Acta Universitatis Matthiae Belii, series Mathematics
Issue 2017, 1–9, ISSN 1338-7111
Online Edition, http://actamath.savbb.sk

Some bounds on the maximum induced matching numbers of certain grids

Deborah Olayide Ajayi
Department of Mathematics, University of Ibadan, Ibadan, Nigeria
olayide.ajayi@mail.ui.edu.ng; adelaideajayi@yahoo.com

Tayo Charles Adefokun
Department of Computer and Mathematical Sciences, Crawford University, Nigeria
tayoadefokun@crawforduniversity.edu.ng

Abstract

An induced matching M in a graph G is a matching in G that is also the edge set of an induced subgraph of G. That is, any edge not in M must have no more than one incident vertex saturated by M. The maximum size $|M|$ of an induced matching M of G is maximum induced matching number of G, which is denoted by $\mathrm{Max}(G)$. In this article, we obtain upper bounds for $\mathrm{Max}(G)$, for $G = G_{n,m}$, grids with $n, m \geq 9$, $m \equiv 1 \mod 4$ and nm odd.

Received 7 April 2016
Accepted in final form 15 July 2017
Published online 14 September 2017
Communicated with Misha Klin.

Keywords induced matching, grids, maximum induced matching number.
MSC(2010) 05C70, 05C15.

1 Introduction

Let G be a graph with edge and vertex sets $E(G)$ and $V(G)$, respectively, and for any $u, v \in V(G)$, let $d(u, v)$ be the distance between u and v. A matching is a set of edges with no shared vertices. The vertices incident to an edge of a matching M are said to be saturated by M, the other vertices are unsaturated by M. A subset $M \subseteq E(G)$ of G is an induced matching of G if for any two edges $e_1 = u_i u_j$ and $e_2 = v_i v_j$ in M, then $d(u_i, v_i) \geq d(u_i, v_j) \geq 2$ and $d(u_j, v_i) \geq d(u_j, v_j) \geq 2$. In other words, M is an induced matching of G if for any two edges e_1, e_2 in M, there is no edge in G incident to both e_1 and e_2. Equivalently, an induced matching is a matching which forms an induced subgraph.

Introduced by Stockmeyer and Vazirani [9] as a special case of the well known matching problem, the concept finds applications, among others, in cryptology where certain communication channels between two ends are classified [2].

The size $|M|$ of an induced matching M is the number of edges in the induced matching. Denote by $\mathrm{Max}(G)$ the maximum size of an induced matching in G. A maximum induced matching M in G is an induced matching with $\mathrm{Max}(G)$ edges. We refer to $\mathrm{Max}(G)$ as the maximum induced matching number (or strong matching number) of G. Unlike in the case with finding the maximum matching number of a graph, which can be obtained in polynomial time [4], obtaining $\mathrm{Max}(G)$ in general, is $NP-$hard, even for classes of graphs such as the regular bipartite graphs [3].

In [10], it was observed that for any G with maximum degree $\Delta(G)$,

$$\text{Max}(G) \geq \frac{|V(G)|}{2(2\Delta(G)^2 + 2\Delta(G) + 1)} \tag{1.1}$$

The above bound is certainly not a sharp one and therefore, Joos in [6], presented the following bound:

$$\text{Max}(G) \geq \frac{|V(G)|}{(\lceil \frac{\Delta(G)}{2} \rceil + 1)(\lfloor \frac{\Delta(G)}{2} \rfloor + 1)} \tag{1.2}$$

Joos also showed that it holds for $\Delta(G) = 1000$ and this, according to him, could be reviewed down to 200. In the end, he conjectured that this bound holds for $\Delta(G) \geq 3$ with the exception of certain graphs that he listed. Inspired by the work in [6], Nguyen [8] showed that the conjecture is true for $\Delta(G) = 4$ as long as G is not one of the excepted graphs.

The maximum induced matching number of many graphs can be obtained efficiently just as in the cases of chordal graphs [2], bounds of bounded cliques width, intersection graphs [1], circular arc graphs [5] among others.

A grid $G_{n,m}$ is obtained by the Cartesian product of any two paths of lengths n and m, where $n, m \geq 2$ are integers, representing rows and columns of the grid, respectively. We introduce an odd grid as a grid whose path factors are of odd order. Marinescu-Ghemaci in [7], obtained $\text{Max}(G)$ values for all grids with even nm, and some cases where nm is odd. She also gave useful lower and upper bounds. Particularly, she showed that for any odd grid $G_{n,m}$, $\text{Max}(G_{n,m}) \leq \lfloor \frac{nm+1}{4} \rfloor$.

This paper improves Marinescu-Ghemachi's upper bound for $G_{n,m}$, n, m odd, $m \equiv 1 \mod 4$. The results provide new upper bounds for some cases whose lower bounds are established in [7] and thus, in a number of situations, precise values of $\text{Max}(G_{n,m})$ were obtained. These may also prove useful in probing some of the unresolved conjectures made in [7].

2 Definitions and Preliminary Results

Grid, $G_{n,m}$, as defined in this work, is the Cartesian product of paths P_n and P_m with n and m being positive integers, where P_n and P_m have disjoint vertex sets $V(P_n) = \{u_1, u_2, \cdots, u_n\}$ and $V(P_m) = \{v_1, v_2, \cdots, v_m\}$, respectively. Unless explicitly stated, n is any odd integer, $m \equiv 1 \mod 4$ and $2 \leq n \leq m$. We introduce the following notations: $V_i = \{u_1v_i, u_2v_i, \cdots, u_nv_i\} \subset V(G_{n,m})$ and $U_i = \{u_iv_1, u_iv_2, \cdots, u_iv_m\} \subset V(G_{n,m})$; for edge set $E(G_{n,m})$ of $G_{n,m}$, if $u_iv_j\ u_kv_j \in E(G_{n,m})$ and $u_iv_j\ u_iv_k \in E(G_{n,m})$, we write $u_{\{i,k\}}v_j \in E(G_{n,m})$ and $u_iv_{\{j,k\}} \in E(G_{n,m})$ respectively.

Recall that a vertex v is said to be saturated by an induced matching M if it is a member of an edge in M and unsaturated by M, otherwise. We say v is saturable if either v is saturated by M or v is unsaturated by M, but satisfies $d(v, u) \geq 2$ for every saturated vertex u in $V(G)$. This implies that an unsaturated vertex can become saturated if there is no saturated vertex within distance 2. If v can not be saturated, then we say v is an isolated vertex. A boundary vertex is a vertex on any of V_1, V_n, U_1, U_m. A saturable vertex v in subgraph G_a of G, which is not saturated by induced matching M_a of G_a can still be saturated by M of G in case v is on the boundary of G_a. However, if v is not on the boundary of G_a, then v is isolated. The sets of all saturated vertices, saturable vertices and isolated vertices in a graph G are denoted by $V_s(G), V_{sb}(G)$ and $V_{is}(G)$, respectively. Clearly, $|V_s(G)|$ is even and $V_s(G) \subseteq V_{sb}(G)$.

The following results about grid $G_{n,m}$ are from [7]:

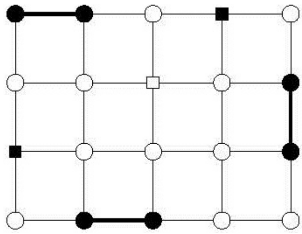

Figure 1. Saturable vertices as black squares and an isolated vertex as white square in an induced matching of $G_{4,5}$

Lemma 1. *Let $m, n \geq 2$ be two positive integers.*

1. *If $m \equiv 2 \mod 4$ and n odd, then, $|V_{sb}(G_{n,m})| = \frac{mn+2}{2}$ and $|V_{sb}(G_{n,m})| = \frac{mn}{2}$ otherwise;*

2. *For $m \geq 3$, m odd, $|V_{sb}(G_{n,m})| = \frac{nm+1}{2}$, for $n \in \{3, 5\}$.*

Theorem 2. *For $G_{n,m}$ where $2 \leq n \leq m$, let $|M| = Max(G_{n,m})$. Then, for n even, $Max(G_{n,m}) = \lceil \frac{mn}{4} \rceil$, for $n \in \{3, 5\}$, $m \equiv 1 \mod 4$, $Max(G_{n,m}) = \frac{n(m-1)}{4} + 1$ and for $m \equiv 3 \mod 4$, $Max(G_{n,m}) = \frac{n(m-1)+2}{4}$.*

Remark 3. For $m \equiv 1 \mod 4$, $|V_{sb}(G_{3,m})| = 2(\text{Max } (G_{3,m})) = |V_s(G_{n,m})|$.

Theorem 4. *For m, n odd integers, $Max(G_{n,m}) \leq \lfloor \frac{mn+1}{4} \rfloor$.*

The obvious implication of Theorem 4, based on the proof, is that $|V_{sb}G_{n,m}| \leq \frac{nm+1}{2}$, for n, m odd.

3 Results

We start our results by stating a few observations.

Remark 5. Let $G_{3,3}$ be a 3×3 grid with induced matching M. Clearly, by Lemma 1, $|V_{sb}(G_{3,3})| = 5$. Suppose $u_{\{1,2\}}v_2 \in M$, then $|M| = 1$. However, there are non-adjacent saturable vertices u_3v_1 and u_3v_3.

Lemma 6. *Let M be an induced matching of $G_{3,m}$. If $u_{\{1,2\}}v_2 \in M$. Then, $|M| \neq Max(G_{3,m})$.*

Proof. Suppose $u_{\{1,2\}}v_2 \in M$ and let $G_a = G_{3,m-3} \subset G_{3,m}$, be a subgrid of $G_{3,m}$ induced by $V(G_{3,m}) \setminus \{V_1, V_2, V_3\}$, where $V_1, V_2, V_3 \subset V(G_{3,m})$, $m \geq 3$.

Case I: Suppose $m \equiv 3 \mod 4$, $m \geq 7$. Then $m = 4k + 3$ for some positive integers k. By Lemma 1 and Theorem 2, $|V_s(G_a)| = 6k$. Now, with $u_{\{1,2\}}v_2 \in M$, u_3v_1 is saturable, by its position and $V_s(G_a) = V_{sb}(G_a)$. Since $m - 3$ is even, then either u_3v_3 remains isolated (or unsaturated) or if it is forced to be saturated with u_3v_4, a saturable vertex in $V(G_a)$ becomes isolated (or unsaturated). So without loss of generality, we may assume u_3v_3 is unsaturated. Therefore $G_{3,m}$ contains $3 + 6k$ saturable vertices and then, $|M| \leq 3k + 1$, which is a contradiction since by Theorem 2, $\text{Max}(G_{3,m}) = 3k + 2$. Note that $m = 3$, has been covered by Remark 5.

Case II: Suppose $m \equiv 1 \mod 4$. From Lemma 1, Theorem 2 and following similar argument as in Case I, we see that if $u_{\{1,2\}}v_2 \in M$ then one vertex in $V_{sb}(G_3, m)$ will become isolated and therefore, $|V_{sb}(G_{3,m})| = 6k+1$. Hence $|M| \leq 3k$ and a contradiction since $\text{Max}(G_{3,m}) = 3k + 1$ if $m = 4k + 1$.

Case III: If $m \equiv 0 \mod 4$, then $m-3 \equiv 1 \mod 4$, and by Remark 3, $G_{3,m-3} = G_a$, $|V_s(G_a)| = |V_{sb}(G_a)|$. Now $|V_s(G_a)| = \frac{3(m-3)+1}{2} = 6k-4$. By following the argument in the previous cases, we see that $|V_{sb}(G_a)| = 6k-1$. Therefore, $|M| \leq 3k-1$, which is less than $3k$.

Case IV: If $m \equiv 2 \mod 4$, then $m-3 \equiv 3 \mod 4$ and therefore, $|V_{sb}(G_a)| = \frac{3(4k+2-3)+1}{2} = 6k-1$, which is odd. Therefore, there exists a saturable vertex in $V(G_a)$, which can pair with u_3v_3 and thus the two vertices become saturated. This way, we have the total saturable vertices in $G_{3,m}$, which has $u_{\{1,2\}}v_2 \in M$, to be $6k+3$ which implies that $|M| \leq 3k+1$. But from the results in Theorem 2, if $m \equiv 2 \mod 4$, $\text{Max}(G_{3,m}) = 3k+2$. This is a contradiction and the claim holds. □

Remark 7. Suppose we have $u_{\{1,2\}}v_2 \in M$ in a grid $G_{3,m}$, then the following holds for $|V_{sb}(G_{3,m})|$, from Lemma 6.

m	$\lvert V_{sb}(G_{3,m})\rvert$
$4k$	$6k-1$
$4k+1$	$6k+1$
$4k+2$	$6k+3$
$4k+3$	$6k+3$

Lemma 8. *If $m \equiv 1 \mod 4$ and there exists M, an induced matching of $G_{3,m}$, such that $u_{\{1,2\}}v_j, u_{\{1,2\}}v_{j+2} \in M$, then $|M| \neq Max(G_{3,m})$.*

Proof. Let $u_{\{1,2\}}v_j, u_{\{1,2\}}v_{j+2} \in M$, where M is an induced matching of $G_{3,m}$.

Case I: Suppose that $j+1 \equiv 3 \mod 4$, then $m-(j+1) \equiv 2 \mod 4$. Since $u_{\{1,2\}}v_j \in M$, by Lemma 6, suppose there exist an induced matching M' in $G_a = G_{3,j+1}$, induced by $V_1, V_2, \cdots, V_{j+1}$, with $u_{\{1,2\}}v_j \in M'$, then $|M'| \neq \text{Max}(G_a)$. By Remark 7, given a non-negative integer l, $|V_{sb}(G_a)| = 6l+3$, which being odd, contains a saturable vertex $v' = u_3v_{j+1}$ which is not a member of $V_s(G_a)$. In fact, $v' \in V_{is}(G_a)$, since $u_{\{1,2\}}v_{j+1} \in M$. Since $m-(j+1) \equiv 2 \mod 4$, then given a subgrid $G_b = G_{3,m-(j+1)}$, induced by $V_{j+2}, V_{j+3}, \cdots, V_m$ $|V_s(G_b)| = 6(k-l)-2$. Therefore, $|V_{sb}(G_{3,m})| = 6k$ and hence, $|M| \leq 3k$, which is less that $3k+1$.

Case II: If $j+1 \equiv 1 \mod 4$, by following the argument in Case I, $v' \in V_{is}(G_{3,m})$. Now, $j+1 = 4l+1$, and by the isolation of v', and Remark 7, $|V_{sb}(G_{3,m})| = 6l+|V_{sb}(G_b)|$. Meanwhile, $m-(j+1) \equiv 0 \mod 4$ and therefore, $|V_{sb}(G_b)| = 6k$. Thus, $|M| \leq 3k$.

Case III: If $j+1$ is even, we follow similar arguments as the earlier cases. □

Lemma 9. *Suppose that $m \equiv 1 \mod 4$ and M is an induced matching of $G_{3,m}$. If $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+3} \in M$, then $|M| \neq Max(G_{3,m})$.*

Proof. For some positive integers k, let $m = 4k+1$. Now suppose that $j \equiv 3 \mod 4$. This implies that $j+3 \equiv 2 \mod 4$ and $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+3} \in M$. Let $G_a = G_{3,j+1}$, $G_b = G_{3,m-(j+1)} \subset G_{3,m}$, induced by $V_1, V_2, \cdots, V_{j+1}$ and $V_{j+2}, V_{j+3}, \cdots, V_m$, respectively. By earlier result and remark, $|V_{sb}(G_a)| = 6l-1$ since $j+1 = 4l$. Also, $|V_{sb}(G_b)| = 6(k-l)+1$, since $m-(j+1) = 4(k-l)+1$. Certainly, $|V_{sb}(G_{3,m})| = 6k$. Thus $|M| \leq 3k$. Therefore, $|M| \neq 3k$. For $j \equiv 1 \mod 4$, $j+3 \equiv 3 \mod 4$. Let $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+3} \in M$. By earlier lemma and result, we have that $|V_{sb}(G_a)| = 6l+3$. Since $j+1 = 4l+2$. Also, $|V_{sb}(G_b)| = 6[(k-l)-1]+3$. Therefore, $|V_{sb}(G_{3,m})| = 6k$ and therefore, $|M| = 3k$, which is a contradiction. □

Remark 10. By following similar argument as in the last result, it is easy to see that if M contains $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+4}$, then $|M| \neq \text{Max}(G_{3,m})$. Therefore, suppose M' is a maximum induced matching of $(G_{3,m})$ and $u_{\{1,2\}}u_j, u_{\{1,2\}}u_{j+8} \in M'$, then there is no $u_{\{1,2\}}u_{j+k} \in M$, such that $2 \leq k \leq 6$.

Theorem 11. *For $m = 4k+1$, there exist at least $2k$ saturated vertices in $U_1 \subset V(G_{3,m})$.*

Proof. Let $G_a = G_{2,m} \subset G_{3,m}$, induced by U_2, U_3, a subgrid of $G_{3,m}$. From earlier results, $|V_{sb}(G_a)| = 4k + 2$. Now, $|V_{sb}(G_{3,m})| = 6k + 2$. Therefore, V_1 has at least $2k$ saturable vertices. □

Theorem 12. *Suppose $u_{\{1,2\}}u_j, u_{\{1,2\}}v_{j+8} \in M$, in a $G_{3,m}$ and $m \equiv 1 \mod 4$.*

(a) There exists four other saturated vertices from u_1v_j to u_1v_{j+8}.

(b) There exists at most one saturated vertex between u_2v_j and u_2v_{j+8}.

(c) There exists at most five saturated vertices from v_3v_j to u_3v_{j+8}.

Proof. (a) From Remark 10, let $|M| = \mathrm{Max}(G_{3,m})$, and $u_{\{1,2\}}u_j, u_{\{1,2\}}v_{j+8} \in M$, then there is no $u_{\{1,2\}}v_k \in E(G_{3,m})$, $1 \leq k \leq 7$ such that $u_{\{1,2\}}v_k \in M$. Thus, suppose there exists another saturated vertex such that $u_1v_{\{j+k,j+k-1\}} \in M$. Then there exist at least four saturated vertices from u_1v_1 to u_1v_m. Now suppose that there is no other saturated vertex on U_1, then by the Theorem 11, the grid $G_a = G_{2,m} \subset G_{3,m}$, induced by vertices $\{u_2v_j, u_2v_{j+1}, \cdots, u_2v_{j+8}\}$ and $\{u_3v_j, u_3v_{j+1}, \cdots, u_3v_{j+8}\}$ must contain ten saturated vertices (including u_2v_j and u_2v_{j+8}). Clearly, vertices $u_2v_{j+1}, u_2v_{j+7}, u_3v_j$ and u_3v_{j+8} can not be saturated by G_a. It is clear, therefore, that G_a only has eight saturable vertices, which is a contradiction. Thus, there exists two more saturated vertices in U_1, and hence the claim.

Parts (b) and (c) follow from (a). □

Remark 13. (a) Since there are five saturated vertices between u_3v_j and u_3v_{j+8} and there exist only one saturated vertex between u_2v_j and u_2v_{j+8}, then there are edges $e_1, e_2 \in E(G_b)$, G_b induced by $u_3v_j, u_3v_{j+1}, \cdots u_3v_{j+8}$.

(b) Suppose $m = 4k + 1$, k being positive integers, then, at $G_c = G_{1,m} \subset G_{3,m}$, induced by either V_1 or V_3, there exists at least k edges of G_c in the maximum induced matching M of $G_{3,m}$.

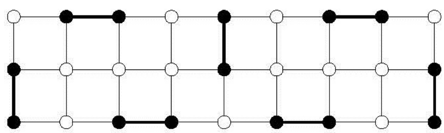

Figure 2. A $G_{3,9}$ grid with $Max(G_{3,9}) = 7$

Next, we consider the grid $G_{4,m}$, $m \equiv 1 \mod 4$.

Lemma 14. *Let $|M| = Max(G_{4,m})$ and let U_4 contain $\frac{m-1}{2}$ saturated vertices. Then, for any edge $e_1 \in E(G_a)$, $G_a = G_{1,m} \subset G_{4,m}$, induced by $U_4 \subset V(G_{4,m})$, $e_1 \notin M$.*

Proof. Let $u_4v_i, u_4v_{i+1} \in U_4 \subset V(G_{4,m})$, be saturated vertices. By hypothesis, there are $\frac{m-5}{2}$ other saturated vertices in U_4. Suppose $G_a = G_{2,m} \subset G_{4,m}$, induced by U_1, U_2. The $|V_s(G_a)| = |V_{sb}(G_a)| = m + 1$. Let $G_b = G_{2,m} \subset G_{4,m}$, induced by U_3, U_4. Since $|V_s(G_a)| = m + 1$, then $|V_s(G_b)| \leq m - 1$ where $|V_s(G_{4,m})| = 2m$. By hypothesis, suppose there exist $\frac{m-1}{2}$ saturated vertices in U_4, then there are also at most $\frac{m-1}{2}$ saturated vertices in U_3. Without loss of generality, suppose for all the $\frac{m-5}{2}$ other saturated vertices in U_s, there exist adjacent saturated vertices in U_3, then we have that $|V_s(G_b)| \geq m-3$.

Claim: There are at most $\frac{m-5}{2}$ saturated vertices in U_3.

Reason: Since there are at most $\frac{m-1}{2}$ saturated vertices in U_4, then suppose v_k is saturable in U_3, v_k is not incident to a saturable vertex in U_4. Also, since $V_s(G_a) = V_{sb}(G_a)$, then there is no saturable vertex in U_2 to which v_k is incident to form an edge in M. Thus, suppose there exists a vertex v_{k-1}, adjacent to $v_k \in U_3$, v_{k-1} will be adjacent to a saturated vertex in U_2 since there can not be two adjacent vertices both of which are not saturated in U_2. Thus, v_k is isolated.

Finally, $|V_s(G_{4,m})| \leq 2m-2$ which implies that $|M| \leq m-1$. However by Theorem 2, $\text{Max}(G_{4,m}) = m$. □

Corollary 15. *Let $G_{n,m}$ be a grid with $n \equiv 0 \mod 4, m \equiv 1 \mod 4$ and U_n contains $\frac{m-1}{2}$ saturable vertices, with $|M| = Max(G_{n,m})$, then no two saturated vertices, say, $v', v'' \in U_n$ such that $v'v'' \in M$.*

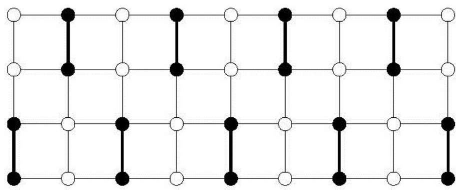

Figure 3. A $G_{4,9}$ grid with $Max(G_{4,9}) = 11$

Next we observe some results in the grid $G_{5,m}$, where $m \equiv 1 \mod 4$.

Lemma 16. *Let $G_{5,5}$ be a grid and suppose that $|M| = Max(G_{5,5})$, and $u_{\{1,2\}}v_1 \in M$, then*

(a) there exists at least another saturated vertex in V_1 (either u_4v_1, u_5v_1 or both).

(b) suppose $G_a = G_{5,5} \backslash V_1$, and M_a is some induced matching in G_a, then $|M_a| \neq Max(G_a)$.

(c) there exists some $e \in M$ with $e = u_iu_j$, such that $u_i, u_j \in V_5$.

Proof. (a) By Theorem 2, $|M| = 6$. Suppose that $u_{\{1,2\}}v_1 \in M$ and that no other vertex in V_1 is saturated. Clearly, u_1v_2 and u_2v_2 can not be saturated. Therefore, there can only be two saturated vertices in V_2. Suppose $u_{[4,5]}v_2 \in M$. Now we show that in $G_{5,5} \backslash \{V_1, V_2\}$, only three edges belong to M. Let vertices u_1v_3, u_1v_4, u_1v_5 and u_2v_3, u_2v_4, u_2v_5, induce $G_b = G_{2,3} \subset G_{5,5}$. Now, G_b has a maximum of four saturable vertices. Also let G_c be a subgraph of $G_{5,5}$, induced by u_3v_3, u_3v_4, u_3v_5; u_4v_4, u_4v_5 and u_5v_4, u_5v_5. The subgraph G_c also has a maximum of four saturable vertices. However, if we consider the positions G_b and G_c, it is clear that at least a saturable vertex in G_b is adjacent to a saturable vertex in G_c, which implies that $|V_s(G_b \cup G_c)| \leq 6$. Thus $|V_s(G_{5,5})| \leq 10$, and therefore $|M| \neq \text{Max}(G_{5,5})$ and hence a contradiction. By following similar argument, it can be seen also that $|M| = 5$ if we consider $u_{\{3,4\}v_2} \in M$.

(b) Suppose that $G_a = G_{5,5} \backslash \{V_1\} \subset G_{5,5}$. Assume that V_1 contains only three saturated vertices. There exists a saturated vertex $u_2 \in V_2$ such that given some vertex $u_1 \in V_1$, $v_1 v_2 \in M$. Now, it is clear that $v_1 v_2 \notin E(G_a)$ therefore, $|V_{sb}(G_a) \backslash v_2| = 9$, implying that $|M_a| \leq 4$, while $\text{Max}(G_a) = 5$.

(c) By (a) and (b) above, $|M_a| = 4$. Suppose that $u_3 v_{\{2,3\}} \notin M$, then, for $G_{5,5} \backslash \{V_1, V_2\}$, there must be at least two saturated vertices on V_5. Let $G'' = G_{5,2} = G_{5,5} \backslash \{V_1, V_2, V_5\} \subset G_{5,5}$. Clearly $G_{5,2}$ will contain at most six saturated vertices. Suppose that u_a, u_b are saturated in V_5, and $u_a u_b \notin E(G_{5,5})$, then u_a and u_b form two edges with adjacent vertices $v_a, v_b \in V_4$. However, it can be seen that if this is so, there would be, at most, only four saturable vertices in G'', apart from v_a and v_b, and at least one of which is an isolated vertex. Thus, there could only be four saturated vertices in G'', which is a contradiction. Suppose $u_3 v_{\{2,3\}} \in M$. It is easy to see by observation that it is impossible to have the matching in subgraph $G_f \subset G_{5,5}$, induced by $\{u_1 v_3, u_1 v_4, u_1 v_5, u_2 v_4, u_2 v_5, u_3 v_5, u_4 v_4, u_4 v_5, u_5 v_3, u_5 v_4, u_5 v_5\} \in V(G_{5,5})$ since it contains at most three edges in M without any of them being made up of adjacent vertices in V_5. □

Lemma 17. *Let M be a maximum induced matching of $G_{5,9}$. If $v_{\{1,2\}} u_1 \in M$, and there is at most one more saturated vertex v_1 on V_1. Then, $v_1 = u_5 v_1$.*

Proof. It is obvious that if $u_{\{1,2\}} v_1 \in M$, then $u_3 v_1 \notin V_s(G_{5,9})$. Suppose that $v_1 = u_4 v_1$, then since it is the only lone saturated vertex on V_1, then, $u_4 v_{\{1,2\}} \in M$. For G_a, some $G_{5,2}$ grid, induced by V_1, V_2, clearly, there is no other saturable vertex in V_2. Now, let $G_b = G_{5,9} \backslash G_a$. The subgrid G_b is a $G_{5,7}$ grid and $\text{Max}(G_b) = 8$. Thus, $|M| = 10$, and hence, not $\text{Max}(G_{5,9})$. Hence a contradiction. □

Lemma 18. *Suppose that there are at most two saturated vertices on $V_1 \subset V(G_{5,5})$, and that they are adjacent. Then, there are at least three saturated vertices on $V_5 \subset V(G_{5,5})$, two of which are adjacent.*

Proof. Suppose M is the maximal induced matching of $G_{5,5}$, and that $u_a u_b \in M$, where $u_j v_1, u_{j+1} v_1 \in U_1$. (It should be noted that since there are two saturated vertices in V_1 and are adjacent, then by Lemma 16(a), none of j and $j+1$ is either 1 or 5). Suppose that $G_b \subset G_{5,5}$, induced by V_2, V_3, V_4 and that V_5 only contains two saturated vertices $u_i v_5, u_{i+1} v_5$ and are adjacent, (meaning also that neither i nor $i+1$ is 0 or 5.) Let $j \neq i$ and obviously, without loss of generality, set $i = 2$, while j clearly becomes 3. Let $G_d, G_e \subset G_b$ be two subgraphs of G_b induced by vertex sets $\{u_1 v_2, u_1 v_3, u_1 v_4, u_2 v_3, u_2 v_4\}$ and $\{u_3 v_3, u_4 v_2, u_4 v_3, u_5 v_2, u_5 v_3, u_5 v_4\}$. $E(G_d), E(G_e)$ can only have a member each in M and if $u_{\{3,4\}} v_3 \in M$, then no member of $E(G_d)$ belongs M. Then $|M| \leq 4$, which is a contradiction. If $i = j = 1$, let G_d be induced by $\{u_1 v_2, u_1 v_3, u_1 v_4, u_2 v_3\}$ and G_e by $\{u_2 v_3, u_4 v_2, u_4 v_3, u_4 v_4, u_5 v_2, u_5 v_3, u_5 v_4\}$. Following similar argument as above, $E(G_d)$ and $E(G_e)$ have maximum of three members in M, and thus $|M| = 5$, which is a contradiction. Suppose V_5 contains three saturated vertices, such that none is adjacent to another. Clearly the saturated vertices are $u_1 v_5, u_3 v_5$ and $u_5 v_5$. Since they are saturated, then $u_1 v_{\{3,4\}}, u_3 v_{\{3_4\}}, u_5 v_{\{3,4\}} \in M$. Therefore by this, only two vertices on V_3 is saturable. Obviously $|M| \leq 5$. □

Remark 19. Let $G_a \subseteq G_{n,m}$ be a $G_{5,9}$ grid induced by $V_1, V_2, \cdots, V_9$, an induced matching M. From Lemma 16 and Lemma 18, we see that if $u_{\{1,2\}} v_i \in M$, it is possible to have some $M_a \subseteq M$, for which $|M_a| = \text{Max}(G_a)$ as seen in the grid above. Also,

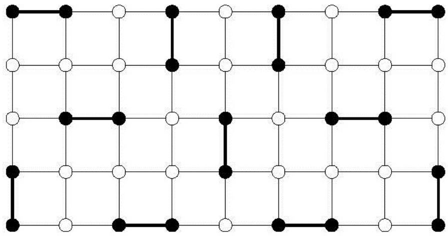

Figure 4. A $G_{5,9}$ grid with $Max(G_{5,9}) = 11$

from Lemma 18, since V_1 has three saturated vertices with two of them being adjacent, then for M_a to be maximal, V_9 will also have three saturated vertices. It is easy to determine, however, that if this scheme extends to $m \geq 13$, then $|M| \neq \text{Max}(G_{n,m})$ since $|M| \leq 11 + 4j + 5k$, $j \geq k$, and $j - k \leq 1$.

Lemma 20. *Let M be a matching of $G_{5,m}$, $m \equiv 1 \mod 4$, $m \geq 13$, with $u_{\{1,2\}}v_1 \in M$, then $|M| \neq Max(G_{5,m})$.*

Proof. Let $m - 10 = q$. Clearly, $q \equiv 3 \mod 4$. For $q \geq 7$, let $G_{5,q} = G_a$, induced by $V_{11}, V_{12}, \cdots V_m$. We already know that $|V_{sb}(G_a)| = \frac{5q+1}{2}$, while $|V_s(G_a)| = 2\left(\frac{5(q-1)-2}{4}\right)$. Therefore, there are two saturable vertices, say, v_1, v_2 on $V(G_a)$. Suppose that $v_1, v_2 \in V_{11}$. Note that v_1, v_2 are not adjacent, else $v_1v_2 \in M$. Let $G_b \subset G_{5,m}$ be a $G_{5,9}$, induced by $V_1, V_2, \cdots, V_9$. Suppose that $u_{\{1,2\}}v_1 \in M$, then there are two adjacent saturated vertices in V_9 and at least a saturated boundary vertex, by Lemma 18 and Remark 19. This implies that there are two adjacent saturable vertices on V_{10}, say u_1, u_2. Now, there could be at most one edge in M from $\{u_1, u_2, v_1, v_2\}$. Hence, $|M| = \text{Max}(G_b) + \text{Max}(G_a) + 1 = \frac{45+5q}{4}$. Since $q = m - 10$, we have $|M| = \frac{5(m-1)}{4}$, which is less than $\text{Max}(G_{5,m})$ by an edge, and hence a contradiction. For $q = 3$, the result is similar to that obtained by careful observation of the positions u_1, u_2 and the possible isolated vertex or vertices on G_a. □

Corollary 21. *Let M be a matching of $G_{5,m}$, $m \geq 13$, with $u_{\{1,2\}}v_i \in M$, $1 \leq i \leq m$, then $|M| \neq Max(G_{5,m})$.*

Lemma 22. *Let $U_1 \subset V(G_{5,m})$. Then there are at least $\frac{m+1}{2}$ saturable vertices in U_1.*

Proof. This follows similar arguments as in the proof of Theorem 11. □

Remark 23. We note that for $m = 4k+1$, $\frac{m+1}{2}$ is odd, also from Lemma 20, the number of saturated vertices on U_1 is even and there cannot be any isolated vertex on U_1 if M is a maximal induced matching of $G_{5,m}$. Therefore, for $G_1 \subset G_{5,m}$, induced by U_1, there exists at least $k+1$ edges of G_1 in M, for $m \geq 13$. For $m = 9$, there are at least k edges in G_1 as seen in the last figure.

Theorem 24. *Let $G_{n,m}$ be a grid with $m \equiv 1 \mod 4$, $m \geq 13$ and let M be the maximum induced matching of $G_{n,m}$. Then*

$$Max(G_{n,m}) \leq \begin{cases} \left\lfloor \frac{2mn-m-1}{8} \right\rfloor & \text{if } n \equiv 1 \mod 4; \\ \left\lfloor \frac{2mn-m+3}{8} \right\rfloor & \text{if } n \equiv 3 \mod 4 \end{cases}$$

Proof. Let $n = 4l + 1$, l a positive integer, and let $r = n - 5$, that is, $r \equiv 0 \mod 4$. Suppose G_a is a $G_{r,m}$ induced by $V_1, V_2, \cdots, V_r$. By Theorem 22, at least there are $\frac{m+1}{2}$, saturated vertices on U_r and these are $u_r v_2, u_r v_4, \cdots, u_r v_{m-1}$. Let G_b be a $G_{5,m}$ grid, induced by $V_{r+1}, V_{r+2}, \cdots, V_n$ and G_c be a $G_{1,m}$ grid, induced by U_{r+1}. By Remark 23, there are $k+1$ edges of $E(G_c)$ in M. Clearly a saturated vertex induced by M on each of the $k+1$ edges is adjacent to some saturated vertices in U_r, implying that only one of the two saturated vertices belongs to $V_{sb}(G_{n,m})$. Hence, $V_{sb}(G_{n,m}) \leq \frac{nm+1}{2} - k + 1$. Now, $k+1 = \frac{m-1}{4} + 1$. Therefore, $V_{sb}(G_{n,m}) \leq \frac{2nm-m-1}{4}$ and hence, $|M| \leq \left\lfloor \frac{2nm-m-1}{8} \right\rfloor$. For $n = 4l + 3$, set $s = n - 3$, that is, $s \equiv 0 \mod 4$. By Remark 23, and following the argument above, $V_{sb}(G_{n,m}) = \frac{mn+1}{2} - k$ with $k = \frac{m-1}{4}$, $\text{Max}(G_{n,m}) \leq \left\lfloor \frac{2mn-m+3}{8} \right\rfloor$. □

Remark 25. For the grid $G_{n,m}$, it should be noted that the results in the last theorem depend mainly on the value of m.

Remark 26. Following similar argument as Theorem 24, for $m = 9$, $\text{Max}(G_{n,9}) \leq \left\lfloor \frac{18n-6}{8} \right\rfloor$.

Acknowledgements

We would like to thank the anonymous referee for very helpful comments and suggestions.

References

[1] K. Cameron. Induced matching in intersection graphs. *Discrete Math.* **278** (2004), 1–9.

[2] K. Cameron, R. Sritharan and Y. Tang. Finding a maximum induced matching in weakly chordal graphs. *Discrete Math.* **266** (2003), 133-142.

[3] K. K. Dabrowski, M. Demange and V. Lozin. New results on maximum induced matchings in bipartite graphs and beyond. *Theoretical Computer Science,* **478** (2013), 33–40.

[4] J. Edmonds, Paths, trees, and flowers, *Canad. J. Math.* **17** (1965), 449–467 .

[5] M. C. Golumbic and R.C Laskar, Irredundancy in circular arc graphs. *Discrete Applied Math.* **44** (2013), 79–89.

[6] F. Joos, Induced matching in graphs of bounded maximum degree. *arXiv:* (2014) 1406.2440.

[7] R. Marinescu-Ghemeci, Maximum induced matchings in grids. *Optimization Theory, Decision Making and Operations Research Applications. Springer New York*, (2013) 177–187.

[8] V. H. Nguyen, Induced matching in graphs of degree at most 4. *arXiv:* (2014) 1407.7604.

[9] L. J. Stockmeyer and V.V. Vazirani. $NP-$completeness of some generalizations of the maximum matching problem. *Information Processing Letters*, **15**(1) (1982), 14–19.

[10] M. Zito, "Induced matching in regular graphs and trees". Lecture Notes in Computer Sci. 1665, Springer, Berlin, 1999