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Initial Coefficient Bounds for a Comprehensive Subclass of Sakaguchi Type Functions

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Abstract
In this paper, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions defined in the open unit disk. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

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1 Introduction and Definitions

Let A denote the class of analytic functions in the unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

that have the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

Further, by S we shall denote the class of all functions in A which are univalent in U.

The Koebe one-quarter theorem [8] states that the image of U under every function f from S contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse f^{-1} which satisfies

$$f^{-1}(f(z)) = z, \quad (z \in U)$$

and

$$f\left(f^{-1}(w)\right) = w, \quad \left(|w| < r_0(f), \ r_0(f) \geq \frac{1}{4}\right),$$

where

$$f^{-1}(w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots.$$

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A function $f(z) \in A$ is said to be bi-univalent in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the class of bi-univalent functions defined in the unit disk U.

If the functions f and g are analytic in U, then f is said to be subordinate to g, written as

$$f(z) \prec g(z), \qquad (z \in U)$$

if there exists a Schwarz function $w(z)$, analytic in U, with

$$w(0) = 0 \text{ and } \quad |w(z)| < 1 \qquad (z \in U)$$

such that

$$f(z) = g(w(z)) \qquad (z \in U).$$

Lewin [15] studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient $|a_2|$. Subsequently, Netanyahu [17] showed that $max\,|a_2| = \frac{4}{3}$ if $f(z) \in \Sigma$. Brannan and Clunie [5] conjectured that $|a_2| \leq \sqrt{2}$ for $f \in \Sigma$. Brannan and Taha [4] introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses of univalent functions consisting of strongly starlike, starlike and convex functions. They introduced bi-starlike functions and obtained estimates on the initial coefficients. Bounds for the initial coefficients of several classes of functions were also investigated in ([1], [3], [7], [9], [13], [14], [16], [19], [21], [22], [23]).

Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only a few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([2], [6], [10], [11], [12]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N}\backslash\{1,2\}$; $\mathbb{N} = \{1,2,3,...\}$) is still an open problem.

Motivated by the earlier work of Sakaguchi [20] on the class of starlike functions with respect to symmetric points denoted by S_S consisting of functions $f \in A$ satisfy the condition $\text{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right) > 0$, $(z \in U)$, we introduce a new subclass of the function class Σ of bi-univalent functions, and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in this new subclass.

Definition 1. Let $h : U \longrightarrow \mathbb{C}$, be a convex univalent function such that $h(0) = 1$ and $h(\bar{z}) = \overline{h(z)}$, for $z \in U$ and $\text{Re}(h(z)) > 0$. A function $f \in \Sigma$ is said to be in the class $S_\Sigma^\lambda(\beta, s, t, h)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad e^{i\beta}\frac{[(s-t)z]^{1-\lambda} f'(z)}{[f(sz) - f(tz)]^{1-\lambda}} \prec h(z)\cos\beta + i\sin\beta, \quad z \in U \tag{1.2}$$

and

$$e^{i\beta}\frac{[(s-t)w]^{1-\lambda} g'(w)}{[g(sw) - g(tw)]^{1-\lambda}} \prec h(w)\cos\beta + i\sin\beta, \quad w \in U \tag{1.3}$$

where $g(w) = f^{-1}(w)$, $s, t \in \mathbb{C}$ with $s \neq t$, $|t| \leq 1$, $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $\lambda \geq 0$.

Remark 2. If we set $h(z) = \dfrac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, in the class $S_\Sigma^\lambda(\beta, s, t, h)$, we have $S_\Sigma^\lambda\left(\beta, s, t, \frac{1+Az}{1+Bz}\right)$ and defined as

$$f \in \Sigma, \quad e^{i\beta}\frac{[(s-t)z]^{1-\lambda} f'(z)}{[f(sz) - f(tz)]^{1-\lambda}} \prec \frac{1+Az}{1+Bz}\cos\beta + i\sin\beta, \quad z \in U$$

and

$$e^{i\beta}\frac{[(s-t)w]^{1-\lambda}g'(w)}{[g(sw)-g(tw)]^{1-\lambda}}\prec\frac{1+Aw}{1+Bw}\cos\beta+i\sin\beta,\quad w\in U$$

where $g(w)=f^{-1}(w)$, $s,t\in\mathbb{C}$ with $s\neq t$, $|t|\leq 1$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\lambda\geq 0$.

Remark 3. If we set $h(z)=\dfrac{1+(1-2\alpha)z}{1-z}$, $0\leq\alpha<1$, in the class $S_\Sigma^\lambda(\beta,s,t,h)$, we have $S_\Sigma^\lambda(\beta,s,t,\alpha)$ and defined as

$$f\in\Sigma,\quad \mathrm{Re}\left\{e^{i\beta}\frac{[(s-t)z]^{1-\lambda}f'(z)}{[f(sz)-f(tz)]^{1-\lambda}}\right\}>\alpha\cos\beta,\quad z\in U$$

and

$$\mathrm{Re}\left\{e^{i\beta}\frac{[(s-t)w]^{1-\lambda}g'(w)}{[g(sw)-g(tw)]^{1-\lambda}}\right\}>\alpha\cos\beta,\quad w\in U$$

where $g(w)=f^{-1}(w)$, $s,t\in\mathbb{C}$ with $s\neq t$, $|t|\leq 1$, $\beta\in(-\frac{\pi}{2},\frac{\pi}{2})$ and $\lambda\geq 0$.

Lemma 4. *(see [18]) Let the function $\phi(z)$ given by*

$$\phi(z)=\sum_{n=1}^{\infty}B_nz^n$$

be convex in U. Suppose also that the function $h(z)$ given by

$$h(z)=\sum_{n=1}^{\infty}h_nz^n$$

is holomorphic in U. If $h(z)\prec\phi(z)$, $z\in U$, then $|h_n|\leq|B_1|$, $n\in\mathbb{N}=\{1,2,3,...\}$.

2 Coefficient Estimates

Theorem 5. *Let f given by (1.1) be in the class $S_\Sigma^\lambda(\beta,s,t,h)$. Then*

$$|a_2|\leq\sqrt{\frac{2|B_1|\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}},\tag{2.1}$$

and

$$|a_3|\leq\frac{2|B_1|\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}.\tag{2.2}$$

Proof. Let $f\in S_\Sigma^\lambda(\beta,s,t,h)$, g be the analytic extension of f^{-1} to U and $s,t\in\mathbb{C}$ with $s\neq t$, $|t|\leq 1$ and $\lambda\geq 0$. It follows from (1.2) and (1.3) that there exists $p,q\in P$ such that

$$e^{i\beta}\frac{[(s-t)z]^{1-\lambda}f'(z)}{[f(sz)-f(tz)]^{1-\lambda}}=p(z)\cos\beta+i\sin\beta,\qquad(z\in U)\tag{2.3}$$

and

$$e^{i\beta}\frac{[(s-t)w]^{1-\lambda}g'(w)}{[g(sw)-g(tw)]^{1-\lambda}}=q(w)\cos\beta+i\sin\beta,\qquad(w\in U)\tag{2.4}$$

where $p(z)\prec h(z)$ and $q(w)\prec h(w)$ have the forms

$$p(z)=1+p_1z+p_2z^2+\cdots$$

and

$$q(w) = 1 + q_1 w + q_2 w^2 + \cdots,$$

respectively. It follows from (2.3) and (2.4), we deduce

$$e^{i\beta}\left[(\lambda-1)(s+t)+2\right]a_2 = p_1\cos\beta, \tag{2.5}$$

$$e^{i\beta}\left\{\left[(\lambda-1)\left(s^2+t^2+st\right)+3\right]a_3 - \tfrac{\lambda(\lambda-1)}{2}(s+t)^2a_2^2 + (\lambda-1)(s+t)\left[2+(\lambda-1)(s+t)\right]a_2^2\right\}$$
$$= p_2\cos\beta, \tag{2.6}$$

and

$$-e^{i\beta}\left[(\lambda-1)(s+t)+2\right]a_2 = q_1\cos\beta, \tag{2.7}$$

$$e^{i\beta}\left\{2\left[(\lambda-1)\left(s^2+t^2+st\right)+3\right] - \tfrac{\lambda(\lambda-1)}{2}(s+t)^2 + (\lambda-1)(s+t)\left[2+(\lambda-1)(s+t)\right]\right\}a_2^2$$
$$-e^{i\beta}\left[(\lambda-1)\left(s^2+t^2+st\right)+3\right]a_3 = q_2\cos\beta. \tag{2.8}$$

From (2.5) and (2.7) we obtain

$$p_1 = -q_1,$$

By adding (2.6) to (2.8), we get

$$e^{i\beta}\left\{2(\lambda-1)(s+t)\left[2+(\lambda-1)(s+t)\right] + 2\left[(\lambda-1)\left(s^2+st+t^2\right)+3\right] - \lambda(\lambda-1)(s+t)^2\right\}a_2^2$$
$$= (p_2+q_2)\cos\beta. \tag{2.9}$$

Since $p, q \in h(U)$, applying Lemma 4, we have

$$|p_m| = \left|\frac{p^{(m)}(0)}{m!}\right| \le |B_1|, \quad m \in \mathbb{N} \tag{2.10}$$

and

$$|q_m| = \left|\frac{q^{(m)}(0)}{m!}\right| \le |B_1|, \quad m \in \mathbb{N}. \tag{2.11}$$

Applying (2.10), (2.11) and Lemma 4 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_2| \le \sqrt{\frac{2|B_1|\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}}.$$

Subtracting (2.8) from (2.6) we have

$$e^{i\beta}\left\{2\left[(\lambda-1)(s^2+t^2+st)+3\right]a_3 - 2\left[(\lambda-1)(s^2+t^2+st)+3\right]a_2^2\right\} = (p_2-q_2)\cos\beta. \tag{2.12}$$

or, equivalently

$$a_3 = \frac{e^{-i\beta}(p_2+q_2)\cos\beta}{2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2} + \frac{e^{-i\beta}(p_2-q_2)\cos\beta}{2[(\lambda-1)(s^2+t^2+st)+3]}.$$

Applying (2.10), (2.11) and Lemma 4 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \le \frac{2|B_1|\cos\beta}{\left|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2\right|}.$$

This completes the proof of Theorem 5. □

3 Corollaries and Consequences

Corollary 6. *Let f given by (1.1) be in the class* $S_{\Sigma}^{\lambda}\left(\beta, s, t, \frac{1+Az}{1+Bz}\right)$. *Then*

$$|a_2| \leq \sqrt{\frac{2(A-B)\cos\beta}{|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2|}}$$

and

$$|a_3| \leq \frac{2(A-B)\cos\beta}{|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2|}.$$

where $-1 \leq B < A \leq 1,\ s,t \in \mathbb{C}$ *with* $s \neq t,\ |t| \leq 1$ *and* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2}),\ \lambda \geq 0$.

Corollary 7. *Let f given by (1.1) be in the class* $S_{\Sigma}^{\lambda}(\beta, s, t, \alpha)$. *Then*

$$|a_2| \leq \sqrt{\frac{4(1-\alpha)\cos\beta}{|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2|}},$$

and

$$|a_3| \leq \frac{4(1-\alpha)\cos\beta}{|2(\lambda-1)(s+t)[2+(\lambda-1)(s+t)]+2[(\lambda-1)(s^2+st+t^2)+3]-\lambda(\lambda-1)(s+t)^2|}.$$

where $0 \leq \alpha < 1,\ s,t \in \mathbb{C}$ *with* $s \neq t,\ |t| \leq 1$ *and* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

If we get $\lambda = 0$ in Theorem 5,

Corollary 8. *Let f given by (1.1) be in the class* $S_{\Sigma}^{0}(\beta, s, t, h)$. *Then*

$$|a_2| \leq \sqrt{\frac{|B_1|\cos\beta}{|3-2s-2t+st|}},$$

and

$$|a_3| \leq \frac{|B_1|\cos\beta}{|3-2s-2t+st|}.$$

If we get $\lambda = 0$ in Corollary 6,

Corollary 9. *Let f given by (1.1) be in the class* $S_{\Sigma}^{0}\left(\beta, s, t, \frac{1+Az}{1+Bz}\right)$. *Then*

$$|a_2| \leq \sqrt{\frac{(A-B)\cos\beta}{|3-2s-2t+st|}}$$

and

$$|a_3| \leq \frac{(A-B)\cos\beta}{|3-2s-2t+st|}$$

where $-1 \leq B < A \leq 1,\ s,t \in \mathbb{C}$ *with* $s \neq t,\ |t| \leq 1$ *and* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

If we get $\lambda = 0$ in Corollary 7,

Corollary 10. *Let f given by (1.1) be in the class* $S_{\Sigma}^{0}(\beta, s, t, \alpha)$. *Then*

$$|a_2| \leq \sqrt{\frac{2(1-\alpha)\cos\beta}{|3-2s-2t+st|}},$$

and

$$|a_3| \leq \frac{2(1-\alpha)\cos\beta}{|3-2s-2t+st|}.$$

where $0 \leq \alpha < 1,\ s,t \in \mathbb{C}$ *with* $s \neq t,\ |t| \leq 1$ *and* $\beta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

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