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Maximal designs and configurations - a survey

Alexander Rosa
McMaster University, Hamilton, ON, Canada
and
Matej Bel University, Banská Bystrica, Slovakia
rosa@mcmaster.ca

Abstract
It is the aim of this article to provide a unified view for, and a survey of, a class of problems that occur often in combinatorics, graph theory and related areas but also in "real life".

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Introduction

It is the aim of this article to provide a unified view for, and a survey of, a class of problems that occur often in combinatorics, graph theory and related areas but also in "real life".

We want to discuss a situation which typically is as follows. Given is a finite set $\mathcal{F}$ of objects called *figures*, and a symmetric irreflexive relation R on $\mathcal{F}$ (the *compatibility rule*) which specifies when two figures are compatible. An $(\mathcal{F}, R)$-configuration or simply a *configuration* is a set of pairwise compatible figures. A configuration C is *maximal* if there is no $f \in \mathcal{F},\ f \notin C$ such that $f \cup C$ is also a configuration. In other words, maximality is here with respect to inclusion.

More generally, the compatibility rule R is a function from a subset of the power set of $\mathcal{F}$ into $\{0,1\}$ but we will restrict ourselves to examples which are all of the simpler type above.

The *size* of a configuration is the number of its figures. An $(\mathcal{F}, R)$-configuration is *maximum* if it is maximal and contains the largest possible number of figures. Maximum configurations are sometimes called *maximum packings* or just *packings*.

Our interest will be mainly in the possible sizes of maximal $(\mathcal{F}, R)$-configurations, i.e. in the *spectrum* $Sp(\mathcal{F}, R)$ defined by

$$Sp(\mathcal{F}, R) = \{m\colon \text{there exists a maximal } (\mathcal{F}, R)\text{-configuration of size } m\}.$$

To determine the spectrum $Sp(\mathcal{F}, R)$, one usually needs to determine first the size of the smallest maximal, and maximum configurations, that is, the size of the smallest and the largest element of $Sp(\mathcal{F}, R)$.

We may envisage a procedure under which one tries to build maximal (maximum) configurations of given kind in a naive way: given any configuration, try to enlarge it by adjoining another figure subject to the compatibility rule, then another one, and so on, until this is no longer possible, i.e. you get "stuck". The elements of the spectrum represent sizes of all possible outcomes of such a process.

What follows is a (non-exhaustive) survey of problems falling under this framework, both completely solved (unfortunately, very few), partially solved (a few more) and those that remain largely open. This article may be viewed as an expanded and updated version of [61].

Our first example is a problem that has been solved completely.

1 Maximal sets of 1-factors

The figures here are 1-factors of the complete graph K_{2n} on a given set of $2n$ vertices; two such 1-factors are compatible if they are edge-disjoint. Let $M(2n)$ be the spectrum for maximal sets of 1-factors, i.e.

$$M(2n) = \{m\colon \text{there exists a maximal set of } m \text{ edge-disjoint 1-factors in } K_{2n}\}.$$

As a corollary to Dirac's Theorem (see, e.g., [70]) one obtains immediately

$$M(2n) \subseteq \{n, n+1, \ldots, 2n-1\}.$$

Trivially, $2n-2 \notin M(2n)$ since the complement of the union of $2n-2$ one-factors is itself a 1-factor. Furthermore, $n \notin M(2n)$ if n is even [16]. On the other hand, when k is odd and $k \in \{n, n+1, \ldots, 2n-1\}$ then $k \in M(2n)$, as shown by the following simple construction.

Let $Z_k \cup \{a_i : i = 1, 2, \ldots, 2n-k\}$ be the set of vertices of K_{2n} and let the 1-factor F be defined by

$$\begin{aligned} F = \{&\{a_1, 0\}, \{a_2, 1\}, \{a_3, k-1\}, \ldots, \{a_{2n-k-1}, \tfrac{1}{2}(2n-k-1)\}, \\ &\{a_{2n-k}, k - \tfrac{1}{2}(2n-k-1)\}, \{\tfrac{1}{2}(2n-k-1)+1, k - \tfrac{1}{2}(2n-k-1)-1\}, \\ &\{\tfrac{1}{2}(2n-k-1)+2, k - \tfrac{1}{2}(2n-k-1)-2\}, \ldots, \{\tfrac{1}{2}(k-1), \tfrac{1}{2}(k+1)\}\} \end{aligned}$$

(the edges in the last two lines are used only when $k \neq n$).

Developing F modulo k yields a maximal set of 1-factors, since the complement of the union of these 1-factors contains an odd component K_{2n-k}.

The case of even k turned out to be much more difficult. It was shown in [60] that for k even, $k \in M(2n)$ if and only if $\frac{1}{3}(4n+4) \leq k \leq 2n-4$.

Explicitly, we have for small values of n;

$$\begin{aligned} &M(4) = \{3\}, M(6) = \{3, 5\}, M(8) = \{5, 7\}, M(10) = \{5, 7, 9\}, \\ &M(12) = \{7, 9, 11\}, M(14) = \{7, 9, 11, 13\}, M(16) = \{9, 11, 12, 13, 15\}, \ldots, \\ &M(30) = \{15, 17, 19, 21, 22, 23, 24, 25, 27\} \text{ and so on.} \end{aligned}$$

Although the spectrum for maximal sets of 1-factors has thus been completely determined, several further problems arise when one puts additional conditions on the 1-factors comprising the set in question. Among several possible variations of the above problem that have been treated, at least to some degree, in the literature, is the one concerning maximal *perfect* sets of 1-factors. In this variation of the problem, two 1-factors are compatible if they are edge-disjoint and their union is a hamiltonian cycle. Let $M_{perf}(2n)$ be the spectrum of maximal perfect sets of 1-factors. It is currently not known whether the maximum possible value $2n-1$ is a member of $M_{perf}(2n)$ for all n since to determine this is equivalent to determining whether there exists a perfect 1-factorization of K_{2n} for all n. The latter remains a difficult unsolved problem (cf., e.g., [56]).

It is easily verified that $M_{perf}(4) = \{3\}, M_{perf}(6) = \{3, 5\}, M_{perf}(8) = \{5, 7\}$ but the determination of $M_{perf}(2n)$ becomes much more difficult for larger orders. Petrenjuk [58], [59] determined the sets $M_{perf}(2n)$ for $2n = 10, 12$: $M_{perf}(10) = \{5, 6, 7, 9\}$, $M_{perf}(12) = \{6, 7, 8, 9, 10, 11\}$. It is established in [56] that $M_{perf}(14) = \{7, 8, 9, 10, 11, 12, 13\} \cup I$ where either $I = \emptyset$ or $I = \{6\}$.

When n is an odd prime then $n \in M_{perf}(2n)$ but not much else seems to be known about $M_{perf}(2n)$.

One may, of course, consider also the situation where the union of any two 1-factors in a set $\mathcal{F}$ of disjoint 1-factors is isomorphic to a fixed 2-regular factor Q, not necessarily a hamiltonian cycle. Such a set has been called *Q-uniform* or simply *uniform*. Let $M_Q(2n) = \{s$: there exists a maximal Q-uniform set of s 1-factors of $K_{2n}\}$. The following results concerning small maximal uniform sets have been established in [56] with the aid of computer (below Q is represented just as a partition of $2n$).

$$M_{4+4}(8) = \{7\}, M_{6+4}(10) = \{3, 9\},$$
$$M_{4+4+4}(12) = \{3\}, M_{6+6}(12) = \{3, 5, 11\}, M_{8+4}(12) = \{6, 9\},$$
$$M_{6+4+4}(14) = \{3, 5, 7\}, M_{8+6}(14) = \{5, 6, 7\}, M_{10+4}(14) = \{5, 6, 7, 8\},$$
$$M_{4+4+4+4}(16) = \{7, 15\}, M_{6+6+4}(16) = M_{8+4+4}(16) = \{3, 4, 5, 7\},$$
$$\{5, 6, 7, 8, 9, 10\} \subseteq M_{8+8}(16).$$

2 Maximal sets of 2-factors

The figures here are 2-factors in the complete graph on a given set of n vertices; two 2-factors are compatible if they are edge-disjoint.

Let $M^{(2)}(n) = \{m$: there exists a maximal set of m edge-disjoint 2-factors of $K_n\}$. Petersen's theorem about the existence of a 2-factor in any regular graph of even degree (cf. [70]) implies that for odd n,

$$M^{(2)}(n) = \{\frac{1}{2}(n-1)\}.$$

The situation is somewhat more involved for n even. This is due to the fact that for odd d, there exist regular graphs of degree d without proper regular factors. König [53] calls such graphs *primitive*. An obvious extension of König's example for $d = 3$ yields a primitive graph of odd degree d $(d > 1)$ with $(d+1)^2$ vertices. It is shown in [42] that this is the minimum number of vertices a primitive graph of odd degree d can have. This implies that the spectrum $M^{(2)}(n)$ for n even is the following interval:

$$M^{(2)}(n) = \{\lfloor\frac{1}{2}(n-\sqrt{n})\rfloor, \lfloor\frac{1}{2}(n-\sqrt{n})\rfloor + 1, \ldots, \frac{1}{2}(n-2)\}.$$

In the next two examples, the figures are still 2-factors but of a restricted type.

3 Maximal sets of hamiltonian cycles

The figures here are *connected* 2-factors of K_n, that is, hamiltonian cycles; two hamiltonian cycles are compatible if they are edge-disjoint.

Let

$$MH(n) = \{m\colon \text{there exists a maximal set of } m \text{ hamiltonian cycles in } K_n\}.$$

Put

$$Dir(n) = \{\lfloor \frac{1}{4}(n+3) \rfloor, \lfloor \frac{1}{4}(n+3) \rfloor + 1, \dots, \lfloor \frac{1}{2}(n-1) \rfloor\}.$$

It follows directly from Dirac's theorem and a result of Nash-Williams (cf. [70]) that $MH(n) \subseteq Dir(n)$. One would like to show that, in fact, equality takes place here. To achieve this, consider the following.

Let n be even, $n = 2k$, and let m be a positive integer, $2m \leq k$. Let G be a regular graph of degree $2k - 4m$ with $2k - 2m$ vertices, and let $H = \bar{K}_{2m} \nabla\, G$. Similarly, let n be odd, $n = 2k + 1$, m be a positive integer, $2m + 1 \leq k$, and let G be a regular graph of degree $2k - 4m - 1$ with $2k - 2m$ vertices, and let $H = \bar{K}_{2m+1} \nabla\, G$ (here ∇ denotes the join, cf. [70]).

In order to show that $MH(n) = Dir(n)$, it clearly suffices to show that the graph H, with G suitably chosen, has a hamiltonian decomposition. Indeed, the complement $\bar{H}$ of H is disconnected, and so the set of hamiltonian cycles in any hamiltonian decomposition is maximal. The corresponding proof that H has a hamiltonian decomposition for G suitably chosen is given in [42].

The above provides another example of a problem with completely determined spectrum.

4 Maximal sets of Δ-factors

The figures are Δ-factors of K_n, that is, 2-factors whose each component is a triangle (sometimes also called triangle-factors); two Δ-factors are compatible when they are edge-disjoint. Clearly, here we must have $n \equiv 0 \pmod 3$.

Let $\Delta(n) = \{m\colon$ there exists a maximal set of Δ-factors of $K_n\}$.

A classical result of Corrádi and Hajnal [12] states that a graph with $n = 3k$ vertices and minimum degree at least $2k$ has a Δ-factor. Thus a maximal set of Δ-factors on $3k$ vertices must contain at least $\frac{k}{2}$ triangle-factors. This implies

$$\Delta(n) \subseteq \{\lceil \frac{n}{6} \rceil, \lceil \frac{n}{6} \rceil + 1, \dots, \lfloor \frac{n-1}{2} \rfloor\}.$$

It is easily seen that $\Delta(3) = \Delta(6) = \{1\}$, $\Delta(9) = \{4\}$.

For every odd k, there is a maximal (in fact, maximum) set of $\frac{3k-1}{2}$ Δ-factors in K_{3k}. For every even $k \geq 6$, there is maximal set of $\frac{3k-2}{2}$ Δ-factors in K_{3k}. This just restates the fact that for every $n \equiv 3 \pmod 6$ there exists a Kirkman triple system $KTS(n)$ of order n, and for every $n \equiv 0 \pmod 6$, $n \geq 18$, there exists a nearly Kirkman system $NKTS(n)$ of order n [13].

Furthermore, it is not difficult to establish that $2 \notin \Delta(12)$, while $5 \notin \Delta(12)$ folows from the nonexistence of a nearly Kirkman triple system of order 12. Thus $\Delta(12) = \{3, 4\}$.

On the other hand, it is not easy to establish that $\Delta(15) = \{4, 5, 6, 7\}$ (see [30]). More precisely, it is difficult to show $3 \notin \Delta(15)$; no computer-free proof of this fact is known to us. (In [30], *all* maximal sets of Δ-factors in K_{15} are enumerated.)

It is proved in [60] that $\Delta(18) = \{4, 5, 6, 7, 8\}$ (this involved showing $3 \notin \Delta(18)$), $\Delta(21) = \{4, 5, 6, 7, 8, 9, 10\}$, and $\Delta(24) = \{4, 5, 6, 7, 8, 9, 10, 11\}$. It is also shown that $\{6, 7, 8, 9, 10, 11, 12, 13\} \subseteq \Delta(27)$ but whether or not $5 \in \Delta(27)$ remains undecided. Similarly, $[6, 14] \subseteq \Delta(30)$ but whether $5 \in \Delta(30)$ is undecided.

It was conjectured in [60] that the spectrum $\Delta(n)$ contains the interval $[\lceil\frac{n}{6}\rceil, \frac{n-1}{2}]$, and proved that $[\lceil\frac{n}{6}\rceil, \lceil\frac{n}{4}\rceil] \in \Delta(n)$. Several further constructions for maximal sets of Δ-factors are given in [60] but especially for k in the interval $[\frac{n}{4}, \frac{n}{3}]$, new ideas appear to be needed. Also, for $k = \lceil\frac{n}{6}\rceil$ when $n \equiv 0, 9$ or $12 \pmod{18}$, not a single maximal set of k Δ-factors is known to exist (and $\lceil\frac{n}{6}\rceil \notin \Delta(n)$ for $n = 9, 12, 18$). So, e.g., whether or not $6 \in \Delta(33)$ remains an open problem.

5 Maximal partial latin squares and latin cubes

The figures are elements of $N \times N \times N$, i.e. ordered triples from a set N of n elements; two such triples are compatible if they agree in at most one coordinate. We can take $N = \{1, 2, \ldots, n\}$.

It is somewhat more convenient to think of a partial latin square as an $n \times n$ array whose cells are either empty or contain an element of N such that no element occurs in a cell of any row or column more than once. A partial latin square is then maximal if no further nonempty cell can be filled without violating this condition.

Let $ML(n)$ be the spectrum of maximal partial latin squares of order n, that is,

$$ML(n) = \{m\colon \text{there exists a maximal partial latin square of order } n \text{ with exactly } m \text{ nonempty cells}\}.$$

The set $ML(n)$ was investigated in [44]. Clearly, if $t < \frac{n^2}{2}$ or if $t = n^2 - 1$ then $t \notin ML(n)$. It is shown in [44] that when either

(i) $t = \frac{n^2}{2} + k, 1 \le k \le \frac{n}{2}$ where k is odd and n is even, or

(ii) $t = \lceil\frac{n^2}{2}\rceil + k, 1 \le k \le \frac{n-1}{2}$ where k is odd and n is odd,

we also have $t \notin ML(n)$.

It was also shown in [44] that the spectrum $ML(n)$ contains all integers t in the interval $[\frac{n^2}{2}, n^2 - 2]$ except possibly when (1) $t = \frac{n^2}{2} + k$, n even, k odd, $\frac{n}{2} < k \le n - 1$, or when (2) $t = \frac{n^2+1}{2} + k$, n odd, k odd, $\frac{n-1}{2} \le k \le n - 1$. It is conjectured in [44] that these possible exceptions are in fact true exceptions.

Recently, in [5] an analogous question was studied for partial latin cubes. Here the figures are elements of $N \times N \times N \times N$, i.e. ordered quadruples from a set N of n elements. Two such quadruples are compatible if they agree in at most two coordinates. One can picture a partial latin cube as a set of layers where each layer is a partial latin square, and no element occurs in the same row or column of distinct layers. A partial latin cube is then maximal if no further cell can be filled without violating this.

Let $ML^{(3)}(n) = \{m\colon$ there exists a maximal partial latin cube of order n with exactly m nonempty cells$\}$.

Neither $n^3 - 1$ nor $n^3 - 2$ can belong to $ML^{(3)}(n)$. In [5] it is shown that, unlike for maximal partial latin squares, there exist maximal partial latin cubes with substantially less than half of its n^3 cells filled. In fact, while any maximal partial latin cube must contain at least $t > (1 - \frac{1}{\sqrt{2}}n^3 > 0.29289n^3$ nonempty cells, there exist maximal partial latin cubes with $\frac{n^3}{3} + O(n^2)$ nonempty cells. For instance, when $n \equiv 1 \pmod 3$, there exists a maximal partial latin cube with at most $\frac{n^3+9n^2-6n-4}{3}$ nonempty cells.

A large portion of spectrum is determined in [5]: when n is even, $n \ge 10$ then $[\frac{n^3}{2}, n^3 - 3] \subseteq ML^{(3)}(n)$, and when n is odd, $n \ge 21$ then $[\frac{n^3+n}{2}, n^3 - 3] \subseteq ML^{(3)}(n)$. But for less than "half-full" maximal partial latin cubes, gaps remain (cf. also [55]).

In the same paper [5], the spectra $ML^{(3)}(n)$ for $n = 2, 3, 4$ are determined almost completely, with only three values in the case of $n = 4$ remaining in doubt. In particular, it is shown that $ML^{(3)}(2) = \{4, 5, 8\}$,
$ML^{(3)}(3) = \{9, 12, 14, 15, \ldots, 24, 27\}$, $ML^{(3)}(4) = \{31, 32, \ldots, 61, 64\} \cup S$ where $S \subseteq \{28, 29, 30\}$.

6 Row-maximal latin rectangles and maximal latin parallelepipeds

Here the figures are permutations of n symbols, say $1, 2, \ldots, n$; two such permutations are compatible if they are discordant, i.e. do not agree in any position.

In 1945, M. Hall Jr. proved [41] that if $r < n$ then any $r \times n$ latin rectangle can be extended to a $(r + 1) \times n$ latin rectangle. His proof is a nice application of Philip Hall's Theorem on systems of distinct representatives.

It follows that the spectrum of row-maximal $r \times n$ latin rectangles

$$MLR(n) = \{r\colon \text{there exists a row-maximal } r \times n \text{ latin rectangle}\}$$

consists of a single element, namely n.

The situation changes dramatically as one tries to extend M. Hall's result to three dimensions. Now the figures are $(n \times n)$ latin squares; they are compatible if they are disjoint. A latin $(n \times n \times r)$-parallelepiped is maximal if it cannot be extended to a latin $(n \times n \times (r + 1))$-latin parallelepiped. Let $MLC(n) = \{r\colon$ there exists a maximal $n \times n \times r$-latin parallelepiped$\}$.

Horák [43] was the first to show that for all $n = 2^k$, there exist infinitely many Latin $(n \times n \times (n - 2))$-parallelepipeds that cannot be completed to a Latin cube of order n and are therefore maximal. In [31], [50] further results on maximal $(n \times n\ (n - 2))$-latin parallelepipeds were obtained. (Clearly, any latin $(n \times n\ (n-1))$-latin parallelepiped can be extended to a latin cube of order n.)

Subsequently Kochol [51], [49], [52] proved that for any r, n such that $\frac{n}{2} < r \leq n - 2$ there exists a *noncompletable* $n \times n\ r$ latin parallelepiped. In [8] both noncompletable and nonextendible (that is, maximal) latin parallelepipeds are investigated. A maximal $5 \times 5 \times 2$ and a $6 \times 6 \times 3$ latin parallelepiped is produced, and a construction is given showing that for all even $m > 2$, there exists a maximal $(2m - 1) \times (2m - 1) \times (m - 1)$-latin parallelepiped. In particular, that shows the existence of a maximal $7 \times 7 \times 3$-latin parallelepiped.

The above are first examples of maximal latin parallelepipeds that are less than "half-full". But clearly, lots of work remains towards determining the spectrum $MLC(n)$.

7 Row-maximal orthogonal latin rectangles

The figures are pairs of permutations of degree n. Two pairs (P_1, P_1') and (P_2, P_2') are compatible if (P_1, P_2) and (P_1', P_2') are both discordant, and the two $2 \times n$ latin rectangles $\binom{P_1}{P_2}$ and $\binom{P_1'}{P_2'}$ are orthogonal.

Let $MOR(r, n)$ be a pair of row-maximal orthogonal latin $r \times n$ rectangles. Let the spectrum for row-maximal orthogonal latin ($r \times n$ rectangles be $MOR(n) = \{r\colon$ there exists a $MOR(r, n)\}$.

For small values of n, we have

$$MOR(1) = MOR(2) = \{1\}, MOR(3) = \{3\}, MOR(4) = \{3, 4\},$$
$$MOR(5) = \{3, 5\}, MOR(6) = \{3, 4, 5\}, MOR(7) = \{3, 4, 5, 6, 7\},$$
$$MOR(8) = \{3, 4, 5, 6, 7, 8\}.$$

Several partial results are obtained in [45] towards settling the following conjecture.

Conjecture. For $n \geq 7, MOR(n) = \{r : \frac{n}{3} < r \leq n\}$.

In particular, it is shown in [45] that $MOR(r,n)$ exists if $n \geq 7$ and

(i) $\frac{n}{3} < r \leq \frac{n}{2}$, except possibly when $(r,n) = (6,12)$

(ii) $\frac{7}{9}n \leq r \leq n$

(iii) $\frac{2n-1}{3} \leq r \leq n-1, r$ odd

(iv) $\frac{3}{7}n < r < \frac{3}{4}n,\ r \equiv 3 \pmod 6,\ n \equiv 1 \pmod 2$

(v) $\frac{3}{5}n < r \leq \frac{3}{4}n,\ r \equiv 3 \pmod 6,\ n \equiv 0 \pmod 2$

(vi) $\frac{n}{2} \leq r \leq \frac{3n+2}{4},\ r \equiv 0 \pmod 2,\ n \equiv 0 \pmod 2$.

On the other hand, there exist no $MOR(r,n)$ for $r \leq \frac{n}{4}$.

Several recursive constructions for row-maximal orthogonal latin rectangles are given in [45]. These, together with the above results, suffice to show, for instance, that $\{r : 11 \leq r \leq 30\} \subseteq MOR(30)$ where the set on the left conincides with the conjectured spectrum. But in general, quite a few undecided cases remain.

8 Maximal sets of mutually orthogonal Latin squares

This is an extremely important topic, because of its connections to the existence of finite projective planes.

The figures here are latin squares of order n on N; two latin squares are compatible if they are orthogonal. Recall that two latin squares $A = (a_{ij}), B = (b_{ij})$ are orthogonal if $|\{(a_{ij}, b_{ij}) : i,j = 1,\ldots,n\}| = n^2$, that is, when A and B are superimposed, each ordered pairs (a,b) with $a \in A$, $b \in B$ will appear exactly once.

Let $L(n)$ be the spectrum of sizes of maximal sets of mutually (pairwise) orthogonal latin squares (MOLS) of order n, i.e.

$$L(n) = \{r\colon \text{there exists a maximal set of } r \text{ MOLS of order} n\}.$$

The maximum number of MOLS of order n cannot exceed $n-1$, and equals $n-1$ whenever n is a prime power. Thus $L(n) \subseteq \{1,2,\ldots,n-1\}$. To determine $L(n)$ in its entirety would involve, among other things, to settle the existence question for finite projective planes of order n. Worse yet, even *max* $L(n)$ remains undetermined for all values of n other than prime powers or $n = 6$. Nevertheless, any progress towards determining $L(n)$ is very desirable.

A latin square without an orthogonal mate is called a *bachelor square*. It has been now determined that bachelor squares exist for all $n > 3$ [29], [69]. Thus $1 \in L(n)$ for all $n > 3$. A latin square which has an orthogonal mate but is not contained in any set of three mutually orthogonal squares is called *monogamous* (cf. [17]). A monogamous latin square is known to exist for all orders $n > 6$ except possibly when $n = 2p$ for some prime $p \geq 7$. Thus $2 \in L(n)$ for all $n > 6$ except possibly when $n = 2p$ for some prime $p \geq 7$.

To determine the set $L(n)$ even for relatively small values of n is not an easy task. For example, whether or not $4 \in L(8)$ had been an open question for good forty years before it was recently settled [24].

The set $L(n)$ has now been determined for all $n \leq 9$. For $n \leq 7$ this has been done by Drake [22]; the last two outstanding values for $n = 8$ and $n = 9$ have been settled in [24]. In particular, we have

$$L(3)=\{2\}, L(4)=\{1,3\}, L(5)=\{1,4\}, L(6)=\{1\}, L(7)=\{1,2,6\},$$
$$L(8)=\{1,2,3,7\}, L(9)=\{1,2,3,4,5,8\}.$$

We have further $\{1,2\} \subseteq L(10)$, $\{1,2,3,4,10\} \subseteq L(11)$, $\{1,2,3,5\} \subseteq L(12)$, $\{1,2,3,4,6,12\} \subseteq L(13)$,$\{1,2,3,4,7,8,11,15\} \subseteq L(16)$ (cf. [1]), [3]).

As for general results, a theorem of Bruck [7] implies that for $n > 4$, $n-2 \notin L(n)$, $n-3 \notin L(n)$. If $n = q_1.q_2. \ldots . q_r$ is the prime power factorization of n then $min\ (q_i - 1) \in L(n)$.

If $p \geq 7$ is a prime, $p \equiv 3 \pmod 4$, then $\frac{p-3}{2} \in L(p)$. If $p \geq 13$ is a prime, $p \equiv 1 \pmod 4$, then $\frac{p-1}{2} \in L(p)$. If q is a prime power then $q^2 - q - 1 \in L(q^2)$, and if $q = p^r$, $p \geq 5$, then $q^2 - q - 2,\ q^2 - q \in L(q^2)$ (cf. [1]). For many additional results on maximal sets of MOLS, see [1], [3], [25], [27], [28], [38], [47], [48] and references therein.

9 Maximal partial Steiner triple systems

The figures are 3-subsets (triples) of a given v-set; two triples are compatible if they intersect in at most one element.

Alternatively, the figures are triangles in the complete graph K_v; two triangles are compatible if they are edge-disjoint.

Let $S^{(3)}(v)$ be the spectrum for maximal partial Steiner triple systems (STS), i.e.

$$S^{(3)}(v) = \{m\colon \text{there exists a maximal partial STS of order } v \text{ with exactly } m \text{ triples}\}.$$

The largest element $M^{(3)}(v)$ of $S^{(3)}(v)$ was determined already in the 1840's by Kirkman [?] (and since then repeatedly by many others) :

$$\begin{aligned} M^{(3)}(v) &= v(v-1)/6 & v &\equiv 1,3 \pmod 6 \\ &= [v(v-1)-8]/6 & v &\equiv 5 \pmod 6 \\ &= v(v-2)/6 & v &\equiv 0,2 \pmod 6 \\ &= [v(v-2)-2]/6 & v &\equiv 4 \pmod 6. \end{aligned}$$

The smallest element $m^{(3)}(v)$ of $S^{(3)}(v)$ was determined in 1974 by Novák [57]: $m^{(3)}(v) = (v^2 + \delta_v)/12$ where

$$\begin{aligned} \delta_v &= -2v+36 & v &\equiv 0,8 \pmod{12} \\ &= -1 & v &\equiv 1,5 \pmod{12} \\ &= -2v & v &\equiv 2,6 \pmod{12} \\ &= 3 & v &\equiv 3 \pmod{12} \\ &= -2v+4 & v &\equiv 4 \pmod{12} \\ &= 11 & v &\equiv 7,11 \pmod{12} \\ &= 15 & v &\equiv 9 \pmod{12} \\ &= -2v+16 & v &\equiv 10 \pmod{12} \end{aligned}$$

The spectrum $S^{(3)}(v)$ for odd v was determined completely by Severn [65].

Let $R(v)$ be the interval $\{m^{(3)}(v), M^{(3)}v)\}$. It was shown in [65] that

$$\begin{aligned} S^{(3)}(v) &= R(v) \setminus \{M^{(3)} - 1\} & &\text{if } v \equiv 1,3 \pmod 6 \\ &= R(v) & &\text{if } v \equiv 5 \pmod 6. \end{aligned}$$

For even v, the situation is slightly more complicated. The spectrum $S^{(3)}(v)$ in this case has been determined "almost completely" by [65] who left only a few open cases.

These have been settled in [14] so that the spectrum $S^{(3)}(v)$ has now been completely determined:

For v even, $S^{(3)}(v) = R(v) \setminus Q(v)$, where

$$Q(v) = \{r \colon m^{(3)}(v) < s < Y(v) \text{ and } s - \frac{v-1}{2} \equiv 1 \pmod 2),$$

and

$$\begin{aligned} Y(v) &= 12k^2 + 2k && \text{if } v = 12k \\ &= 12k^2 + 6k + 1 && \text{if } v = 12k+2 \\ &= 12k^2 + 10k + 2 && \text{if } v = 12k+4 \\ &= 12k^2 + 14k + 5 && \text{if } v = 12k+6 \\ &= 12k^2 + 18k + 6 && \text{if } v = 12k+8 \\ &= 12k^2 + 22k + 11 && \text{if } v = 12k+10 \end{aligned}$$

(cf. [14], [13]).

10 Maximal partial 4-cycle systems

The figures here are quadrangles (cycles with 4 edges, 4-cycles); two quadrangles are compatible when they are edge-disjoint.

Let $S^{(4)}(n) = \{r \colon$ there exists a maximal set of quadrangles with exactly r quadrangles$\}$. Let the smallest and largest element of $S^{(4)}(n)$ be $m^{(4)}(n)$ and $M^{(4)}(n)$, respectively.

The numbers $M^{(4)}(n)$ have been determined completely in [64] (cf. also [32]). For odd n, when $n \equiv 1 \pmod 8$, there exists a 4-cycle system of order n, thus $M^{(4)}(n) = \frac{n(n-1}{8}$, the number of 4-cycles in such a system. For $n \equiv 3, 5, 7 \pmod 8$, there exists a maximum packing of 4-cycles whose leave is a triangle, a 2-regular graph with 6 edges (and thus either a 6-cycle, or two vertex-disjoint triangles, or a "bowtie"), or a pentagon, respectively [64]. Thus we have

$$\begin{aligned} M^{(4)}(n) &= \lfloor \tfrac{n(n-1)}{8} \rfloor && \text{if } n \equiv 1 \text{ or } 3 \pmod 8 \\ &= \lfloor \tfrac{n(n-1)}{8} \rfloor - 1 && \text{if } n \equiv 5 \text{ or } 7 \pmod 8. \end{aligned}$$

In order to determine the spectrum $S^{(4)}(n)$, it is necessary to know the values of $m^{(4)}(n)$ but therein lies the difficulty: to determine the maximum number of edges in an n-vertex graph without 4-cycles is a difficult unsolved problem [33], [34], [37], [10], [71].

Nevertheless, the values $ex(n; C_4)$, the largest number of edges in a graph with n vertices without a 4-cycle, has been determined exactly for all $n \leq 31$ [37], [71] which makes it possible to determine $min^{(4)}(n)$, and also the whole spectrum $S^{(4)}(n)$ for certain small values of n. No exact formula for $ex(n; C_4)$ appears to be known although it is known that $ex(n; C_4) < \frac{1}{4}n(1 + \sqrt{4n-3})$ when $n > 3$, and asymptotically $ex(n; C_4) \simeq \frac{1}{2}n^{\frac{3}{2}}$.

[The value of $ex(n; C_4)$ has also been determined exactly for $n = q^2 + q + 1$ when q is either a power of 2 [33] or when q is a prime power greater than 13 [34]].

While knowing the maximum number of edges in an n-vertex graph is, in turn, a necessary step in determining $m^{(4)}(n)$, what is actually needed is the maximum number of edges in an n-vertex eulerian and antieulerian graph without 4-cycles, according as n is odd and even, respectively. This numbers are usually somewhat smaller than the former; for example, the maximum number of edges in a 9-vertex graph without 4-cycles is 13 [10], that in a 9-vertex *eulerian* graph is 12. Similarly, for example, the maximum number of edges in a 10-vertex graph without 4-cycles is 16 [10], that in an *antieulerian* graph is 13.

Clearly, $\frac{n(n-1}{8} - 1 \notin S^{(4)}(n)$ when $n \equiv 1 \pmod 8$.

The bounds given in [10] plus ad hoc considerations enable one to determine the spectrum $S^{(4)}(n)$ for small values of n. In particular, we have
$S^{(4)}(4) = S^{(4)}(5) = \{1\}, S^{(4)}(6) = \{3\}, S^{4)}(7) = \{3,4\}, S(4)(8) = \{5,6\}$
$S^{(4)}(9) = \{6,7,9\}, S^{(4)}(10) = \{8,9,10\}, S^{(4)}(11) = \{10,11,12,13\}$,
$S^{(4)}(12) = \{12,13,14,15\}$, $S^{(4)}(13) = \{15,16,17,18\} \cup I$ where $I = \emptyset$ or $I = \{14\}$. At this time, I am unable to decide whether $14 \in S^{(4)}(13)$ or not.

11 Maximal partial 5-cycle systems

The figures here are pentagons (cycles with 5 edges, 5-cycles); two pentagons are compatible if they are edge-disjoint.

Let

$$S^{(5)}(n) = \{r\colon \text{there exists a maximal set of pentagons with exactly } r \text{ pentagons}\}.$$

Let the smallest and largest element of $S^{(5)}(n)$ be $m^{(5)}(n)$ and $M^{(5)}(n)$, respectively.

The numbers $M^{(5)}(n)$ have been determined completely in [63]:

$$\begin{aligned} M^{(5)}(n) &= \lfloor \tfrac{e_n}{5} \rfloor && \text{if } n \not\equiv 7,9 \pmod{10} \\ &= \lfloor \tfrac{e_n}{5} \rfloor - 1 && \text{if } n \equiv 7,9 \pmod{10} \end{aligned}$$

where $e_n = \frac{n(n-1)}{2}$ or $\frac{n(n-2)}{2}$ according as n is odd or even.

To determine $m^{(5)}(n)$ turned out to be much more difficult. The first step in this was to obtain bounds on $m^{(5)}(n)$ by determining extremal graphs not containing a pentagon. While for $n \geq 7$ the maximal size of a graph with n vertices without a pentagon is $\lfloor \frac{n^2}{4} \rfloor$, for a nonbipartite graph the maximal size is $f(n) = \lfloor \frac{n^2}{4} \rfloor - n + 4$, a slight improvement [63]. Furthermore, a nonbipartite eulerian graph (all degrees even) without a pentagon with an odd number of vertices $n \geq 11$ has at most $\lfloor \frac{n^2}{4} \rfloor - n + 3$ edges. It follows that for a maximal size $E(n)$ of an eulerian graph without a pentagon we have

$$\begin{aligned} E(n) &= \tfrac{n^2}{4} && \text{if } n \equiv 0 \pmod 4 \\ &= \tfrac{(n-1)^2}{4} && \text{if } n \equiv 1 \pmod 4 \\ &= \tfrac{n^2-4}{4} && \text{if } n \equiv 2 \pmod 4 \\ &= \tfrac{n^2-2n-3}{4} && \text{if } n \equiv 3 \pmod 4. \end{aligned}$$

When $n \equiv 0 \pmod 4$ and $n \equiv 2 \pmod 4$, the extremal graph is $K_{\frac{n}{2},\frac{n}{2}}$ and $K_{\frac{n+1}{2},\frac{n-3}{2}}$, respectively. When $n \equiv 1 \pmod 4$ and $n \equiv 3 \pmod 4$, one of the extremal graphs is $K_{\frac{n-1}{2},\frac{n-1}{2}}$ and $K_{\frac{n+1}{2},\frac{n-3}{2}}$, respectively.

Similarly, let $A(n)$ be the maximal size of an antieulerian (all degrees odd) graph without a pentagon. Then

$$\begin{aligned} A(n) &= \tfrac{n^2-4}{4} && \text{if } n \equiv 0 \pmod 4 \\ &= \tfrac{n^2}{4} && \text{if } n \equiv 2 \pmod 4, \end{aligned}$$

and the extremal graphs are $K_{\frac{n+2}{2},\frac{n-2}{2}}$ and $K_{\frac{n}{2},\frac{n}{2}}$, respectively.

Let

$$\begin{aligned} \Delta_n &= \lceil \tfrac{n(n-1)}{2} - \tfrac{E(n)}{5} \rceil && \text{if } n \text{ is odd,} \\ &= \lceil \tfrac{n(n-1)}{2} - \tfrac{A(n)}{5} \rceil && \text{if } n \text{ is even.} \end{aligned}$$

It is shown in [63] that

$$\begin{array}{rcll} m^{(5)}(n) & \geq & \Delta_n & \text{if } n \geq 11 \\ & \geq & \Delta_n + 1 & \text{if } n \equiv 13, 14, 15, 16, 17, 18 \pmod{20} \\ & \geq & \Delta_n + 2 & \text{if } n \equiv 4, 8 \pmod{20},\ n \geq 24. \end{array}$$

Equality in the above for all $n \geq 11$ is then established by three special constructions (see [63]).
Clearly, the spectrum $S^{(5)}(n)$ is a subset of the interval $[m^{(5)}(n), M^{(5)}(n)]$. This spectrum has not been determined completely yet, except when $n \equiv 3 \pmod{40}$. In [15], the following conjecture on the shape of the spectrum $S^{(5)}(n)$ was formulated.

Conjecture. For any $n \geq 6$, there is a number z_n (for $n \geq 45$, $z_n - m^{(5)}(n) \geq \frac{n}{5} - 5$) such that

(i) if $t \in [m^{(5)}(n), z_n]$ then $t \in S^{(5)}(n)$ if and only if t has the same parity as $m^{(5)}(n)$;

(ii) if $t \in [z_n, M^{(5)}(n)]$ then $t \in S^{(5)}(n)$.

It is shown in [15] that (i) holds for all $n \geq 45$. This has required determining the maximum number of edges in a pentagon-free nonbipartite eulerian (antieulerian) graph.

The conjecture has been proved in full only for $n \equiv 3 \pmod{40}$ (see [15]). If $n = 40k + 3$, $k \geq 2$, then $m^{(5)}(40k + 3) = 80k^2 + 12k + 1$, $M^{(5)}(40k + 3) = 160k^2 + 20k$, $z_{40k+3} = m^{(5)}(40k + 3) + 8k - 1$, and
$S^{(5)}(40k+3) = \{80k^2+12k+1, 80k^2+12k+3, \ldots, 80k^2+20k-1, 80k^2+12k+1, 80k^2+20k+2, \ldots, 160k^2+20k\}$.

For $n \not\equiv 3 \pmod{40}$, part (ii) of the Conjecture remains open.

To determine the spectra in the following two sections appears quite difficult.

12 Maximal sets of disjoint Steiner triple systems

The figures are Steiner triple systems on a given v-set; they are compatible if they are disjoint, i.e. they have no triple in common. Here, of course, we must have $v \equiv 1$ or $3 \pmod 6$.

Let $DS(v) = \{m$: there exists a maximal set of m pairwise disjoint STS(v)s$\}$. It is well known that $DS(7) = \{2\}$, a result by Cayley that goes back to the middle of the 19th century. For $v > 7$, the largest element of $DS(v)$ was determined in [54], [68]: $max\ DS(v) = v - 2$.

The only other general results are:

(1) for $v \geq 7$, every Steiner triple system of order v has a disjoint mate, thus $1 \notin DS(v)$ [67],

(2) $v - 4 \in DS(v)$ for $v = 5.2^i - 1, i \geq 1$,

(3) $v - 5 \in DS(v)$ for $v = 2^{i+2} - 1, 5.2^i - 1, i \geq 1$ [12].

Cooper [9] determined $DS(9)$ (follows also from [12]): $DS(9) = \{4, 5, 7\}$. He also determined the isomorphism classes of all maximal sets of disjoint STS(9)s. At this point, for no other values of v has $DS(v)$ been determined.

13 Maximal sets of orthogonal Steiner triple systems

A property stronger than disjointness is the orthogonality property. Two Steiner triple systems $(V, \mathcal{B}_1), (V, \mathcal{B}_2)$ are *orthogonal* if they are disjoint, and, moreover, whenever $\{x, y, a\}, \{w, z, a\} \in \mathcal{B}_1$, $\{x, y, b\}, \{w, z, c\} \in \mathcal{B}_2$ then $b \neq c$. In other words, whenever two pairs of elements occur with the same third element in triples in one of the systems, they must occur with different third elements in the triples of the other system. Orthogonal STSs were originally introduced for the purpose of constructing Room squares.

For all $v \equiv 1, 3 \pmod 6$, $v \neq 9$, there exists a pair of orthogonal STS(v) [11]; there exists no such pair for $v = 9$. Moreover, it is shown in [18] that a set of three pairwise orthogonal STS(v) exists for all $v \equiv 1, 3 \pmod 6$, except when $v \leq 15$, and except possibly for 24 values of v, all of which are $\equiv 3 \pmod 6$, and smallest of which is $v = 21$. Many multiple sets of pairwise orthogonal STS(v) were constructed in [39] where it is shown, among other things, that for any positive integer t there exists a set of t pairwise orthogonal STS(v) provided $v \equiv 1 \pmod 6$ and v is sufficiently large. No maximality of these sets seems to have been investigated though, and while some of the sets constructed may indeed be maximal, it appears hard to either verify or disprove maximality.

Concerning *maximal* sets of orthogonal STS(v), let $OM(v) = \{r\colon$ there exists a maximal set of r orthogonal STS(v)$\}$. It is known that $OM(7) = \{2\}, OM(9) = \{1\}, OM(13) = \{1, 2\}, OM(15) = \{1, 2\}$ [36] but hardly anything else. It is believed that $max\ OM(v) \leq \frac{v-1}{2}$ but no nontrivial upper bound on $max\ OM(v)$ has been proved.

14 Row-maximal Room rectangles

Let N be a $2n$-set; a *Room* $(r, 2n)$*-rectangle* on N is an $r \times 2n - 1$ array $(r \leq 2n - 1)$ whose cells are either empty or contain a 2-subset of N. Each element of N occurs in exactly one cell of each row and in at most one cell of each column, and no 2-subset appears more than once in the array. A Room $(2n - 1, 2n)$-rectangle is called a *Room square* (of order $2n$, or of side $2n - 1$). A Room $(r, 2n)$ rectangle is row-maximal if no further row can be added to it to produce a Room $(r + 1, 2n)$-rectangle.

The figures are pairs (f, α) where f is a 1-factor of K_{2n} on a given $2n$-set N, and α is an injection from f into $\{1, 2, \ldots, 2n - 1\}$. Two figures $(f_1, \alpha_1), (f_2, \alpha_2)$ are compatible if $\alpha_1^{-1}(i) \cap \alpha_2^{-1}(i) = \emptyset$ whenever $\alpha_1^{-1}(i) \neq \emptyset$ and $\alpha_2^{-1}(i) \neq \emptyset$. Less formally, the figures are rows with $2n - 1$ cells of which $n - 1$ are empty such that the n nonempty cells contain a partition of N into 2-subsets; two such rows are compatible if no element occurs in any of the $2n - 1$ columns more than once.

Here we have the following result.

Let

$$MRR(2n) = \{r\colon \text{there exist a row-maximal Room } (r, 2n)\text{-rectangle}\}.$$

A row-maximal Room $(r, 2n)$-rectangle exists if

(i) $(r, 2n) = (1, 4)$

(ii) $n \leq r \leq 2n - 1$ except when $(r, 2n) \in \{(2, 4), (3, 4), (5, 6)\}$.

Indeed, (i) is trivial while (ii) follows from the fact that there exists no Room square of order 4 or 6 (i.e. there exists no Room $(3, 4)$-rectangle or Room $(5, 6)$-rectangle, and there exists no Howell design $H(2, 4)$ [19]. It remains to be observed that while a Howell design $H(5, 8)$ does not exist, either, a row-maximal Room $(5, 8)$-rectangle does:

12	34	56	78	–	–	–
–	–	–	13	24	57	68
67	–	14	–	58	23	–
–	15	–	26	–	48	37
38	–	27	–	16	–	45

(For a general reference on Room squares and Howell designs, see, e.g., [19]).

15 Packings of dominoes

A different kind of a problem on packing dominoes onto a square $n \times n$ board was considered in [40]. It is trivial to see that the maximum number $M^{(d)}(n)$ of dominoes (1×2 tiles) that may be packed onto an $n \times n$ board (without overlap) is $\frac{n^2}{2}$ if n is even, and $\frac{n^2}{2} - 1$ when n is odd. The authors of [40] were interested in the *minimum* number $m^{(d)}(n)$ of dominoes that can be placed on an $n \times n$ board in such a way that no further domino can be placed on it without an overlap. It is shown in [40] that $m^{(d)}(n) = \frac{n^2}{3}$ if $n \equiv 0 \pmod 3$, and $m^{(d)}(n) > \frac{n^2}{3} + \frac{n}{111}$ otherwise, provided n is large.

While $m^{(d)}(n) = \lfloor \frac{n^2+2}{3} \rfloor$ for $2 \leq n \leq 12$, the exact value of $m^{(d)}(n)$ for $n \not\equiv 0 \pmod 3$ is not known. The best upper bound known is $m^{(d)}(n) < \frac{n^2}{3} + \frac{n}{12} + 1$. In any case, in any maximal packing of dominoes roughly at least two thirds of the cells must be covered.

Let $S^{(d)}(n) = \{r$: there exists a maximal packing of the $n \times n$ board with exactly m dominoes$\}$. The constructions given in [40] allow one to deduce that

$$S^{(d)}(n) = \left\{ \frac{n^2}{3}, \frac{n^2}{3} + 1, \ldots, \lfloor \frac{n^2}{2} \rfloor \right\} \text{ when } n \equiv 0 \pmod 3, \text{ and}$$

$$\left\{ \frac{n^2}{3} + \frac{n}{12} + 1, \frac{n^2}{3} + \frac{n}{12} + 2, \ldots, \lfloor \frac{n^2}{2} \rfloor \right\} \subseteq S^{(d)}(n) \text{ for } n \not\equiv 0 \pmod 3.$$

The case of maximal packing of "dominoes" on triangular and hexa(gonal) boards is also considered in [40]. For example, a "domino" for a hexaboard is a pair of two neighbouring hexagonal cells. Hexa board itself has a triangular shape and consists of n rows containing a total of $\binom{n}{2}$ hexagonal cells.

While clearly one can cover the entire n-hexaboard (a board with n rows) by $\lfloor \frac{n(n+1)}{4} \rfloor$ dominoes (except for one hexagonal cell when $\binom{n}{2}$ is odd), in this case one is also able to determine the minimum number of dominoes in a maximal packing; this minimum equals $\lfloor \frac{n(n+1)}{6} \rfloor$. Thus this case turns out to be easier than that of the regular $n \times n$ board (cf. [40]).

Conclusion and some open problems

This survey cannot, and does not attempt to, encompass all situations where the spectrum problem for maximal designs and configurations arises - this would anyway be virtually impossible. There are many further examples of problems of the kind similar to those explored above. To name just a few further examples of problems that have been studied to various degrees of depth in the literature, maximal sets of orthogonal hamiltonian cycles [46], maximal sets of orthogonal hamiltonian decompositions [46], maximal sets of disjoint 1-factorizations [2], [9] maximal sets of orthogonal 1-factorizations (or, equivalently, dimension-maximal Room cubes) [2], maximal k-cliques [21], [23], [26], [35], maximal partial projective planes [20], and several others come to mind.

A concept closely related to maximal configurations is that of *premature* configurations (see, e.g., [4], [62]). While maximal configurations are not extendible, premature configurations are not completable (to maximum configurations) but may themselves not be maximal. Although in the context of some of the problems discussed above, premature configurations have been explored in the literature, for example, premature sets of 1-factors (cf. Section 1), premature sets of latin parallelepipeds (cf. Section 6), or premature sets of MOLS (cf. Section 8), we refrained in this article from discussing premature configurations in more detail.

The spectrum problems treated in Sections 1, 2, 3, 9 and 14 have been solved completely. In the remaining sections, many open problems remain. Some open problems that I would like to see solved, or at least seriously attacked, are:

(i) Maximal partial Steiner systems $\mathrm{S}(2,4,v)$.
Here the figures are 4-subsets of a given v-set; two 4-subsets are compatible if they intersect in at most one element. Let $S^4(v)$ be the spectrum for maximal partial Steiner systems $\mathrm{S}(2,4,v)$, and let $m^4(v)$ and $M^4(v)$ be the smallest and largest element of $S^4(v)$, respectively. The numbers $M^4(v)$ have been determined by Brouwer [6] (cf. also [66]. The numbers $m^4(v)$ and the spectrum $S^4(v)$ have not been determined yet.

(ii) Maximal partial Room squares.
Here I am not aware of any results in this direction.

It is hoped that by bringing together the most up-to-date results on these and potentially many other similar or related problems a renewed interest will be generated.

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