x)B_j(t)$, then the coefficients k_{ij} can be calculated from the following relation*

$$k_{ij} = \frac{1}{i!j!}\int_0^1\int_0^1 \frac{\partial^{i+j}}{\partial x^i \partial t^j}k(x,t)dxdt, \quad i,j = 0, 1, \dots, N.$$

Corollary 6. *Assume that the function $k(x,t) \in \mathcal{C}^{N,N}(\Omega \times \Omega)$ is approximated by Bernoulli polynomials as argued in Corollary 5. Then the coefficients k_{ij} decay as follows*

$$k_{mn} \leq \frac{\bar{K}_{i,j}}{i!j!}, \quad i,j = 0,1,\ldots,N,$$

where $\bar{K}_{i,j}$ denotes the maximum of $\frac{\partial^{i+j}}{\partial x^i \partial t^j} k(x,t)$ in the unit square $\Omega \times \Omega$.

Proof. Since it is trivial we omit the proof. □

Theorem 7. *[43] Suppose $k(x,t) \in \mathcal{C}^{N,N}(\Omega \times \Omega)$ and $P_N[k](x,t)$ be its approximation in terms of Bernoulli polynomials. Then the error bound would be obtained as follows*

$$E(k) = \|k(x,t) - P_N[k](x,t)\|_\infty \leq C\hat{K}N(2\pi)^{-N},$$

where C is a positive constant independent of N and $\hat{K}$ is such that

$$\left\| \frac{\partial^{i+j}}{\partial x^i \partial t^j} k(x,t) \right\|_\infty \leq \hat{K}, \quad i,j = 0,1,\ldots,N.$$

2.3 Operational matrices of integration

In this section, the Bernoulli operational matrices of integration and the product will be derived.

Theorem 8. *Let $B(x)$ be the Bernoulli vector defined in (2.3). Then*

$$\int_0^x B(t)dt \simeq PB(x), \tag{2.16}$$

where P is the $(N+1) \times (N+1)$ operational matrix of integration defined by

$$P = \begin{bmatrix} -B_1 & 1 & 0 & \ldots & 0 \\ \frac{-B_2}{2} & 0 & \frac{1}{2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{-B_N}{N} & 0 & 0 & \ldots & \frac{1}{N} \\ \frac{-B_{N+1}}{N+1} & 0 & 0 & \ldots & 0 \end{bmatrix}. \tag{2.17}$$

Proof. It follows from the first part of (2.2) that

$$\int_0^x B_k(t)dt = \frac{1}{k+1}(B_{k+1}(x) - B_{k+1}), \quad k \geq 0.$$

So, the integration of the vector $B(x)$ is given by

$$\int_0^x B(t)dt = P^* B^*(x), \tag{2.18}$$

where P^* is an $(N+1)\times(N+2)$ matrix having the form

$$P^* = \left[\begin{array}{c|c} P & p \end{array}\right] = \left[\begin{array}{ccccc|c} -B_1 & 1 & 0 & \dots & 0 & 0 \\ \frac{-B_2}{2} & 0 & \frac{1}{2} & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \frac{-B_N}{N} & 0 & 0 & \dots & \frac{1}{N} & 0 \\ \frac{-B_{N+1}}{N+1} & 0 & 0 & \dots & 0 & \frac{1}{N+1} \end{array}\right],$$

and $B^*(x)$ is an $(N+2)\times 1$ vector of the form

$$B^*(x) = \left[\frac{B(x)}{B_{N+1}(x)}\right].$$

If we truncate the term $B_{N+1}(x)$ in the vector $B^*(x)$, i.e., if we drop the vector p in the matrix P^*, relation (2.18) becomes the approximate relation (2.16). □

Note that, the structure of P is simple, since all its elements are zero, except for its first column and its superdiagonal, and hence the Bernoulli basis may be computationally more attractive than other sets of basis functions.

Comparing the structure of the Bernoulli integral operational matrix P (denoted for the moment as P_B) with the corresponding matrices of Walsh P_W, block-pulse P_b, and Laguerre P_L, we may observe that P_B has the following characteristics:

- Using P_B, instead of P_B^* is a rather insignificant approximation, particularly if one considers the fact that $\alpha_N = \frac{1}{N+1}$ diminishes with N. The same approach is applied in the Laguerre case [20], but the approximation there is more significant since the corresponding term α_N in the P matrix is independent of N and is always equal to -1. For the Walsh case, the approximation of the form (2.16) is definitely significant since, for any given N, many non-zero terms in determining P are truncated. Finally, the case of block-pulse functions appears not to involve this type of approximation. This fact may be of great importance, since it could considerably reduce the overall approximation error.
- The accuracy in relation (1.2) depends on two factors, namely, the dimension $(N+1)$ of the basis vector $\Psi(x)$ and the particular $\Psi(x)$ used. From the remarks of the previous paragraph it appears that relation (1.2) could be more accurate if Bernoulli functions were used rather than Walsh or Laguerre functions.

It is to be noted that, using equations (2.13) and (2.16), the integral of any function $g(x)$ can be approximated as

$$\int_0^x g(t)dt \simeq \int_0^x G^T B(t)dt \simeq G^T P B(x).$$

We also need to evaluate the product of $B(x)$ and $B^T(x)$, which is called the product matrix of Bernoulli polynomials. For this purpose, we first approximate the functions $x^k B_i(x)$, for $i,k = 0,1,\dots,N$, in terms of $B(x)$. By using (2.13), we can write

$$x^k B_i(x) \simeq B^T(x) e_{k,i}, \tag{2.19}$$

where $e_{k,i}$ is the Bernoulli polynomial coefficients vector defined as

$$e_{k,i} = [e_0^{k,i}, e_1^{k,i}, \dots, e_N^{k,i}]^T. \tag{2.20}$$

Using Eqs. (2.8) and (2.19), we obtain

$$e_{k,i} \simeq Q^{-1} \int_0^1 x^k B(x) B_i(x) dx = Q^{-1} \begin{bmatrix} \int_0^1 x^k B_i(x) B_0(x) dx \\ \int_0^1 x^k B_i(x) B_1(x) dx \\ \vdots \\ \int_0^1 x^k B_i(x) B_N(x) dx \end{bmatrix}.$$

Now, for any arbitrary vector $C = [c_0, c_1, \dots, c_N]^T$ in $\mathbb{R}^{N+1}$ we define the notations

$$\tilde{E}_k = E_k C, \quad k = 0, 1, \dots, N,$$
$$\tilde{C} = [\tilde{E}_0, \tilde{E}_1, \dots, \tilde{E}_N],$$

where $\hat{D}$ is the matrix defined by (2.7) and E_k is an $(N+1) \times (N+1)$ matrix with $e_{k,i}$, $i = 0, 1, \dots, N$, as its columns.

Theorem 9. *Let $C = [c_0, c_1, \dots, c_N]^T$ be an arbitrary vector in $\mathbb{R}^{N+1}$. Then*

$$B(x) B^T(x) C \simeq \hat{C} B(x), \tag{2.21}$$

where $\hat{C}$ is the $(N+1) \times (N+1)$ product operational matrix defined by

$$\hat{C} = \hat{D} \tilde{C}^T.$$

Proof. Using (2.6) we obtain

$$\begin{aligned} B(x) B^T(x) C &= \left(\hat{D} T_N(x)\right) B^T(x) C \\ &= \hat{D} \left[B^T(x) C, x B^T(x) C, \dots, x^N B^T(x) C\right]^T \\ &= \hat{D} \left[\sum_{i=0}^{N} c_i B_i(x), \sum_{i=0}^{N} c_i x B_i(x), \dots, \sum_{i=0}^{N} c_i x^N B_i(x)\right]^T, \end{aligned} \tag{2.22}$$

and using (2.19) and (2.20) yield

$$
\begin{aligned}
\sum_{i=0}^{N} c_i x^k B_i(x) &\simeq \sum_{i=0}^{N} c_i \left(B^T(x) e_{k,i}\right) = \sum_{i=0}^{N} c_i \left(\sum_{j=0}^{N} e_j^{k,i} B_j(x)\right) \\
&= B^T(x) \begin{bmatrix} \sum_{i=0}^{N} c_i e_0^{k,i} \\ \sum_{i=0}^{N} c_i e_1^{k,i} \\ \vdots \\ \sum_{i=0}^{N} c_i e_N^{k,i} \end{bmatrix} \\
&= B^T(x)[e_{k,0}, e_{k,1}, \dots, e_{k,N}]C = B^T(x)\tilde{E}_K.
\end{aligned} \tag{2.23}
$$

Combining equations (2.22) and (2.23) gives the result. □

3 Numerical solution of nonlinear VIEs

In this section, we use the operational matrices of the Bernoulli polynomials and a collocation method to numerically solve problem (1.1) with assumption (1.3). So, we consider the following integral equation

$$
u(x) = f(x) + \int_0^x k(x,t) u^m(t) dt, \quad x \in \Omega. \tag{3.1}
$$

If we approximate functions $f(x)$, $u(x)$, $u^m(x)$ and $k(x,t)$ using Bernoulli polynomials, as described by equations (2.13) and (2.15), then we obtain

$$
f(x) \simeq B^T(x) F, \tag{3.2}
$$

$$
u(x) \simeq B^T(x) U, \tag{3.3}
$$

$$
u^m(x) \simeq B^T(x) U^{(m)}, \tag{3.4}
$$

$$
k(x,t) \simeq B^T(x) K B(t), \tag{3.5}
$$

where the vectors $F, U, U^{(m)}$ and matrix K are Bernoulli polynomial coefficients of $f(x)$, $u(x)$, $u^m(x)$ and $k(x,t)$ respectively. We again note that $U^{(m)}$ is a column vector whose elements are nonlinear functions of the elements of the unknown vector U. With substituting approximations (3.2)-(3.5) into (3.1), we get

$$
\begin{aligned}
B^T(x) U &\simeq B^T(x) F + \int_0^x B^T(x) K B(t) B^T(t) U^{(m)} dt \\
&= B^T(x) F + B^T(x) K \int_0^x B(t) B^T(t) U^{(m)} dt.
\end{aligned}
$$

Using (2.21) leads to

$$\begin{aligned} B^T(x)U &\simeq B^T(x)F + B^T(x)K \int_0^x \left(\widehat{U^{(m)}}\right)^T B(t)dt \\ &= B^T(x)F + B^T(x)K\left(\widehat{U^{(m)}}\right)^T \int_0^x B(t)dt. \end{aligned}$$

Now, using (2.16) gives

$$B^T(x)U \simeq B^T(x)F + B^T(x)K\left(\widehat{U^{(m)}}\right)^T PB(x), \tag{3.6}$$

where P is the Bernoulli operational matrix of integration given in (2.17). Collocating equation (3.6) at the $(N+1)$ Newton-Cotes nodes as

$$x_l = \frac{2l+1}{2(N+1)}, \quad l = 0, 1, \ldots, N, \tag{3.7}$$

will result in

$$B^T(x_l)U \simeq B^T(x_l)F + B^T(x_l)\left(K\left(\widehat{U^{(m)}}\right)^T P\right)B(x_l), \quad l = 0, 1, \ldots, N. \tag{3.8}$$

Since $U^{(m)}$ is a column vector whose elements are nonlinear functions of the element of the unknown vector $U = [u_i]_{i=0}^N$, then equation (3.8) is a nonlinear system of $(N+1)$ algebraic equations with $(N+1)$ unknowns u_0, u_1, $\ldots$, u_N. This nonlinear system of algebraic equations can be solved by numerical methods such as Newton's iterative method. If $\bar{U}$ be an approximate solution of this system, then $\bar{U}_m(x) = B^T(x)\bar{U}$ is an approximate solution of equation (3.1).

In the following theorem we shall find an upper bound for the error between the exact solution $u(x)$ and the approximate solution $u_N(x)$ of equation (3.1) with the considered assumptions.

Theorem 10. *Let $u(x)$ be the exact solution and $u_N(x) = B^T(x)\bar{U}$ be the approximated solution of (3.1) where the unknown Bernoulli coefficient vector $\bar{U}$ is determined by solving the nonlinear algebraic system of equations (3.8). Moreover assume that*

(1) $|u(x)| \leq \rho, \quad \forall x \in \Omega,$

(2) $|k(x,t)| \leq \tilde{k}, \quad \forall (x,t) \in \Omega \times \Omega,$

(3) $M\left(\tilde{k} + E(k)\right) < 1$ *in which* $M > 0$ *satisfies*

$$|u^m(t) - u_N^m(t)| \leq M|u(t) - u_N(t)|, \quad \forall t \in \Omega. \tag{3.9}$$

Then we have

$$\|u - u_N\|_\infty \leq \frac{E(f) + \rho^m E(k)}{1 - M\left(\tilde{k} + E(k)\right)}.$$

Proof. If we approximate both the driving term $f(x)$ and kernel $k(x,t)$ in terms of Bernoulli polynomials as described by equations (2.13) and (2.15), then the obtained solution is an approximated polynomial; $u_N(x)$ and we have

$$\begin{aligned} |u(x)-u_N(x)| &= \left|f(x)-f_N(x)+\int_0^x \Big(k(x,t)u^m(t)-k_N(x,t)u_N^m(t)\Big)dt\right| \\ &\le \big|f(x)-f_N(x)\big| + \int_0^x \Big|k(x,t)u^m(t)-k_N(x,t)u_N^m(t)\Big|dt. \end{aligned} \tag{3.10}$$

Moreover, using assumptions (1)-(3) we get

$$\begin{aligned} \Big|k(x,t)u^m(t)-k_N(x,t)u_N^m(t)\Big| &= \Big|k(x,t)\big(u^m(t)-u_N^m(t)\big)+\big(k(x,t)-k_N(x,t)\big)u_N^m(t)\Big| \\ &\le |k(x,t)|\,\big|u^m(t)-u_N^m(t)\big| + \big|k(x,t)-k_N(x,t)\big|\big|u_N^m(t)\big| \\ &\le \tilde{k}M\|u-u_N\|_\infty + E(k)\Big(\big|u^m(t)-u_N^m(t)\big|+\big|u_N^m(t)\big|\Big) \\ &\le M\big(\tilde{k}+E(k)\big)\|u-u_N\|_\infty + \rho^m E(k). \end{aligned} \tag{3.11}$$

Substituting (3.11) in (3.10), and noting that $x \in [0,1]$, we obtain

$$\|u-u_N\|_\infty \le E(f) + \big(\tilde{k}+E(k)\big)\|u-u_N\|_\infty + \rho^m E(k).$$

Then, by assumption (3) we get

$$\|u-u_N\|_\infty \le \frac{E(f)+\rho^m E(k)}{1-M\big(\tilde{k}+E(k)\big)},$$

which completes the proof. □

For a given function $f(x)$ if $f'(x)$ is continuous in $[-1,1]$ except for a finite number of bounded jumps, then $f(x)$ can be expanded in a convergent series as [29, pp. 309]

$$f(x) = \frac{1}{2}c_0 + \sum_{j=1}^{\infty} c_j T_j(x), \tag{3.12}$$

where

$$c_j = \frac{2}{\pi}\int_{-1}^{1} \frac{f(x)T_j(x)}{(1-x^2)^{\frac{1}{2}}}dx,$$

and $T_n(x)$ denotes the Chebyshev polynomial of the first kind of degree n.

Theorem 11. *[17, Theorem 3.12] When a function f has $r+1$ continuous derivatives on $[-1,1]$, where r is a finite number, then $|f(x)-S_n(x)| = \mathcal{O}(n^{-r})$ as $n\to\infty$ for all $x\in[-1,1]$, in which $S_n(x) = \frac{1}{2}c_0 + \sum_{j=1}^{n} c_jT_j(x)$ denotes the partial sum of expansion (3.12).*

We define the residual function $r_N(x)$ on Ω as

$$r_N(x) = u_N(x) - f(x) - \int_0^x k(x,t)u_N^m(t)dt, \tag{3.13}$$

where $u(x)$ is the exact solution of (3.1) and $u_N(x)$ is the approximation of $u(x)$ in terms of Bernoulli polynomials as described by equations (2.13). The next theorem gives an estimation of the residual error.

Theorem 12. *Let r is a finite number and the exact solution $u(x)$ of (3.1) has $r+1$ continuous derivatives on Ω. If $M = \|k(x,t)\|_\infty < \infty$, then $\|r_N\|_\infty = \mathcal{O}(N^{-r})$ as $N \to \infty$.*

Proof. It follows from equations (3.13) and (3.1) that

$$\|r_N\|_\infty \leq (1+M)\|u - u_N\|_\infty. \tag{3.14}$$

Suppose that $u_N(x) = \sum_{n=0}^{N} a_{N,n} B_n(x)$. Since the Bernoulli polynomials can be expressed in terms of Chebyshev polynomials of the first kind [22, Theorem 2.1], then $u_N(x)$ can be expanded as

$$u_N(x) = \sum_{k=0}^{N} b_{N,k} T_k(x),$$

where $b_{N,k}$ can be expressed in terms of $a_{N,n}$, $n,k = 0,\dots,N$. Therefore, by Theorem 11 we have $\|u - u_N\|_\infty = \mathcal{O}(N^{-r})$ as $N \to \infty$ which along with (3.14) completes the proof. □

4 Expressing $U^{(m)}$ in terms of U

For the numerical implementation of the presented method, we need to express the components of the vector $U^{(m)}$ as nonlinear functions of the elements of the vector U, where $U^{(m)}$ and U are the Bernoulli polynomial coefficients vectors of $u(x)$ and $u^m(x)$ respectively. To do this, we state the following lemma.

Lemma 13. *let m be a positive integer and U and $U^{(m)}$ are respectively the Bernoulli polynomial coefficients vectors of $u(x)$ and $u^m(x)$, that are defined on Ω. Also, let Q be the matrix defined in (2.8). Then, we have*

$$U^{(m)} = Q^{-1}(\hat{U}^T)^m e_1, \tag{4.1}$$

where e_1 denotes the first standard unit vector of order $(N+1)$.

Proof. We have

$$QU^{(m)} = \left(\int_0^1 B(x)B^T(x)dx\right)U^{(m)} = \int_0^1 B(x)B^T(x)U^{(m)}dx.$$

Using relations (3.4), (3.3), (2.2) and (2.21), we can write

$$\begin{aligned}
\int_0^1 B(x)B^T(x)U^{(m)}dx &\simeq \int_0^1 B(x)u^m(x)dx \\
&\simeq \int_0^1 B(x)\Big(B^T(x)U\Big)^m dx \\
&= \int_0^1 \Big(B(x)B^T(x)U\Big)\Big(B^T(x)U\Big)^{m-1} dx \\
&\simeq \hat{U}^T \int_0^1 B(x)\Big(B^T(x)U\Big)^{m-1} dx \\
&= \hat{U}^T \int_0^1 \Big(B(x)B^T(x)U\Big)\Big(B^T(x)U\Big)^{m-2} dx \\
&\simeq \Big(\hat{U}^T\Big)^2 \int_0^1 B(x)\Big(B^T(x)U\Big)^{m-2} dx \\
&\vdots \\
&\simeq \Big(\hat{U}^T\Big)^m \int_0^1 B(x)dx \\
&= \Big(\hat{U}^T\Big)^m e_1.
\end{aligned}$$

Since Q is invertible, we obtain (4.1). □

5 Illustrative examples

To demonstrate the applicability and accuracy of our method, we have applied it to several examples. These examples are solved in different references, so the numerical results obtained here can be compared with those of other numerical methods.

In order to analyze the error of the method we introduce notations

$$e_N(x) = u(x) - u_N(x),$$

and

$$\|e_N\|_\infty = \max\Big\{\big|e_N(x_l)\big|, \quad l = 0, 1, \dots, N\Big\},$$

where $u_N(x)$ denotes the approximate solution of order N of integral equation, which is obtained by the method presented in Section 3, and $u(x)$ is the exact solution of integral equation. Also, x_l, $l = 0, 1, \dots, N$, denote the Newton-Cotes nodes defined by (3.7).

Moreover, we define the global error as [26]

$$\epsilon_N = \frac{1}{|u|_{\max}} \sqrt{\frac{1}{N}\sum_{l=0}^{N}\big[e_N(x_l)\big]^2},$$

where $|u|_{\max}$ denotes the maximum absolute value of the exact solution u on Ω.

Experiments were performed on a personal computer using a 2.50 GHz processor and the codes were written in *Mathematica 9*.

Example 14. Consider the nonlinear Volterra integral equation

$$u(x) = 2 - e^x + \int_0^x e^{x-t} u^2(t)dt, \quad x \in [0,1]. \tag{5.1}$$

The exact solution of this equation is $u(x) = 1$. Numerical results obtained by the present method for this example has been shown in the first column of Table 1. Also, Fig. 1 shows the error graph of e_N, for $N = 8$.

Table 1. Computed errors $\|e_N\|_\infty$ for Examples 14-16.

N	Example 14	Example 15	Example 16
1	9.883×10^{-2}	1.138×10^{-2}	3.349×10^{-2}
2	4.703×10^{-2}	5.551×10^{-3}	1.849×10^{-2}
3	9.026×10^{-3}	1.271×10^{-3}	7.851×10^{-3}
4	7.014×10^{-4}	3.847×10^{-4}	2.364×10^{-3}
5	1.938×10^{-4}	1.393×10^{-4}	1.277×10^{-3}
6	1.637×10^{-5}	4.172×10^{-5}	2.925×10^{-4}
7	5.109×10^{-6}	1.397×10^{-5}	1.742×10^{-4}
8	4.010×10^{-7}	4.301×10^{-6}	3.982×10^{-5}

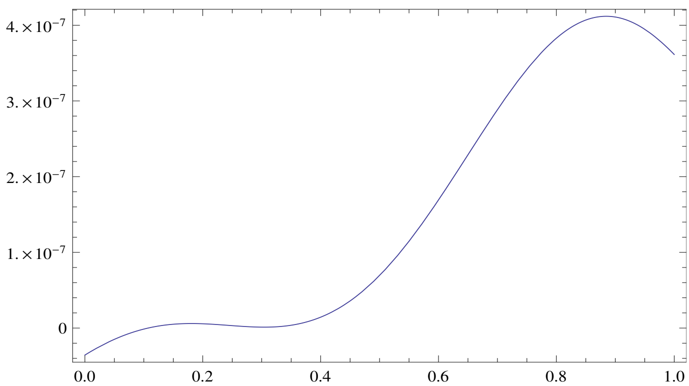

Figure 1. Graph of $e_N(x)$ for Example 14 with $N = 8$.

Example 15. [2, 30] Consider the following nonlinear Volterra integral equation

$$u(x) = \frac{3}{2} - \frac{1}{2}e^{-2x} - \int_0^x \Big(u^2(t) + u(t)\Big)dt, \quad x \in [0,1]. \tag{5.2}$$

The exact solution of this problem is $u(x) = e^{-x}$. The second column of Table 1 illustrates the numerical results obtained by the present method for this example. Also, Fig. 2 shows the error graph of e_N, for $N = 10$.

Integral equation (5.2) is solved in [2] and [30], respectively by Haar wavelets method and triangular functions (TF) method. Comparison of the second column of Table 1 with Fig. 3 of [2] shows better accuracy of our method using fewer number of basis functions and collocation points.

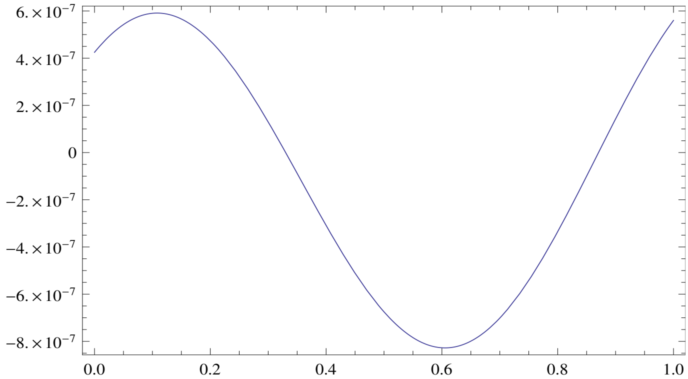

Figure 2. Graph of $e_N(x)$ for Example 15 with $N = 10$.

Example 16. [26] Let us consider the following linear Volterra integral equation

$$u(x) = f(x) + \int_0^x x\cos(t)u(t)dt, \quad x \in [0,1], \tag{5.3}$$

where

$$f(x) = \frac{1}{4}x\cos(2x) + \sin(x) - \frac{1}{2}x.$$

The exact solution of this problem is $u(x) = \sin(x)$. The third column of Table 1 illustrates the numerical results obtained by the present method for this example. Also, Fig. 3 shows the error graph of e_N, for $N = 15$.

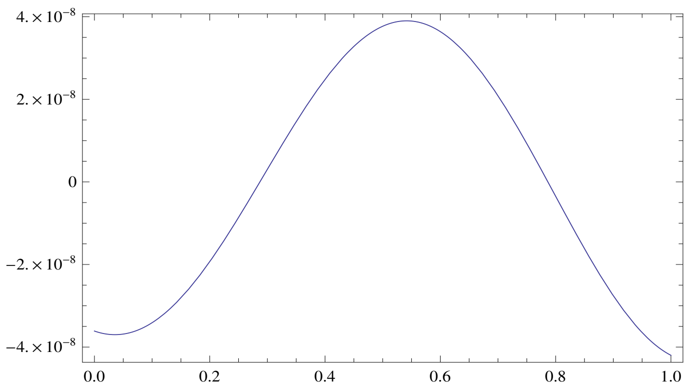

Figure 3. Graph of $e_N(x)$ for Example 16 with $N = 15$.

The random integral quadrature (RIQ) method is used in [26] to approximate the solution of integral equation (5.3) where $0 \le x \le \pi$. In the case of 5 field nodes distributed uniformly and randomly, the global errors obtained by RIQ method are $1.6462E-3$ and $2.4302E-3$ respectively. Also, in the case of 5 collocation points used, the global error obtained by the presented method for Example 16 is $2.0504E-3$ which shows similar accuracies for our method and RIQ method.

Example 17. [3, 33] Consider the nonlinear Volterra integral equation

$$u(x) = x + \cos(x) - 1 + \int_0^x \sin\big(u(t)\big)dt, \quad x \in [0,1], \tag{5.4}$$

with the exact solution $u(x) = x$. Integral equation (5.4) is not in the desired form (1.1), but it can be converted by approximating $\sin(u)$ using a finite number of terms of its Taylor series as

$$\sin(u) = u - \frac{u^3}{3!} + \frac{u^5}{5!} + \cdots + (-1)^d \frac{u^{2d+1}}{(2d+1)!}, \quad d \in \mathbb{Z}^{\geq 0}.$$

Table 2 illustrates the numerical results obtained by the present method for $N = 5$ and different values of d. Also, Fig. 4 shows the error graph of e_N, for $N = 6$ and $d = 3$.

Table 2. Computed errors $\|e_N\|_\infty$ for Example 17.

N	$d=0$	$d=1$	$d=2$	$d=3$
5	3.482×10^{-2}	8.735×10^{-4}	9.372×10^{-6}	2.960×10^{-6}

Integral equation (5.4) is solved in [33] using cubic B-spline wavelets basis. A comparison between the absolute errors obtained by the present method and the method of [33] is done in Table 3. This table shows that our method needs fewer number of basis functions (and therefore fewer number of collocation points) to achieve the desired accuracy.

Table 3. Comparison of absolute errors for Example 17.

x	Present method with 7 basis functions (N=6) and d=3	Algorithm 1 of [33] with 11 basis functions ($m = 4, s_\mu = 3$)
0	7.13684×10^{-7}	4.14485×10^{-6}
0.1	6.29968×10^{-7}	1.61021×10^{-7}
0.2	3.08742×10^{-7}	4.15844×10^{-7}
0.3	1.22505×10^{-7}	7.48669×10^{-7}
0.4	4.93745×10^{-7}	5.50796×10^{-7}
0.5	6.64531×10^{-7}	4.08869×10^{-7}
0.6	5.71379×10^{-7}	4.95687×10^{-7}
0.7	2.50408×10^{-7}	1.71069×10^{-7}
0.8	1.64765×10^{-7}	1.40256×10^{-6}
0.9	4.69861×10^{-7}	1.62347×10^{-6}
0.9	4.41524×10^{-7}	7.31106×10^{-6}

Example 18. [36] Consider the following second kind linear Volterra integral equation

$$3u(x) - \int_0^x (x+t)^2 u(t)dt = f(x), \quad x \in [0,1]. \tag{5.5}$$

The function $f(x)$ was chosen so that the analytical solution of (5.5) is $u(x) = e^x$. Fig. 5 shows the error graph of e_N, for $N = 12$.

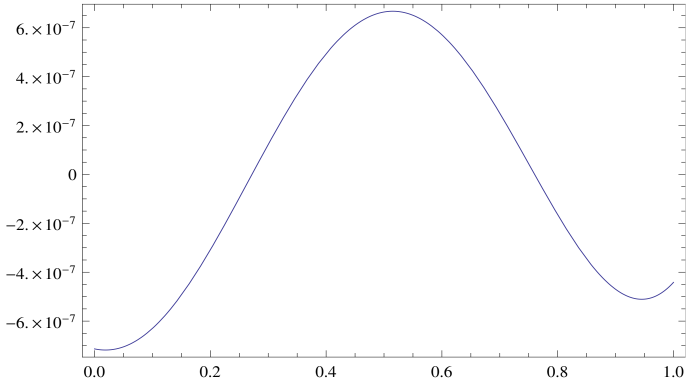

Figure 4. Graph of $e_N(x)$ for Example 17 with $N = 6$ and $d = 3$.

Table 4. Comparison of errors $\|e_N\|_\infty$ for Example 18.

N	Present method	N	Moving least squares (MLS) method [36]	
			Linear ($q = 1$)	Quadratic ($q = 2$)
2	7.09×10^{-2}	5	9.13×10^{-3}	2.39×10^{-4}
4	7.69×10^{-3}	9	2.70×10^{-3}	2.37×10^{-5}
6	4.21×10^{-4}	17	7.84×10^{-4}	7.59×10^{-6}
8	1.93×10^{-5}	33	2.09×10^{-4}	6.14×10^{-6}
10	7.74×10^{-7}	65	5.11×10^{-5}	5.31×10^{-4}
12	2.99×10^{-8}	129	1.37×10^{-5}	2.73×10^{-3}

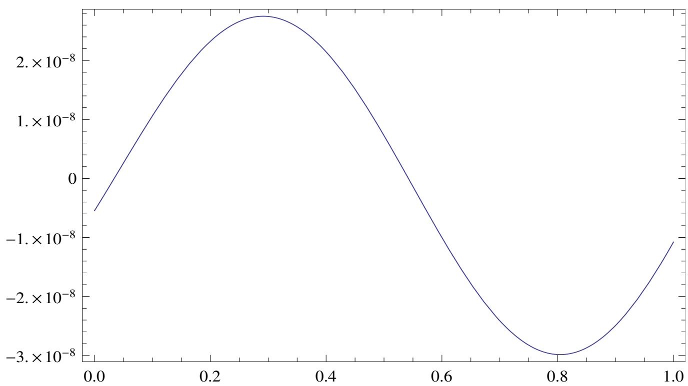

Figure 5. Graph of $e_N(x)$ for Example 18 with $N = 12$.

Integral equation (5.5) was previously considered in [36] by the moving least squares (MLS) method. A comparison between our results and the results of [36] has done in Table 4. The values of N in the first and the third columns of this table show the number of collocation points used for our method and the number of nodal points used for the MLS method, respectively. Based upon the results of Table 4, compared to the

MLS method, our method gives more accurate solutions by solving a very smaller linear system of equations.

Example 19. For our final example we consider the following Volterra integral equation of the second kind

$$u(x) = f(x) + \int_0^x (x-t)u(t)dt, \quad x \in [0,1].$$

The function $f(x)$ was chosen so that the analytical solution of (5.1) is

$$u(x) = \gamma x e^{1-\gamma x},$$

with γ denoting a given (real) parameter. Table 5 illustrates the numerical results obtained by the present method for $\gamma = 1, -1, -2, -3$ and different values of N. In the case of $\gamma = -1$, the numerical results obtained by the present method can be compared with those of Brunner and Yan [10] who used the collocation and iterated collocation methods for the numerical solution of this problem. We see that when γ decreases the total variation of the exact solution $u(x)$ (which is denoted by u_{tv}) increases and the method converges slowly. Also, Fig. 6 shows the error graph of e_N, for $N = 15$ and $\gamma = -1$.

Table 5. Computed errors $\|e_N\|_\infty$ for Example 19.

N	$\gamma = 1(u_{tv} = 1)$	$\gamma = -1(u_{tv} = e^2)$	$\gamma = -2(u_{tv} = 2e^3)$	$\gamma = -3(u_{tv} = 3e^4)$
5	7.204×10^{-4}	2.597×10^{-3}	3.450×10^{-2}	$1.574 \times 10^{+0}$
10	3.126×10^{-7}	1.086×10^{-6}	7.229×10^{-4}	6.114×10^{-2}
15	1.952×10^{-8}	4.549×10^{-9}	5.239×10^{-6}	2.101×10^{-3}

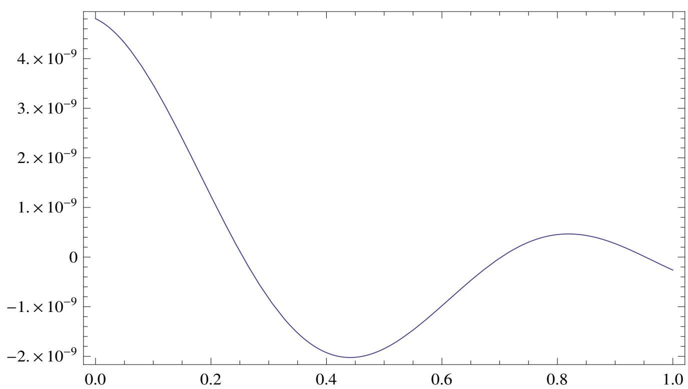

Figure 6. Graph of $e_N(x)$ for Example 19 with $N = 15$ and $\gamma = -1$.

6 Conclusion and comments

In this article we proposed an efficient and simple numerical method for solving a class of nonlinear Volterra integral equations of the form (1.1) and (1.3). For this purpose

the existing functions expanded in terms of Bernoulli polynomials. Then, using the new derived Bernoulli operational matrices and the collocation method, the problem reduced to a nonlinear system of algebraic equations. The obtained results show that this method is competitive with the other ones.
The proposed method has some notable advantages such as:

- The required computational effort to implement the method is small while the accuracy is high (the computations can be carried out on a personal computer).
- As the numerical results show, in the case of smooth solutions, a small number of basis functions ($N \leq 15$) is enough to obtain a high accuracy approximation of the solution $\big($error norm less than $10^{-8}\big)$.

Nevertheless, the method has some limitations and drawbacks, including:

- As we see in Example 19, when the exact solution $u(x)$ of the problem has large total variation, the method converges slowly.
- Since the coefficients of the Bernoulli polynomials grow quite fast in absolute value when N increases, then for large values of N the accuracy of the method is affected badly due to round off errors. This drawback will be also encountered when other classical orthogonal basis such as the shifted Legendre and shifted Chebyshev polynomials is used (since the coefficient of individual terms are greater than the ones of Bernoulli polynomials).

At the end, as it done in Example 17, if the part $N\big(u(x)\big)$ in equation (1.1) is not a polynomial of $u(x)$ but is continuous then the Weierstrass approximation theorem can be used to convert the problem to the desired form.

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