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New sufficient conditions for starlikeness of certain integral operators involving Bessel functions

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Abstract

The purpose of the present paper is to investigate a new integral operator associated with Bessel function. Some sufficient conditions are derived for this integral operator belonging to various subclasses of starlike functions under certain conditions.

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1 Introduction

Let $\mathcal{A}$ denote the class of functions f of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z : z \in \mathbb{C} \; and \; |z| < 1\}$ and satisfy the normalization condition $f(0) = f'(0) - 1 = 0$. Further, we denote by S the subclass of $\mathcal{A}$ consisting of functions of the form (1.1) which are also univalent in U. A function $f(z)$ in $\mathcal{A}$ is said to be starlike of order $\delta(0 \le \delta < 1)$ if the following condition is satisfied

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \delta, \qquad (z \in \mathbb{U}), \tag{1.2}$$

we denote by $S^*(\delta)$ the class of starlike functions of order δ. Clearly $S^*(\delta) \subset S^*(0) = S^*(0 \le \delta < 1)$ and $S^* \subset S$.

The Bessel function of the first kind of order ν is defined by the infinite series

$$J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{z}{2}\right)^{2n+\nu}}{n!\Gamma\left(n+\nu+1\right)}, \tag{1.3}$$

where Γ stands for the Euler-Gamma function $z \in \mathbb{C}$ and $\nu \in \mathbb{R}$. Recently, Szasz and Kupan [17] investigated the univalence of the normalized Bessel function of first kind $g_\nu : U \longrightarrow \mathbb{C}$ defined by

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}). \tag{1.4}$$

Baricz and Frasin [3] have obtained the sufficient condition for the univalence of the various integral operators involving Bessel functions of the first kind of order ν.

Recently, Frasin [5] introduced the following integral operator involving the normalized Bessel function of the first kind

$$F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z) = \int_0^z \prod_{i=1}^{n} \left(\frac{g_{\nu_i}(t)}{t}\right)^{\alpha_i} dt, \tag{1.5}$$

and obtain several sufficient condition for this operator to be convex and strongly convex of given order in the open disc $\mathbb{U}$. Recently analogous to these result Porwal and Breaz [15] studied the sufficient condition for the operator defined by (1.5) for certain class of univalent functions. Very recently, Mishra and Panigrahi [9] obtain some sufficient conditions for starlikeness of certain integral operator. In this paper, motivated with the above mentioned work and work of Kumar [6] we obtain some sufficient condition for the operator defined by (1.5) is in the class S^*.

To prove our main results we shall require the following lemmas:

Lemma 1. *([17]) Let $\nu > \frac{-5+\sqrt{5}}{4}$ and consider the normalized Bessel function of the first kind $g_\nu : \mathbb{U} \longrightarrow \mathbb{C}$ defined by*

$$g_\nu(z) = 2^\nu \Gamma(\nu+1) z^{1-\frac{\nu}{2}} J_\nu(z^{\frac{1}{2}}),$$

where J_ν stands for Bessel function of the first kind, then the following inequality hold for all $z \in \mathbb{U}$.

$$\left|\frac{zg_\nu'(z)}{g_\nu(z)} - 1\right| \le \frac{\nu+2}{4\nu^2+10\nu+5}.$$

Lemma 2. *([16]) If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{\delta+1}{2(\delta-1)}, \qquad (z \in \mathbb{U}),$$

for some $2 \le \delta < 3$,
or

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < \frac{5\delta-1}{2\delta+1}, \qquad (z \in \mathbb{U}),$$

for some $1 \le \delta \le 2$, then $f \in S^*$.

Lemma 3. *([16]) If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > -\frac{\delta+1}{2\delta(\delta-1)}, \qquad (z \in \mathbb{U}),$$

for some $\delta \leq -1$,
or

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{3\delta+1}{2\delta(\delta+1)}, \qquad (z\in\mathbb{U}),$$

for some $\delta > 1$, *then* $f\in S^*(\frac{\delta+1}{2\delta})$.

Throughout this paper, we frequently use the notation $F_{\nu_1,\ldots,\nu_n,\alpha_1,\ldots,\alpha_n}(z) = F_{\nu_i,\alpha_i}(z)$.

2 Main Results

Theorem 4. *Let n be a natural number such that* $\nu_1, \nu_2, ..., \nu_n > (\frac{-5+\sqrt{5}}{4})$. *Consider the function* $g_{\nu_i} : \mathbb{U}\to\mathbb{C}$ *defined by*

$$g_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i+1)z^{1-\frac{\nu_i}{2}}J_{\nu_i}(z^{\frac{1}{2}}).$$

Let $\nu = min\{\nu_1, \nu_2, ..., \nu_n\}$ *and suppose that the inequality*

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i \leq \frac{3-\delta}{2(\delta-1)} \tag{2.1}$$

is satisfied. Then the function $F_{\nu_i,\alpha_i}(z)$ *defined by (1.5) is in the class* S^* *for some* $2 \leq \delta \leq 3$.

Proof. First we observe that, since for all $i\in\{1, 2, \ldots, n\}$, we have $g_{\nu_i}\in\mathcal{A}$ i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i,\alpha_i}(z) = \prod_{i=1}^{n}\left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}.$$

Taking logarithmic differentiation

$$\frac{F''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)} = \sum_{i=1}^{n}\alpha_i\left(\frac{g'_{\nu_i}(z)}{g_{\nu_i}(z)} - \frac{1}{z}\right)$$

$$1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)} = 1+\sum_{i=1}^{n}\alpha_i\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right)$$

$$\begin{aligned}\Re\left\{1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} &= 1+\sum_{i=1}^{n}\alpha_i\Re\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right)\\ &\leq 1+\sum_{i=1}^{n}\alpha_i\left|\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right|\\ &\leq 1+\sum_{i=1}^{n}\alpha_i\left(\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}\right).\end{aligned} \tag{2.2}$$

For all $z\in\mathbb{U}$ and $\nu_1, \nu_2, ..., \nu_n > \frac{-5+\sqrt{5}}{4}$. Since the function $\phi : \left(\frac{-5+\sqrt{5}}{4}, \infty\right) \to \mathbb{R}$, defined by,

$$\phi(x) = \frac{x+2}{4x^2+10x+5}$$

is decreasing and consequently for all $i \in \{1, 2, ..., n\}$. We have

$$\frac{\nu_i + 2}{4{\nu_i}^2 + 10\nu_i + 2} \leq \frac{\nu + 2}{4\nu^2 + 10\nu + 5}.$$

Using this result, inequality (2.7) can be written as

$$\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} \leq 1 + \frac{\nu + 2}{4\nu^2 + 10\nu + 5}\sum_{i=1}^{n}\alpha_i.$$

Since

$$1 + \frac{\nu + 2}{4\nu^2 + 10\nu + 5}\sum_{i=1}^{n}\alpha_i < \frac{\delta + 1}{2(\delta - 1)}.$$

$$\frac{\nu + 2}{4\nu^2 + 10\nu + 5}\sum_{i=1}^{n}\alpha_i < \frac{\delta + 1}{2(\delta - 1)} - 1$$

$$= \frac{\delta + 1 - 2\delta + 2}{2(\delta - 1)}$$

$$= \frac{3 - \delta}{2(\delta - 1)}$$

Hence from Lemma 2 $F_{\nu_i,\alpha_i}(z) \in S^*$ for some $2 \leq \delta \leq 3$.

Thus the proof of Theorem 4 is established. □

Theorem 5. *Let n be a natural number and $\nu_1, \nu_2, ..., \nu_n > \left(\frac{-5+\sqrt{5}}{4}\right)$. Consider the function $g_{\nu_i} : \mathbb{U} \to \mathbb{C}$ defined by*

$$g_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i + 1)z^{1-\frac{\nu_i}{2}}J_{\nu_i}(z^{\frac{1}{2}}).$$

Let $\nu = max\{\nu_1, \nu_2, ..., \nu_n\}$ be a positive real numbers and suppose that the inequality

$$\frac{\nu + 2}{4\nu^2 + 10\nu + 5}\sum_{i=1}^{n}\alpha_i \leq \frac{2\delta^2 - \delta + 1}{2\delta(\delta - 1)}, \tag{2.3}$$

is satisfied then $F_{\nu_i,\alpha_i}(z)$ defined by (1.5) is in the class $S^*(\frac{\delta+1}{2\delta})$ for some $\delta \leq -1$.

Proof. First we observe that since for all $i \in \{1, 2, 3, \ldots, n\}$, we have $g_{\nu_i} \in \mathcal{A}$ i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i,\alpha_i}(z) = \prod_{i=1}^{n}\left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}.$$

Taking logarithmic differentiation,

$$\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} = \sum_{i=1}^{n}\alpha_i\Re\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right) + 1.$$

From Lemma 1, we have

$$\left|\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}-1\right|\leq\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}.$$

Since $\Re(z)\leq|z|$

$$\Re\left\{1-\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right\}\leq\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}$$

$$\Re\left\{\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right\}\geq 1-\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}$$

$$\Re\left\{1+\frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\}\geq\sum_{i=1}^{n}\alpha_i\left(1-\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}\right)+1-\sum_{i=1}^{n}\alpha_i$$

$$=-\left(\frac{\nu+2}{4\nu^2+10\nu+5}\right)\sum_{i=1}^{n}\alpha_i+1$$

$$\geq-\frac{\delta+1}{2\delta(\delta-1)}$$

$$-\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i\geq-1-\frac{\delta+1}{2\delta(\delta-1)}$$

$$\frac{\nu+2}{4\nu^2+10\nu+2}\sum_{i=1}^{n}\alpha_i\leq 1+\frac{\delta+1}{2\delta(\delta-1)}$$

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i\leq\frac{2\delta^2-2\delta+\delta+1}{2\delta(\delta-1)}$$

$$=\frac{2\delta^2-\delta+1}{2\delta(\delta-1)}$$

Hence from Lemma 3, $F_{\nu_i,\alpha_i}(z)\in S^*(\frac{\delta+1}{2\delta})$ for some $\delta\leq-1$.
Thus the proof of Theorem 5 is established.

□

Theorem 6. *Let n be a natural number such that $\nu_1,\nu_2,\ldots,\nu_n>\left(\frac{-5+\sqrt{5}}{4}\right)$. Consider the function $g_{\nu_i}:\mathbb{U}\to\mathbb{C}$ defined by*

$$g_{\nu_i}(z)=2^{\nu_i}\Gamma(\nu_i+1)z^{1-\frac{\nu_i}{2}}J_{\nu_i}(z^{\frac{1}{2}}).$$

Let $\nu=min\{\nu_1,\nu_2,.......,\nu_n\}$ and suppose that the inequality

$$\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i\leq\frac{5\delta-1}{2(\delta+1)} \tag{2.4}$$

is satisfied. Then the function $F_{\nu_i,\alpha_i}(z)$ defined by (1.5) is in the class S^* for some $1\leq\delta\leq2$.

Proof. First we observe that, since for all $i\in\{1,2,\dots,n\}$. We have $g_{\nu_i}(z)\in\mathcal{A}$ i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i,\alpha_i}(z) = \prod_{i=1}^{n}\left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}$$

$$\left(\frac{F''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right) = \sum_{i=1}^{n}\alpha_i\left(\frac{g'_{\nu_i}(z)}{g_{\nu_i}(z)} - \frac{1}{z}\right)$$

$$\begin{aligned}\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} &= 1 + \sum_{i=1}^{n}\alpha_i\Re\left(\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right)\\ &\le 1 + \sum_{i=1}^{n}\alpha_i\left|\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right|\\ &\le 1 + \sum_{i=1}^{n}\alpha_i\left(\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5}\right)\\ &\le 1 + \frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i.\end{aligned}$$

For all $z\in\mathbb{U}$ and $\nu_1,\nu_2,........,\nu_n > \frac{-5+\sqrt{5}}{4}$.
Since the function,
$\phi : \left(\frac{-5+\sqrt{5}}{4},\infty\right)\to\mathbb{R}$ defined by

$$\phi(x) = \frac{x+2}{4x^2+10x+5}$$

is decreasing and consequently for all $i\in\{1,2,......,n\}$, we have

$$\frac{\nu_i+2}{4{\nu_i}^2+10\nu_i+5} \le \frac{\nu+2}{4\nu^2+10\nu+5}$$

Since,

$$1 + \frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i \le \frac{5\delta-1}{2(\delta+1)}$$

$$\begin{aligned}\frac{\nu+2}{4\nu^2+10\nu+5}\sum_{i=1}^{n}\alpha_i &\le \frac{5\delta-1}{2(\delta+1)} - 1\\ &= \frac{5\delta-1-2(\delta+1)}{2\delta+1}\\ &= \frac{5\delta-1-2\delta-2}{2(\delta+1)}\\ &= \frac{3(\delta-1)}{2(\delta+1)}.\end{aligned}$$

Hence from Lemma 2 , $F_{\nu_i,\alpha_i}(z)\in S^*$ for some $1 \le \delta\le 2$
Thus the proof of Theorem 6 is established. □

Theorem 7. *Let n be a natural number and $\nu_1, \nu_2......, \nu_n > \left(\frac{-5+\sqrt{5}}{4}\right)$. Consider the function $g_{\nu_i} : \mathbb{U} \to \mathbb{C}$ defined by*

$$g_{\nu_i}(z) = 2^{\nu_i}\Gamma(\nu_i + 1)z^{1-\frac{\nu_i}{2}} J_{\nu_i}(z^{\frac{1}{2}}).$$

Let $\nu = max\,\{\nu_1, \nu_2, ..., \nu_n\}$ be positive real numbers and suppose that the inequality

$$\frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^{n} \alpha_i \leq \frac{\delta + 1 - 2\delta^2}{2\delta(\delta + 1)}$$

is satisfied then $F_{\nu_i,\alpha_i}(z)$ defined by (1.5) is in the class $S^*(\frac{\delta+1}{2\delta})$ for some $\delta > 1$.

Proof. First we observe that since for all $i \in \{1, 2, \ldots, n\}$. We have $g_{\nu_i} \in \mathcal{A}$ i.e.

$$g_{\nu_i}(0) = g'_{\nu_i}(0) - 1 = 0.$$

From (1.5) we have

$$F'_{\nu_i,\alpha_i}(z) = \prod_{i=1}^{n} \left(\frac{g_{\nu_i}(z)}{z}\right)^{\alpha_i}.$$

Taking logarithmic differentiation

$$\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} = \sum_{i=1}^{n} \alpha_i \Re\left\{\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right\} + 1$$

$$\left|\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)} - 1\right| \leq \frac{\nu_i + 2}{4\nu_i{}^2 + 10\nu_i + 5}$$

$$\Re\left\{1 - \frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right\} \leq \frac{\nu_i + 2}{4\nu_i{}^2 + 10\nu_i + 5}$$

$$\Re\left\{\frac{zg'_{\nu_i}(z)}{g_{\nu_i}(z)}\right\} \geq 1 - \frac{\nu_i + 2}{4\nu_i{}^2 + 10\nu_i + 5}$$

$$\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} \geq \sum_{i=1}^{n} \alpha_i \left(1 - \frac{\nu_i + 2}{4\nu_i{}^2 + 10\nu_i + 5}\right) + 1 - \sum_{i=1}^{n} \alpha_i$$

$$\Re\left\{1 + \frac{zF''_{\nu_i,\alpha_i}(z)}{F'_{\nu_i,\alpha_i}(z)}\right\} \geq \left(1 - \frac{\nu + 2}{4\nu^2 + 10\nu + 5}\right) \sum_{i=1}^{n} \alpha_i + 1 - \sum_{i=1}^{n} \alpha_i.$$

Here $\phi(x) = \frac{x+2}{4x^2+10x+5}$ is decreasing. So for all $i \in \{1, 2, \ldots, n\}$.

$$\left(1 - \frac{\nu + 2}{4\nu^2 + 10\nu + 5}\right) \sum_{i=1}^{n} \alpha_i + 1 - \sum_{i=1}^{n} \alpha_i \geq \frac{3\delta + 1}{2\delta(\delta + 1)}$$

$$-\frac{\nu + 2}{4\nu^2 + 10\nu + 5} \sum_{i=1}^{n} \alpha_i \geq \frac{3\delta + 1}{2\delta(\delta + 1)} - 1$$

$$\geq \frac{3\delta + 1 - 2\delta(\delta + 1)}{2\delta(\delta + 1)}$$

$$= \frac{3\delta + 1 - 2\delta^2 - 2\delta}{2\delta(\delta+1)}$$
$$= \frac{\delta + 1 - 2\delta^2}{2\delta(\delta+1)}.$$

Hence from Lemma (2), $F_{\nu_i,\alpha_i} \in S^*(\frac{\delta+1}{2\delta})$ for some $\delta > 1$.
Thus the proof of Theorem 7 is established. □

Remark 8.

1. One may also obtain the analogukes of these results for various subclasses of analytic functions e.g. Convex functions, Strongly convex functions and Strongly starlike functions, Spiral-like functions etc.

2. The results of this paper can also be extended by using the integral operator studied by Breaz *et al.* [4], Merkes and Wright [7], Miller *et al.* [8], Pesker [10], Porwal and Kumar [13] and Porwal and Singh [14].

3. Recently Baricz [1] introduced generalized Bessel functions of first kind which is a natural generalization of Bessel function, Modified Bessel function, Spherical Bessel function and Modified spherical Bessel function and give wide applications in Geometric Function Theory. For detailed study one may refer [2], (see also [11], [12]).

4. It is interesting to find the analogues of these results for harmonic starlike and convex functions.

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