Acta Universitatis Matthiae Belii, series Mathematics
Issue 2015, 72–79, ISSN 1338-7111
Online Edition, http://actamath.savbb.sk

Weyl-Type Theorems For Restrictions Of Closed Linear Unbounded Operators

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Abstract

In this paper, T is a closed linear unbounded operator on an infinite dimensional complex Hilbert space H. We relate the study of Weyl-type theorems and properties for T to the study of Weyl-type theorems and properties for some restriction of T. Sufficient conditions are given for which T satisfies various Weyl-type theorems and properties if and only if $\mathcal{R}(T^n)$ is closed for some n $\in \mathbb{N}$ and some restriction of T satisfies the corresponding theorem or property.

Received 2 February 2015
Accepted in final form 6 November 2015
Published online 18 Januar 2016
Communicated with Peter Maličký.

Keywords Weyl-type theorems, restriction operators, semi-Fredholm operators.
MSC(2010) Primary: 47A10, 47A11 Secondary: 47A53, 47A55.

1 Introduction

Let H be an infinite dimensional complex Hilbert space and let $B(H)$ and $C(H)$ be the set of all bounded linear operators and set of all closed linear operators from domain $\mathcal{D}$(T) $\subseteq H$ to H, respectively. By $\mathcal{N}$(T) and $\mathcal{R}$(T) we denote the null space and range of T, respectively. We call an operator T $\in C(H)$ *upper semi-Fredholm* (respectively, *lower semi-Fredholm*) if $\mathcal{R}$(T) is closed and nullity of T, α(T) = dim $\mathcal{N}$(T) $< \infty$ (respectively, defect of T, β(T) = codim $\mathcal{R}$(T) $< \infty$). A *semi-Fredholm operator* is either an upper or lower semi-Fredholm operator. If T is both upper and lower semi-Fredholm, that is, if α(T) and β(T) both are finite, then T is called a *Fredholm operator*. By $SF_+(H)$ (respectively, $SF_-(H)$) we denote the class of all upper (respectively, lower) semi-Fredholm operators. For T $\in SF_+(H) \cup SF_-(H)$, index of T is defined as ind(T) = α(T) - β(T). An operator $T \in C(H)$ is called *Weyl* if it is Fredholm of index 0 and the *Weyl spectrum* of T is defined as σ_w(T) = {$\lambda \in \mathbb{C}$: T - λI is not Weyl}. We have the following notations :

$$SF_+^-(H) = \{T \in C(H) \ : \ T \in SF_+(H) \text{ and } ind(T) \leqslant 0\}$$

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$$SF_{-}^{+}(H) = \{T \in C(H) \ : \ T \in SF_{-}(H) \text{ and } ind(T) \geqslant 0\}$$

and these operators generate the following spectrum

$$\sigma_{SF_{+}^{-}}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SF_{+}^{-}(H)\}$$

$$\sigma_{SF_{-}^{+}}(T) = \{\lambda \in \mathbb{C} \ : \ T - \lambda I \notin SF_{-}^{+}(H)\}.$$

The *ascent* p(T) and *descent* q(T) of an operator T $\in C(H)$ are defined as follows:

$$p(\mathrm{T}) = inf\{n : \mathcal{N}(\mathrm{T}^n) = \mathcal{N}(\mathrm{T}^{n+1})\}$$

$$q(T) = inf\{n : \mathcal{R}(\mathrm{T}^n) = \mathcal{R}(\mathrm{T}^{n+1})\}.$$

Let σ(T), σ_a(T), σ_s(T) and ρ(T) denote the spectrum, approximate spectrum, surjective spectrum and the resolvent set of T $\in C(H)$, respectively, and let $\sigma_{des}(\mathrm{T}) = \{\lambda \in \mathbb{C}$: q(T - λI) $\not< \infty\}$ denote the *descent spectrum* of T. Evidently, $\sigma_{des}(\mathrm{T}) \subseteq \sigma(\mathrm{T})$.

By isoσ(T) and isoσ_a(T) we denote the isolated points of σ(T) and σ_a(T), respectively. It is well known that the resolvent operator $R_\lambda(\mathrm{T}) = (\mathrm{T} - \lambda\mathrm{I})^{-1}$ is an analytic operator-valued function for all $\lambda \in \rho$(T) and the points of isoσ(T) are either poles or essential singularities of R_λ(T). For T $\in C(H)$, $\lambda \in$ isoσ(T) is said to be a *pole of order p* if p = p(T - λI) $< \infty$ and q(T - λI) $< \infty$ ([6]). Also $\lambda \in \sigma_a$(T) is said to be a *left-pole* if p = p(T - λI) $< \infty$ and $\mathcal{R}(\mathrm{T} - \lambda\mathrm{I})^{p+1}$ is closed. Let π_o(T) and π_o^a(T) denote the set of all poles of finite multiplicity and left poles of finite multiplicity, respectively.

An important property of closed linear operators in Fredholm theory is the single valued extension property (SVEP). This property was first introduced by Dunford [3]. We mainly concern with the SVEP at a point, localized version of SVEP, introduced by Finch [4], and relate it to the finiteness of the ascent of a closed linear operator. Let $T : \mathcal{D}(T) \subset H \to H$ be a closed linear mapping and let λ_o be a complex number. The operator T has the *single valued extension property* (SVEP) at λ_o, if $f = 0$ is the only solution to $(T - \lambda I)f(\lambda) = 0$ that is analytic in every neighborhood of λ_o. Also T has SVEP, if it has this property at every point λ_o in the complex plane.

Evidently, T $\in C(H)$ has SVEP at every $\lambda \in \rho$(T). Moreover, by identity theorem for analytic functions, it is easily seen that T has SVEP at every boundary point (in particular, every isolated point) of σ(T). Also, from the definition of localized SVEP,

$$\lambda \in \mathrm{iso}\sigma_a(T) \Longrightarrow \mathrm{T} \text{ has SVEP at } \lambda, \quad \text{and by duality,}$$
$$\lambda \in \mathrm{iso}\sigma_s(T) \Longrightarrow \mathrm{T}^* \text{ has SVEP at } \lambda.$$

The above implications become equivalences whenever T is a bounded semi-Fredholm operator [1, Chapter 3]. For the case T $\in C(H)$, we prove this equivalence in the following theorem. We first give a definition and lemma needed for the proof of the theorem:

Definition 1. [5, Ch IV, §1] Let T $\in C(H)$. Let A be an operator such that $\mathcal{D}$(T) $\subset \mathcal{D}$(A) and $\|\mathrm{Au}\| \leqslant \mathrm{a}\|\mathrm{u}\| + \mathrm{b}\|\mathrm{Tu}\|$, u $\in \mathcal{D}$(T) where a,b are non-negative constants. Then we say that *A is relatively bounded with respect to T* or *T-bounded* and the *T-bound* of A is inf b.

Lemma 2. *[5, Ch. IV, Theorem 5.31] Let $T \in C(H)$ be semi-Fredholm and let A be a T-bounded operator in H. Then $S = T + \lambda A \in C(H)$, S is semi-Fredholm and α(S) as well as β(S) are constant for sufficiently small $|\lambda| > 0$.*

The *reduced minimum modulus* of $\mathrm{T} \in C(H)$ is defined by

$$\gamma(\mathrm{T}) = \mathit{inf}\,\{\|Tx\| : x \in \mathcal{D}(T) \cap \mathcal{N}(T)^{\perp}, \|x\| = 1\}.$$

Then, it is known that $\mathcal{R}(\mathrm{T})$ is closed iff $\gamma(\mathrm{T}) > 0$ for every $\mathrm{T} \in C(H)$.

Theorem 3. *Let $T \in C(H)$ be a semi-Fredholm operator and $\lambda_o \in \mathbb{C}$. Then the following are equivalent:*

(i) T has SVEP at λ_o

(ii) $\sigma_a(T)$ does not cluster at λ_o

(iii) $p(T - \lambda_o I) < \infty$.

Proof. We shall assume that $\lambda_o = 0$.

(i) $\Leftrightarrow$ (iii) follows from [4, Theorem 15]

(i) $\Rightarrow$ (ii) Suppose T has SVEP at zero. Since T is semi-Fredholm operator, so is T^n for all $n \in \mathbb{N}$. Then $\mathcal{R}(\mathrm{T}^n)$ is closed for all n, so that $\mathrm{T}^{\infty}(H) = \bigcap_{n=1}^{\infty} \mathcal{R}(\mathrm{T}^n)$ is closed.

Since T is semi-Fredholm, so that $\mathcal{R}(\mathrm{T})$ is closed, there exists an $\epsilon > 0$ such that $\gamma(\mathrm{T}) > \epsilon$. Consider λ in $0 < |\lambda| < \epsilon$. Then $\|\lambda x\| = |\lambda|\|x\| < \epsilon\|x\| < \gamma(\mathrm{T})\|x\|$, for all $x \in H$. By lemma 2, T - λI is a closed semi-Fredholm operator, so that $\mathcal{R}(\mathrm{T} - \lambda\mathrm{I})$ is closed for all $0 < |\lambda| < \epsilon$. Thus we have that if $0 < |\lambda| < \epsilon$, then $\lambda \in \sigma_a(\mathrm{T})$ iff $\lambda \in \sigma_p(\mathrm{T})$.

If $0 \neq x \in \mathcal{N}(\mathrm{T} - \lambda\mathrm{I})$ then $x = \frac{1}{\lambda}Tx = T(\frac{x}{\lambda}) \in \mathcal{R}(\mathrm{T})$. Also, $\mathrm{T}^2 x = \mathrm{T}(\lambda x) = \lambda \mathrm{T} x = \lambda^2 x$. This implies $x = \frac{1}{\lambda^2}\mathrm{T}^2 x \in \mathcal{R}(\mathrm{T}^2)$. Continuing like this, we get $x \in \mathrm{T}^{\infty}(H)$. Thus, $\mathcal{N}(\mathrm{T} - \lambda\mathrm{I}) \subseteq \mathrm{T}^{\infty}(H)$ for all $\lambda \neq 0$. This implies that every non-zero eigenvalue of T belongs to $\sigma(\mathrm{T}|_{T^{\infty}(H)})$.

Suppose that 0 is a cluster point of $\sigma_a(\mathrm{T})$. There exists a sequence (λ_n) of non-zero eigenvalues of T such that $\lambda_n \to 0$ as $n \to \infty$. Then $\lambda_n \in \sigma(\mathrm{T}|_{T^{\infty}(H)})$ so that $0 \in \sigma(\mathrm{T}|_{T^{\infty}(H)})$, as the spectrum of an operator is closed. Since T has SVEP at 0, so does $\mathrm{T}|_{T^{\infty}(H)}$. From [1, Lemma 1.9], $\mathrm{T}|_{T^{\infty}(H)}$ is onto. By [4, Theorem 2], $\mathrm{T}|_{T^{\infty}(H)}$ is injective so that $0 \notin \sigma(\mathrm{T}|_{T^{\infty}(H)})$, which is a contradiction. Therefore, $\sigma_a(\mathrm{T})$ does not cluster at 0.

(ii) $\Rightarrow$ (i) holds for all closed linear operators. ☐

Remark 4. By duality, we have that if $\mathrm{T} \in C(H)$ is semi-Fredholm, then the following statements are equivalent:

(i) T^* has SVEP at λ_o

(ii) $\sigma_s(\mathrm{T})$ does not cluster at λ_o

(iii) $\mathrm{q}(\mathrm{T} - \lambda_o\mathrm{I}) < \infty$.

Let $\mathrm{E}_o(\mathrm{T})$ and $\mathrm{E}_o^a(\mathrm{T})$ denote the set of all eigenvalues of finite multiplicities in iso$\sigma(\mathrm{T})$ and iso$\sigma_a(\mathrm{T})$, respectively. If $\mathrm{T} \in C(H)$, then T satisfies:

(i) Weyl's theorem if $\sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \mathrm{E}_o(\mathrm{T})$.

(ii) Browder's theorem if $\sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \pi_o(\mathrm{T})$.

(iii) a-Browder's theorem if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) = \pi_o^a(\mathrm{T})$.

(iv) a-Weyl's theorem if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) = \mathrm{E}_o^a(\mathrm{T})$.

(v) property (w) if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) = \mathrm{E}_o(\mathrm{T})$.

(vi) property (b) if $\sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T}) = \pi_o(\mathrm{T})$.

(vii) property (ab) if $\sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \pi_o^a(\mathrm{T})$.

(viii) property (aw) if $\sigma(\mathrm{T}) \setminus \sigma_w(\mathrm{T}) = \mathrm{E}_o^a(\mathrm{T})$.

Weyl's theorem and a-Weyl's theorem for restriction of bounded linear operators have been recently studied in [2]. In this paper, it is shown that with certain sufficient conditions on $\mathrm{T} \in C(H)$, several Weyl-type theorems and properties hold for T, if and only if there exists an n such that $\mathcal{R}(\mathrm{T}^n)$ is closed and Weyl type theorems and properties holds for $\mathrm{T}_n = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^n)}$.

In the second section, we consider Weyl's theorem and Browder's theorem and certain conditions have been given for which the study of Weyl's (respectively, Browder's) theorem for $\mathrm{T} \in C(H)$, can be reduced to the study of Weyl's (respectively, Browder's) theorem for some restriction of T. Also, sufficiency theorems are given for the case of a-Weyl's theorem and a-Browder's theorem. In the third section, properties (w), (b), (aw) and (ab) are considered. An example is given at the end of the third section to illustrate all the theorems proved.

Following lemma will be used throughout the paper:

Lemma 5. *[2, Lemma 2.1] Let $T \in C(H)$ and T_n be the restriction of T to the subspace $\mathcal{R}(T^n)$. Then, for all $\lambda \neq 0$, we have:*

(i) $\mathcal{N}((T_n - \lambda I)^m) = \mathcal{N}((T - \lambda I)^m)$, *for all m;*

(ii) $\mathcal{R}((T_n - \lambda I)^m) = \mathcal{R}((T - \lambda I)^m) \cap \mathcal{R}(T^n)$, *for all m;*

(iii) $\alpha(T_n - \lambda I) = \alpha(T - \lambda I)$;

(iv) $p(T_n - \lambda I) = p(T - \lambda I)$;

(v) $\beta(T_n - \lambda I) = \beta(T - \lambda I)$.

2 Weyl-type theorems and Restriction of operators

In this section, we give conditions under which Weyl's theorem (respectively, Browder's theorem) for an operator $\mathrm{T} \in C(H)$ is equivalent to Weyl's theorem (respectively, Browder's theorem) for certain restriction T_n of T. Also, we give certain sufficient conditions in the case of a-Browder's theorem and a-Weyl's theorem. Let $\mathbb{N}$ denote the set of all non-negative integers.

Theorem 6. *Let $T \in C(H)$ and suppose that $0 \notin iso\sigma(T) \cap \sigma_{des}(T)$. Then*

(i) T satisfies Weyl's theorem iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies Weyl's theorem.

(ii) T satisfies Browder's theorem iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies Browder's theorem.

Proof. (i) Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies Weyl's theorem.

Let $\lambda \in \mathrm{E}_o(\mathrm{T})$. Then $\lambda \in \mathrm{iso}\sigma(\mathrm{T})$ so that $\lambda \neq 0$ and and there exists an open disk $\mathbb{D}_\lambda \subseteq \mathbb{C}$ centered at λ such that $\sigma(\mathrm{T}) \cap \mathbb{D}_\lambda = \{\lambda\}$. Since, by Lemma 5, $\sigma(\mathrm{T}) \setminus \{0\}$

$= \sigma(T_n) \setminus \{0\}$ and $\sigma(T_n) \subseteq \sigma(T)$, we have that $\sigma(T_n) \cap \mathbb{D}_\lambda = \{\lambda\}$ so that $\lambda \in$ iso$\sigma(T_n)$. Now, $0 \neq \lambda \in E_o(T_n) = \sigma(T_n) \setminus \sigma_w(T_n)$ and thus, $\lambda \in \sigma(T) \setminus \sigma_w(T)$.

Now, suppose $\lambda \in \sigma(T) \setminus \sigma_w(T)$. If $\lambda = 0$, then since $0 \notin \sigma_{des}(T)$, [1, Theorem 3.4(iv)] implies $0 \in$ iso$\sigma(T)$ which is a contradiction to the hypothesis. Therefore, $\lambda \neq 0$ and $\alpha(T_n - \lambda I) = \alpha(T - \lambda I) = \beta(T - \lambda I) = \beta(T_n - \lambda I) < \infty$ so that $\lambda \in \sigma(T_n) \setminus \sigma_w(T_n) = E_o(T_n)$ and thus $\lambda \in$ iso$\sigma(T_n)$. Then T_n and T_n^* have SVEP at λ and by Theorem 3, $p(T_n - \lambda I) = q(T_n - \lambda I) < \infty$. Hence, $q(T - \lambda I) = p(T - \lambda I) < \infty$ and $\lambda \in E_o(T)$.

Conversely, assume T satisfies Weyl's theorem, then for $n = 0$, $\mathcal{R}(T^0) = H$ is closed and $T_o = T|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = T$ satisfies Weyl's theorem.

(ii) Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies Browder's theorem.

Let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. As proved above, $0 \neq \lambda \in \sigma(T_n) \setminus \sigma_w(T_n) = \sigma(T_n) \setminus \sigma_b(T_n)$, and thus $p(T_n - \lambda I) = q(T_n - \lambda I) < \infty$. Then, $q(T - \lambda I) = p(T - \lambda I) < \infty$. Therefore, $\lambda \in \sigma(T) \setminus \sigma_b(T)$. Hence, $\sigma_b(T) \subseteq \sigma_w(T)$ and since the reverse inclusion holds for every operator in $C(H)$, T satisfies Browder's theorem.

Now, suppose T satisfies Browder's theorem, then for $n = 0$, $\mathcal{R}(T^0) = H$ is closed and $T_o = T|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = T$ satisfies Browder's theorem.

□

Recently, in [7], *property* (w_1) for bounded linear operators, as a variant of Weyl's theorem, was introduced and studied. Further in [8], property (aw_1) was introduced as a variant of a-Weyl's theorem, where we say that an operator T satisfies *property* (aw_1) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subset E_o^a(T)$.

Similarly, we introduce a variant of a-Browder's theorem, viz. *property* (aB_1), where an operator $T \in C(H)$ satisfies property (aB_1) if $\sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subset \pi_o^a(T)$.

For the case of a-Browder's theorem and a-Weyl's theorem, we do not have necessary and sufficient conditions similar to Theorem 6. However, we give the following sufficiency theorems:

Theorem 7. *Let $T \in C(H)$ and suppose that $0 \notin$ iso$\sigma(T) \cap \sigma_{des}(T)$. Then T satisfies property (aB_1) if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies a-Browder's theorem.*

Proof. Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies a-Browder's theorem.

If $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$, then $0 < \alpha(T - \lambda I) < \infty$, $\mathcal{R}(T - \lambda I)$ is closed and $ind(T - \lambda I) \leqslant 0$. Suppose $\lambda = 0$, then $ind(T) \leqslant 0$ and by hypothesis, $0 \notin \sigma_{des}(T)$ implies $q(T) < \infty$ so that $ind(T) \geqslant 0$. Now $ind(T) = 0$ together with [1, Theorem 3.4(iv)] implies $p(T) = q(T) < \infty$ which is a contradiction since $0 \notin$ iso$\sigma(T)$. Therefore, $\lambda \neq 0$ and by Lemma 5, $\lambda \in \sigma_a(T_n) \setminus \sigma_{SF_+^-}(T_n) = \pi_o^a(T_n)$. Thus, $p = p(T - \lambda I) = p(T_n - \lambda I) < \infty$. Also, since $T - \lambda I$ is semi-fredholm, so is $(T - \lambda I)^{p+1}$. Thus, $\mathcal{R}(T - \lambda I)^{p+1}$ is closed and $\lambda \in \pi_o^a(T)$. Hence, T satisfies property (aB_1). □

Theorem 8. *Let $T \in C(H)$ and suppose that $0 \notin$ iso$\sigma_a(T) \cap \sigma_{des}(T)$. Then T satisfies property (aw_1) if there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies a-Weyl's theorem.*

Proof. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies a-Weyl's theorem.

Suppose $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$. If $\lambda = 0$, then $ind(\mathrm{T}) \leqslant 0$ and $0 \notin \sigma_{des}(\mathrm{T})$ implies $\mathrm{q(T)} < \infty$ and thus, $ind(\mathrm{T}) = 0$. Now, $\mathrm{p(T)} = \mathrm{q(T)} < \infty$ and $0 \in \mathrm{iso}\sigma(\mathrm{T})$ which is a contradiction to the hypothesis. Therefore, $0 \neq \lambda \in \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n) = \mathrm{E}_o^a(\mathrm{T}_n)$. Since $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T}_n)$, by Theorem 3, $\mathrm{p(T - \lambda I)} = \mathrm{p(T}_n - \lambda\mathrm{I)} < \infty$ and now $\lambda \in \mathrm{iso}\sigma_a(\mathrm{T})$. Thus, $\lambda \in \mathrm{E}_o^a(\mathrm{T})$ so that T satisfies property (aw_1). □

3 Extended Weyl-type theorems and Restriction of operators

In this section, we give conditions under which property (w) (respectively, property (b), property (aw) and property (ab)) for an operator $\mathrm{T} \in C(H)$ is equivalent to property (w) (respectively, property (b), property (aw) and property (ab)) for certain restriction T_n of T.

Theorem 9. *Let $T \in C(H)$ and suppose that $0 \notin iso\sigma(T) \cap \sigma_{des}(T)$. Then*

(i) T satisfies property (w) iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (w).

(ii) T satisfies property (b) iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (b).

(iii) T satisfies property (ab) iff there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (ab).

Proof. (i) Assume that there exists an $\mathrm{n} \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies property (w).

Let $\lambda \in \mathrm{E}_o(\mathrm{T})$. Then proceeding as in Theorem 6(i), $0 \neq \lambda \in \mathrm{E}_o(\mathrm{T}_n) = \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n)$. By Theorem 3, since $\lambda \in \mathrm{iso}\sigma(\mathrm{T}_n)$, $\mathrm{p(T}_n - \lambda\mathrm{I)} = \mathrm{q(T}_n - \lambda\mathrm{I)} < \infty$. Then, [1, Theorem 3.4(iii)] implies $\beta(\mathrm{T}_n - \lambda\mathrm{I}) = \alpha(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$ so that $0 < \beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ and hence $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$.

Now, suppose $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$. Then, $0 \neq \lambda \in \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n) = \mathrm{E}_o(\mathrm{T}_n)$. Using SVEP for T_n and T_n^* at λ, $\mathrm{p(T}_n - \lambda\mathrm{I)} = \mathrm{q(T}_n - \lambda\mathrm{I)} < \infty$ and thus, $\beta(\mathrm{T}_n - \lambda\mathrm{I}) = \alpha(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$. Now, $\beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ together with $\mathrm{p(T - \lambda I)} = \mathrm{p(T}_n - \lambda\mathrm{I)} < \infty$ imply $\mathrm{q(T - \lambda I)} < \infty$ and thus $\lambda \in \mathrm{E}_o(\mathrm{T})$. Hence, T satisfies property (w).

Conversely, if T satisfies property (w), then for $n = 0$, $\mathcal{R}(\mathrm{T}^0) = \mathrm{H}$ is closed and $\mathrm{T}_o = \mathrm{T}|_{\mathcal{D}(T) \cap \mathcal{R}(T^0)} = \mathrm{T}$ satisfies property (w).

(ii) Assume that there exists an $\mathrm{n} \in \mathbb{N}$ such that $\mathcal{R}(\mathrm{T}^n)$ is closed and T_n satisfies property (b).

Let $\lambda \in \pi_o(\mathrm{T})$. Then $\mathrm{p(T - \lambda I)} = \mathrm{q(T - \lambda I)} < \infty$ together with $\alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ imply $0 < \beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ and thus, $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$.

Now, suppose $\lambda \in \sigma_a(\mathrm{T}) \setminus \sigma_{SF_+^-}(\mathrm{T})$. As in the proof of Theorem 7, we get that $0 \neq \lambda \in \sigma_a(\mathrm{T}_n) \setminus \sigma_{SF_+^-}(\mathrm{T}_n) = \pi_o(\mathrm{T}_n)$. Then, $\mathrm{p(T}_n - \lambda\mathrm{I)} = \mathrm{q(T}_n - \lambda\mathrm{I)} < \infty$ so that $\beta(\mathrm{T}_n - \lambda\mathrm{I}) = \alpha(\mathrm{T}_n - \lambda\mathrm{I}) < \infty$. Now, $\beta(\mathrm{T} - \lambda\mathrm{I}) = \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty$ together with $\mathrm{p(T - \lambda I)} = \mathrm{p(T}_n - \lambda\mathrm{I)} < \infty$ imply $\mathrm{q(T - \lambda I)} < \infty$ and thus $\lambda \in \pi_o(\mathrm{T})$. Hence, T satisfies property (b).

Conversely, if T satisfies property (b), then for $n = 0$, $\mathcal{R}(T^0) = H$ is closed and $T_o = T|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = T$ satisfies property (b).

(iii) Suppose that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (ab).

Let $\lambda \in \pi_o^a(T)$. Then, proceeding as in Theorem 6, $0 \neq \lambda \in \pi_o^a(T_n) = \sigma(T_n) \setminus \sigma_w(T_n)$. and thus $\lambda \in \sigma(T) \setminus \sigma_w(T)$.

Now, suppose $\lambda \in \sigma(T) \setminus \sigma_w(T)$. If $\lambda = 0$, then by [1, Theorem 3.4(iv)], $0 < \alpha(T) = \beta(T) < \infty$ and $0 \notin \sigma_{des}(T)$ together imply that $0 \in iso\sigma(T)$, which is a contradiction to the hypothesis. Therefore, $0 \neq \lambda \in \sigma(T_n) \setminus \sigma_w(T_n) = \pi_o^a(T_n)$ so that $\lambda \in \sigma_a(T_n)$, $p = p(T_n - \lambda I) < \infty$, $\alpha(T_n - \lambda I) < \infty$ and $\mathcal{R}(T_n - \lambda I)^{p+1}$ closed. Now, $\lambda \in \sigma_a(T)$, $p = p(T - \lambda I) < \infty$ and $\alpha(T - \lambda I) < \infty$. Also, $\lambda \in \sigma(T) \setminus \sigma_w(T)$ so that $T - \lambda I$ and hence $(T - \lambda I)^{p+1}$ is a Fredholm operator. Then, $\mathcal{R}(T - \lambda I)^{p+1}$ is closed and so $\lambda \in \pi_o^a(T)$. Hence, T satisfies property (ab).

Conversely, suppose T satisfies property (ab), then for $n = 0$, $\mathcal{R}(T^0) = H$ is closed and $T_o = T|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = T$ satisfies property (ab).

□

Theorem 10. *Let $T \in C(H)$ and suppose $0 \notin iso\sigma_a(T) \cap \sigma_{des}(T)$. Then T satisfies property (aw) iff there exists an $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (aw).*

Proof. Assume that there exists $n \in \mathbb{N}$ such that $\mathcal{R}(T^n)$ is closed and T_n satisfies property (aw).

Suppose $\lambda \in E_o^a(T)$. Then $\lambda \neq 0$, $\lambda \in iso\sigma_a(T)$ and $0 < \alpha(T - \lambda I) < \infty$ so that $0 < \alpha(T_n - \lambda I) < \infty$ and thus $\lambda \in \sigma_a(T_n)$. Since $\lambda \in iso\sigma_a(T)$, there exists a disk $\mathbb{D}_\lambda \subseteq \mathbb{C}$ such that $\mathbb{D}_\lambda \cap \sigma_a(T) = \{\lambda\}$. Then $\mathbb{D}_\lambda \cap \sigma_a(T_n) \subseteq \{\lambda\}$. If $\mathbb{D}_\lambda \cap \sigma_a(T_n) = \phi$, then $\lambda \notin \sigma_a(T_n)$ which is a contradiction. Thus $\mathbb{D}_\lambda \cap \sigma_a(T_n) = \{\lambda\}$ and so $\lambda \in iso\sigma_a(T_n)$. Now, $\lambda \in E_o^a(T_n) = \sigma(T_n) \setminus \sigma_w(T_n)$. Since $\lambda \neq 0$, we get that $\lambda \in \sigma(T) \setminus \sigma_w(T)$.

Now, let $\lambda \in \sigma(T) \setminus \sigma_w(T)$. Then, $0 \neq \lambda \in \sigma(T_n) \setminus \sigma_w(T_n) = E_o^a(T_n)$. Since $\lambda \in iso\sigma_a(T_n)$, by Theorem 3, $p(T - \lambda I) = p(T_n - \lambda I) < \infty$ and $\lambda \in iso\sigma_a(T)$. Therefore, $\lambda \in E_o^a(T)$ and hence, T satisfies property (aw).

Conversely, suppose T satisfies property (aw), then for $n = 0$, $\mathcal{R}(T^0) = H$ is closed and $T_o = T|_{\mathcal{D}(T)\cap\mathcal{R}(T^0)} = T$ satisfies property (aw). □

The following example illustrates all the theorems of this paper:

Example 11. Let $H = l^2$ and let T be defined as follows:

$$T(x_1, x_2, x_3, \dots) = (x_1, 2x_2, 3x_3, 4x_4, 5x_5, \dots)$$

$$\text{where} \quad \mathcal{D}(T) = \left\{(x_1, x_2, x_3, \dots) \in l^2 : \sum_{j=1}^{\infty} |jx_j|^2 < \infty\right\}.$$

Then $c_{oo} = \{(x_n) : x_n \neq 0 \text{ for only finitely many } n\} \subseteq \mathcal{D}(T)$. Since c_{oo} is dense in l^2, so is $\mathcal{D}(T)$. Also $T = T^*$, so that T is a closed linear operator.

Now, $\sigma(T) = \sigma_a(T) = \sigma_p(T) = \{j : j \in \mathbb{N}\}$ and $0 \notin iso\sigma(T) \cap \sigma_{des}(T)$.

For $\lambda = j$, $j \in \mathbb{N}$,

$$\mathcal{N}(T - \lambda I) = span\{e_j\} \Longrightarrow \alpha(T - \lambda I) = 1 < \infty \quad \text{and}$$

$$\mathcal{R}(T - \lambda I) = span\{e_j\}^\perp \Longrightarrow \mathcal{R}(T - \lambda I) \text{ is closed and } \beta(T - \lambda I) = 1 < \infty.$$

Therefore, T - λI is Fredholm operator of index 0 and thus $\sigma_w(\mathrm{T}) = \sigma_{SF_+^-}(\mathrm{T}) = \phi$.

Since $\lambda = j$, for $j \in \mathbb{N}$ is isolated in $\sigma(\mathrm{T}) = \sigma_a(\mathrm{T})$ and $\alpha(\mathrm{T} - \lambda \mathrm{I}) = 1 < \infty$, thus $\mathrm{E}_o(\mathrm{T}) = \mathrm{E}_o^a(\mathrm{T}) = \{\ j : j \in \mathbb{N}\ \}$. Also $\mathrm{p}(\mathrm{T} - \lambda\mathrm{I}) = \mathrm{q}(\mathrm{T} - \lambda\mathrm{I}) = 1$, $\pi_o(\mathrm{T}) = \pi_o^a(\mathrm{T}) = \{j : j \in \mathbb{N}\}$.

Hence:

$$\sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \{j : j \in \mathbb{N}\} = E_o(T),$$

i.e., *Weyl's theorem* and *property (w)* hold for T.

$$\sigma(T) \setminus \sigma_w(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \{j : j \in \mathbb{N}\} = \pi_o(T),$$

i.e., *Browder's theorem* and *property (b)* hold for T.

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_w(T) = \{j : j \in \mathbb{N}\} = E_o^a(T),$$

i.e., *property* (aw_1) and *property (aw)* hold for T.

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \sigma(T) \setminus \sigma_w(T) = \{j : j \in \mathbb{N}\} = \pi_o^a(T),$$

i.e., *property* (aB_1) and *property (ab)* hold for T.

Infact, $\mathcal{R}(\mathrm{T}^n)$ is closed and $\mathrm{T}_n = \mathrm{T}|_{\mathcal{D}(T)\cap\mathcal{R}(T^n)}$ satisfies the corresponding Weyl-type theorems and properties for all $n \in \mathbb{N}$.

Acknowledgements

The second author is supported by the Junior Research fellowship of Council of Scientific and Industrial Research, India, under the University Grants Commission Fellowship scheme (grant no. JRF/AA/139/F-211/2012-13)

References

[1] P. Aiena, "Fredholm and local spectral theory with applications to multipliers", Kluwer Acad. Publishers, 2004.

[2] C. Carpintero, O. García, D. Munoz, E. Rosas and J. Sanabria, Weyl Type Theorems for Restrictions of Bounded Linear Operators, *Extracta Mathematicae* **28(1)** (2013), 127–139.

[3] N. Dunford, Spectral theory I. Resolution of the identity, *Pacific Journal of Math.* **2** (1952), 559–614.

[4] J. K. Finch, The Single Valued Extension Property on a Banach Space, *Pacific Journal of Math.* **58(1)** (1975), 61–69.

[5] T. Kato, "Perturbation theory for linear operators", Springer, Vol. 132, 1995.

[6] D. C. Lay, Spectral Analysis Using Ascent, Descent, Nullity and Defect, *Math. Ann.* **184** (1970), 197–214.

[7] C. Sun, X. Cao and L. Dai, Property (w_1) and Weyl type theorem, *Journal of Mathematical Analysis and Applications* **363(1)** (2010), 1–6.

[8] H. Zariouh and F. Lombarkia, On Browder-type Theorems, *Acta Mathematica Universitatis Comenianae* **83(2)** (2014), 281–289.