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A note on non-unital homomorphisms on C^*-convex sets in $*$-rings

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Abstract

Recently, Ebrahimi et al. [C^*-convexity and C^*-faces in $*$-rings, *Turk. J. Math.* **36** (2012), 131–145] identified the optimal points of continuous unital homomorphisms on some C^*-convex sets of a topological $*$-ring. In this short note, we generalize their results for continuous (non-unital) homomorphisms in a topological $*$-ring. Moreover, for continuous unital homomorphisms, we point out the same conclusion does not hold everywhere that a Krein-Milman type theorem exists. An important issue is so that we do not assume that homomorphisms are unital.

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1 Introduction

The term non-commutative convexity refers to any one of the various forms of convexity in which convex coefficients need not commute among themselves. Formal study of C^*-convexity as a form of non-commutative convexity, was initiated by Loebl and Paulsen in [3], where the notion of C^*-extreme point, as a non-commutative analog of extreme point was also studied.

Recently, Ebrahimi et al. [2] motivated the following general question. Operator algebras are equipped with rich algebraic, geometric and topological structures such that one naturally asks: which of these structures have made a particular theorem work. In the algebraic direction this question has led to evolution of the algebraic theory of operator algebras. Indeed, they defined the notions of C^*-convexity and C^*-extreme point and discussed some illustrative examples of C^*-convex subsets of $*$-rings. For example, they showed that the set $\{x\}$ is C^*-convex, when $x \in Z(R)$, $Z(R)$ is the center of R, and R is a unital $*$-ring (see Example 2.2 of [2]). In this situation, when $x \notin Z(R)$ the set $\{x\}$ is not C^*-convex and the C^*-convex hull of $\{x\}$ is the smallest C^*-convex set containing $\{x\}$ and is denoted by C^*-Co($\{$x$\}$). They investigated some properties of C^*-convex sets and C^*-extreme points and identified some C^*-convex subsets of $*$-rings by applying C^*-convex maps. Moreover, they identified optimal points of some unital homomorphisms on some C^*-convex sets.

In this note, we generalize the theorem appearing as Theorem 4.8 in [2] for continuous (non-unital) homomorphism in a topological $*$-ring. We know that many algebras have no characters, for instance M_n for $n \geq 1$, $\mathcal{B}(H)$, etc. (see for example Exercise IV.1 of [1]) and we also know that the set of real–valued unital homomorphisms on an algebra is a subset of the set of its characters. So, we notice that Corollary 4.9 of [2] discusses on the maximum and minimum of an empty set of functions on M_n .

2 Main results

Throughout this note $\mathcal{R}$ is a unital $*$-ring, that is, a ring with an involution which has an identity element.

Definition 1. A subset K of $\mathcal{R}$ is called C^*-convex if $\sum_{i=1}^n a_i^* x_i a_i \in K$, whenever $x_i \in K$, $a_i \in \mathcal{R}$ and $\sum_{i=1}^n a_i^* a_i = 1_{\mathcal{R}}$.

For example, the positive cone in $\mathcal{R}$ is C^*-convex and in general the segment $[0, a]$ for $a \in \mathcal{R}$ is not C^*-convex, see Example 2.11 of [2].

Definition 2. If K is a C^*-convex subset of $\mathcal{R}$, then $x \in K$ is called a C^*-extreme point for K if the condition

$$x = \sum_{i=1}^n a_i^* x_i a_i, \quad \sum_{i=1}^n a_i^* a_i = 1_{\mathcal{R}}, \; x_i \in K, n \in \mathbb{N},$$

where a_i is invertible in $\mathcal{R}$ implies that all x_i are unitarily equivalent to x in $\mathcal{R}$, that is, there exist unitaries $u_i \in \mathcal{R}$ such that $x_i = u_i^* x u_i$ for all i. The set of all C^*-extreme point of K is denoted by C^*-ext(K).

Recall that a homomorphism $f : \mathcal{R} \to \mathbb{R}$ is unital if $f(1_{\mathcal{R}}) = 1$. The following theorem appears as Theorem 4.8 in [2] for a C^*-convex subset K of $\mathcal{R}$.

Theorem 3. *Suppose $\mathcal{R}$ is a topological $*$-ring, C^*-ext(K) is closed and S is a compact subset of $\overline{C^*}$-Co$(C^*$-ext$(K))$ containing C^*-ext(K). Then every continuous unital homomorphism $f : \mathcal{R} \to \mathbb{R}$ attains its maximum and minimum on S at C^*-extreme points of K. Moreover, maximum and minimum of f on S is equal with its maximum and minimum on C^*-ext(K), respectively.*

As a consequence of this theorem together with the generalized Krein-Milman theorem (Theorem 4.5 of [4]) the following corollary presented as Corollary 4.9 in [2].

Corollary 4. *If $S \subseteq M_n$ is compact, C^*-convex, and the set of all C^*-extreme points of S is closed, then every continuous unital homomorphism $f : M_n \to \mathbb{R}$, attains its maximum and minimum on S at C^*-extreme points of S.*

Remark 5. In the above mentioned theorem, if $\mathcal{R}$ is a simple $*$-ring and $f : \mathcal{R} \to \mathbb{R}$ is a continuous unital homomorphism, then the null space of f is an ideal of $\mathcal{R}$. On the other hand, $\mathcal{R}$ is simple and so it has no nontrivial ideals. Thus, either f is zero or $\mathcal{R}$ is isomorphic to the real numbers. Recall that f is unital, so f is not zero and then it remains that $\mathcal{R}$ is isomorphic to the real numbers, which is of course a trivial case. Since in this case all continuous homomorphisms $f : \mathbb{R} \to \mathbb{R}$ are of the form $f(x) = cx$, where $c \in \mathbb{R}$ is a constant. Moreover, C^*-convex hull and C^*-extreme points of a set in $\mathbb{R}$ are identical with its convex hull and extreme points in the usual sense, respectively. We can observe that the closed convex hull of a subset of $\mathbb{R}$ is a closed interval and the extreme points of a closed interval are its initial and end points. So, for instance, if $c > 0$, then $f(x) = cx$ is increasing and obviously attains its minimum and maximum at the initial and end points of the interval, respectively.

Remark 6. Note that the set of all real–valued unital homomorphisms on M_n are a subset of characters of M_n and note that M_n has no characters (cf. Exercise IV.1 of [1]) and so the above mentioned corollary discusses on the maximum and minimum of an empty set of characters on M_n .

We would remark that for compact C^*-convex subsets of M_n a Krein–Milman type theorem was established by Morenz (Theorem 4.5 of [4]). The authors in [2] claim that the same conclusion (Corollary 4) holds everywhere that a Krein-Milman type theorem exists. For example in the generalized state space of a C^*-algebra with bounded–weak topology such a conclusion holds. However, Remark 6 ensures we can not claim that the same conclusion holds everywhere that a Krein-Milman type theorem exists.

Taking ideas from Remarks 5 and 6, we are going to consider continuous (non-unital) homomorphism on C^*-convex sets. We now state and prove an extended version of Theorem 1.1 to non-unital maps and we show the assumption that f is unital would be dropped. That is, we prove the following theorem.

Theorem 7. *Suppose $\mathcal{R}$ is a topological $*$-ring, C^*-ext(K) is closed and S is a compact subset of $\overline{C^*\text{-Co}(C^*\text{-ext}(K))}$ containing C^*-ext(K). Then every continuous homomorphism $f : \mathcal{R} \to \mathbb{R}$ attains its maximum and minimum on S at C^*-extreme points of K. Moreover, maximum and minimum of f on S is equal with its maximum and minimum on C^*-ext(K), respectively.*

Proof. Suppose that f attains its maximum on S at a point $x \in S$. Then, there exists a net $\{x_\lambda\}$ in C^*-Co$(C^*$-ext$(K))$ such that x_λ converges to x. For every λ, x_λ is C^*-convex combination of points of C^*-ext(K), i.e., $x_\lambda = \sum_{i=1}^{n(\lambda)} a^*_{\lambda,i} x_{\lambda,i} a_{\lambda,i}$, where $n(\lambda)$ is a positive integer, $x_{\lambda,i} \in C^*$-ext(K), and $\sum_{i=1}^{n(\lambda)} a^*_{\lambda,i} a_{\lambda,i} = 1_{\mathcal{R}}$. Define $f(x_{\lambda,i_\lambda}) := \max_{1 \le i \le n(\lambda)} f(x_{\lambda,i})$. Then,

$$\begin{aligned}
f(x_\lambda) &= f(\sum_{i=1}^{n(\lambda)} a^*_{\lambda,i} x_{\lambda,i} a_{\lambda,i}) = \sum_{i=1}^{n(\lambda)} f(a^*_{\lambda,i} x_{\lambda,i} a_{\lambda,i}) \\
&= \sum_{i=1}^{n(\lambda)} f(a^*_{\lambda,i}) f(x_{\lambda,i}) f(a_{\lambda,i}) \le \max_{1 \le i \le n(\lambda)} f(x_{\lambda,i}) \sum_{i=1}^{n(\lambda)} f(a^*_{\lambda,i}) f(a_{\lambda,i}) \\
&= f(x_{\lambda,i_\lambda}) f(\sum_{i=1}^{n(\lambda)} a^*_{\lambda,i} a_{\lambda,i}) \\
&= f(x_{\lambda,i_\lambda}) f(1_{\mathcal{R}}) = f(x_{\lambda,i_\lambda}.1_{\mathcal{R}}) \\
&= f(x_{\lambda,i_\lambda}). \qquad (2.1)
\end{aligned}$$

Since f is multiplicative, in the forth line of (2.1) we use the equality $f(x_{\lambda,i_\lambda}) f(1_{\mathcal{R}}) = f(x_{\lambda,i_\lambda}.1_{\mathcal{R}})$. This shows the assumption that f is unital can be dropped in Theorem 4.8 of [2]. The rest of the proof is similar to that of Theorem 4.8 of [2]. □

However, by removing the assumption that f is unital, we can state a correct version of Corollary 4.9 in [2] as follows:

Corollary 8. *If $S \subseteq M_n$ is compact, C^*-convex, and the set of all C^*-extreme points of S is closed, then every continuous homomorphism $f : M_n \to \mathbb{R}$ attains its maximum and minimum on S at C^*-extreme points of S.*

We recall that every real–valued homomorphism on M_n is zero and it is clear that its maximum and minimum value are zero. Indeed, in this case the above corollary is an obvious corollary. However, unlike Corollary 4, this corollary does not discuss on an empty set of functions on M_n and the discussed set contains at least the zero function.

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