Acta Universitatis Matthiae Belii, series Mathematics
Issue 2017, 77–82, ISSN 1338-7111
Online Edition, http://actamath.savbb.sk

Coefficient inequality for transforms of bounded turning functions

D. Vamshee Krishna*
Department of Mathematics, Gitam Institute of Technology, GITAM University, Visakhapatnam-530 045, A.P., India.
vamsheekrishna1972@gmail.com

D. Shalini
Department of Mathematics, Sri Venkateswara College of Engineering and Technology, Etcherla-532 410, A.P., India.
shaliniraj1005@gmail.com

T. RamReddy
Department of Mathematics, Kakatiya University, Warangal-506 009, T.S., India.
reddytr2@gmail.com

Abstract
The objective of this paper is to obtain sharp upper bound for the second Hankel functional associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of normalized analytic function $f(z)$ when it belongs to bounded turning functions, defined on the open unit disc in the complex plane, with the help of Toeplitz determinants.

Received 4 April 2017
Accepted in final form 9 November 2017
Published online 27 November 2017
Communicated with Vladimír Janiš.

Keywords analytic function, upper bound, second Hankel functional, positive real function, Toeplitz determinants.
MSC(2010) 30C45; 30C50.

1 Introduction

Let A denote the class of analytic functions f of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

defined in the open unit disc $E = \{z : |z| < 1\}$, satisfying the conditions that $f(0) = 0$ and $f'(0) = 1$. Let S be the subclass of A consisting of univalent functions. In 1985, Louis de Branges de Bourcia proved the Bieberbach conjecture, i.e., for a univalent function, its n^{th} Taylor coefficient is bounded by n (see [2]). The bounds for the coefficients of these functions give the information about their geometric properties. In particular, the growth and distortion properties of a normalized univalent function are determined by

*corresponding author

the bound of its second coefficient. The k^{th} root transform for the function f given in (1.1) is defined as

$$F(z) := \left[f(z^k)\right]^{\frac{1}{k}} = z + \sum_{n=1}^{\infty} b_{kn+1} z^{kn+1}. \tag{1.2}$$

Now, we introduce the Hankel determinant for the k^{th} root transform for the function f, for $q, n, k \in \mathbb{N} = \{1, 2, 3, ...\}$, defined as

$$[H_q(n)]^{\frac{1}{k}} = \begin{vmatrix} b_{kn} & b_{kn+1} & \cdots & b_{k(n+q-2)+1} \\ b_{kn+1} & b_{k(n+1)+1} & \cdots & b_{k(n+q-1)+1} \\ \vdots & \vdots & \vdots & \vdots \\ b_{k(n+q-2)+1} & b_{k(n+q-1)+1} & \cdots & b_{k[n+2(q-1)-1]+1} \end{vmatrix} (b_k = 1).$$

In particular for $k = 1$, the above determinant reduces to the Hankel determinant defined by Pommerenke [9] for the function f given in (1.1). For the values $q = 2, n = 1$ and $q = 2, n = 2$, the above Hankel determinant simplifies respectively to

$$[H_2(1)]^{\frac{1}{k}} = \begin{vmatrix} b_k & b_{k+1} \\ b_{k+1} & b_{2k+1} \end{vmatrix} = b_{2k+1} - b_{k+1}^2$$

$$\text{and} \qquad [H_2(2)]^{\frac{1}{k}} = \begin{vmatrix} b_{2k} & b_{2k+1} \\ b_{2k+1} & b_{3k+1} \end{vmatrix} = b_{2k} b_{3k+1} - b_{2k+1}^2. \tag{1.3}$$

Ali et al. [1] obtained sharp bounds for the Fekete-Szegö functional denoted by $|b_{2k+1} - \mu b_{k+1}^2|$ associated with the k^{th} root transform $\left[f(z^k)\right]^{\frac{1}{k}}$ of the function f given in (1.1) and belonging to certain subclasses of S. We refer to $[H_2(2)]^{\frac{1}{k}}$ as the second Hankel determinant for the k^{th} root transform associated with the function f. In the present paper, we consider the Hankel determinant given by $[H_2(2)]^{\frac{1}{k}}$ and obtain sharp upper bound to the functional $|b_{k+1} b_{3k+1} - b_{2k+1}^2|$ for the k^{th} root transform of the function f when it belongs to certain subclass denoted by $\Re$ of S, consisting of functions whose derivative has a positive real part, defined as follows.

Definition 1. Let f be given by (1.1). Then $f \in \Re$, if it satisfies the condition

$$Re f'(z) > 0, \qquad \forall z \in E.$$

The subclass $\Re$ was introduced by Alexander in 1915 and a systematic study of properties of these functions was conducted by MacGregor [7] in 1962, who indeed referred to numerous earlier investigations involving functions whose derivative has a positive real part (also called Bounded turning functions).

Some preliminary Lemmas required for proving our result are as follows:

2 Preliminary Results

Let $\mathcal{P}$ denote the class of functions consisting of p such that

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \tag{2.1}$$

which are analytic (regular) in the open unit disc E and satisfy $\text{Re}\,p(z) > 0$, for any $z \in E$. Here $p(z)$ is called a Caratheódory function [3].

Lemma 2 ([8], [10]). *If $p \in \mathcal{P}$, then $|c_k| \leq 2$, for each $k \geq 1$ and the inequality is sharp for the function $\frac{1+z}{1-z}$.*

Lemma 3 ([4]). *The power series for $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ given in (2.1) converges in the open unit disc E to a function in $\mathcal{P}$ if and only if the Toeplitz determinants*

$$D_n = \begin{vmatrix} 2 & c_1 & c_2 & \cdots & c_n \\ c_{-1} & 2 & c_1 & \cdots & c_{n-1} \\ c_{-2} & c_{-1} & 2 & \cdots & c_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ c_{-n} & c_{-n+1} & c_{-n+2} & \cdots & 2 \end{vmatrix}, n = 1, 2, 3....$$

and $c_{-k} = \overline{c}_k$, are all non-negative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k} z)$, $\sum_{k=1}^{m} \rho_k = 1$, t_k real and $t_k \neq t_j$, for $k \neq j$, where $p_0(z) = \frac{1+z}{1-z}$; in this case $D_n > 0$ for $n < (m-1)$ and $D_n \doteq 0$ for $n \geq m$.

This necessary and sufficient condition found in (see [4]) is due to Carathe*ó*dory and Toeplitz. Without loss of generality, in view of Lemma 2, we consider $c_1 > 0$. On using Lemma 3, for $n = 2$ and $n = 3$ respectively, we have

$$D_2 = \begin{vmatrix} 2 & c_1 & c_2 \\ \overline{c}_1 & 2 & c_1 \\ \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}$$

On expanding the determinant, we get

$$D_2 = [\ 8 + 2Re\{c_1^2 c_2\} - 2 \mid c_2 \mid^2 - 4 \mid c_1 \mid^2\ ] \geq 0,$$

Applying the fundamental principles of complex numbers, the above expression is equivalent to

$$2c_2 = c_1^2 + y(4 - c_1^2), \text{ for some complex value of } y \text{ with } |y| \leq 1. \tag{2.2}$$

$$\text{and } \ D_3 = \begin{vmatrix} 2 & c_1 & c_2 & c_3 \\ \overline{c}_1 & 2 & c_1 & c_2 \\ \overline{c}_2 & \overline{c}_1 & 2 & c_1 \\ \overline{c}_3 & \overline{c}_2 & \overline{c}_1 & 2 \end{vmatrix}.$$

Then $D_3 \geq 0$ is equivalent to

$$|(4c_3 - 4c_1c_2 + c_1^3)(4 - c_1^2) + c_1(2c_2 - c_1^2)^2| \leq 2(4 - c_1^2)^2 - 2|(2c_2 - c_1^2)|^2. \tag{2.3}$$

Simplifying the relations (2.2) and (2.3), we obtain

$$4c_3 = \{c_1^3 + 2c_1(4 - c_1^2)y - c_1(4 - c_1^2)y^2 + 2(4 - c_1^2)(1 - |y|^2)\zeta\} \tag{2.4}$$

for some complex values y and ζ with $|y| \leq 1$ and $|\zeta| \leq 1$ respectively.
To obtain our main result, we refer to the classical method developed by Libera and Zlotkiewicz [6], which has been used widely (see [11, 12, 13, 14, 15]).

3 Main Result

Theorem 4. *If $f \in \Re$ and F is the k^{th} root transformation of f given by (1.2) then*

$$|b_{k+1} b_{3k+1} - b_{2k+1}^2| \leq \frac{4}{9k^2}$$

and the inequality is sharp.

Proof. For $f \in \Re$, by virtue of Definition 1, we have

$$f'(z) = p(z), \qquad \forall z \in E. \tag{3.1}$$

Using the series representation for f and p in (3.1), upon simplification, we obtain

$$a_{n+1} = \frac{c_n}{n+1}, \quad n \in \mathbb{N}. \tag{3.2}$$

For the function f given in (1.1), on computing, we have

$$\left[f(z^k)\right]^{\frac{1}{k}} = \left[z^k + \sum_{n=2}^{\infty} a_n z^{nk}\right]^{\frac{1}{k}} = z + \frac{1}{k}a_2 z^{k+1} + \left\{\frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2\right\}z^{2k+1}$$
$$+ \left\{\frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3\right\}z^{3k+1} + \cdots. \tag{3.3}$$

From the equations (1.2) and (3.3), we get

$$b_{k+1} = \frac{1}{k}a_2 \ ; \quad b_{2k+1} = \frac{1}{k}a_3 + \frac{(1-k)}{2k^2}a_2^2 \ ;$$
$$b_{3k+1} = \frac{1}{k}a_4 + \frac{(1-k)}{k^2}a_2a_3 + \frac{(1-k)(1-2k)}{6k^3}a_2^3. \tag{3.4}$$

Simplifying the expressions (3.2) and (3.4), we get

$$b_{k+1} = \frac{c_1}{2k} \ ; \quad b_{2k+1} = \frac{c_2}{3k} - \frac{(k-1)}{8k^2}c_1^2 \ ;$$
$$b_{3k+1} = \frac{c_3}{4k} - \frac{(k-1)}{6k^2}c_1c_2 + \frac{(k-1)(2k-1)}{48k^3}c_1^3. \tag{3.5}$$

Substituting the values of b_{k+1}, b_{2k+1} and b_{3k+1} from (3.5) in the functional $|b_{k+1}b_{3k+1} - b_{2k+1}^2|$, which simplifies to give

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| = \frac{1}{576k^4}\left|(72c_1c_3 - 64c_2^2)k^2 + 3(k^2-1)c_1^4\right|. \tag{3.6}$$

Substituting c_2 and c_3 values from (2.2) and (2.4) respectively, on the right-hand side of the expression (3.6), we have

$$576k^4|b_{k+1}b_{3k+1} - b_{2k+1}^2| = \Big|\Big[72c_1 \times \frac{1}{4}\{c_1^3 + 2c_1(4-c_1^2)y - c_1(4-c_1^2)y^2$$
$$+2(4-c_1^2)(1-|y|^2)\zeta\} - 64 \times \frac{1}{4}\{c_1^2 + y(4-c_1^2)\}^2\ \Big]\ k^2 + 3(k^2-1)c_1^4\Big|.$$

Then applying the triangle inequality and using the fact $|\zeta| < 1$, will give

$$576k^4|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \big|(5k^2-3)c_1^4 + 36k^2c_1(4-c_1^2) + 4k^2c_1^2(4-c_1^2)|y|$$
$$+2(c_1+2)(c_1+16)k^2(4-c_1^2)|y|^2\big|. \tag{3.7}$$

Choosing $c_1 = c \in [0,2]$, noting that $(c_1+a)(c_1+b) \geq (c_1-a)(c_1-b)$, where $a, b \geq 0$, applying the triangle inequality and replacing $|y|$ by μ on the right-hand side of (3.7), we obtain

$$\begin{aligned} 576k^4|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq & \Big[\big(5k^2-3)c^4 + 36k^2c(4-c^2) + 4k^2c^2(4-c^2)\mu \\ & + 2(c-2)(c-16)k^2(4-c^2)\mu^2 \Big] \\ & = F(c,\mu), \text{ for } 0 \leq \mu = |y| \leq 1. \end{aligned} \tag{3.8}$$

$$\begin{aligned} \text{Here } \ F(c,\mu) = & \big(5k^2-3)c^4 + 36k^2c(4-c^2) + 4k^2c^2(4-c^2)\mu \\ & + 2(c-2)(c-16)k^2(4-c^2)\mu^2. \end{aligned} \tag{3.9}$$

Next, we need to find the maximum value of the function $F(c,\mu)$ on the closed region $[0,2]\times[0,1]$. For this, let us suppose that there exists a maximum value at any point (c,μ) in the interior of the closed region $[0,2]\times[0,1]$ for the function $F(c,\mu)$. Differentiating $F(c,\mu)$ in (3.9) partially with respect to μ, we get

$$\frac{\partial F}{\partial \mu} = 4k^2\big\{c^2 + (c-2)(c-16)\mu\big\}(4-c^2). \tag{3.10}$$

For $0<\mu<1$, for fixed c with $0<c<2$, from (3.10), we observe that $\frac{\partial F}{\partial \mu} > 0$. Therefore, $F(c,\mu)$ becomes an increasing function of μ and hence it cannot have a maximum value at any point (c,μ) in the interior of the closed region $[0,2]\times[0,1]$. The maximum value of $F(c,\mu)$ occurs on the boundary only i.e., when $\mu = 1$. Therefore, for fixed $c \in [0,2]$, we have

$$\max_{0\leq\mu\leq 1} F(c,\mu) = F(c,1) = G(c). \tag{3.11}$$

In view of (3.11), replacing μ by 1 in (3.9), we get

$$G(c) = -(k^2+3)c^4 - 40k^2c^2 + 256k^2, \tag{3.12}$$

$$G'(c) = -4(k^2+3)c^3 - 80k^2c. \tag{3.13}$$

From the expression (3.13), we observe that $G'(c) \leq 0$ for all values of c in the interval $[0,2]$ and for every k. Therefore, $G(c)$ is a monotonically decreasing function of c in the interval $[0,2]$ and hence it attains the maximum value at $c = 0$ only. From (3.12), the maximum value $G(c)$ at $c = 0$ is given by

$$\max_{0\leq c\leq 2} G(0) = 256k^2. \tag{3.14}$$

Simplifying the relations (3.8) and (3.14), we obtain

$$|b_{k+1}b_{3k+1} - b_{2k+1}^2| \leq \frac{4}{9k^2}. \tag{3.15}$$

Choosing $c_1 = c = 0$ and selecting $y = 1$ in (2.2) and (2.4), we find that $c_2 = 2$ and $c_3 = 0$. Substituting the values $c_2 = 2$ and $c_1 = c_3 = 0$ in (3.5) and the obtained values in (3.15), we see that equality is attained, which shows that our result is sharp. For these values, from (2.1), we can derive

$$p(z) = 1 + 2\sum_{n=1}^{\infty} z^{2n} = \frac{1+z^2}{1-z^2}. \tag{3.16}$$

Therefore, in this case the extremal function is

$$f'(z) = 1 + 2\sum_{n=1}^{\infty} z^{2n}.$$

This completes the proof of our Theorem. □

Remark 5. By choosing $k = 1$ in (3.15), the result coincides with that of Janteng et al. [5].

Acknowledgements

The authors are very much thankful to the Referee(s) for their valuable comments and suggestions which helped them in improving the paper.

References

[1] R. M. Ali, S. K. Lee, V. Ravichandran and S. Supramaniam, The Fekete-Szegö coefficient functional for transforms of analytic functions, *Bull. Iranian Math. Soc.,* **35**(2), 119–142 (2009).

[2] L. de Branges de Bourcia, A proof of Bieberbach conjecture, *Acta Mathematica*, **154**(1-2), 137–152 (1985).

[3] P. L. Duren, "Univalent functions", vol. 259 of Grundlehren der Mathematischen Wissenschaften, Springer, New York, USA, 1983.

[4] U. Grenander and G. Szegö, "Toeplitz forms and their applications", 2nd ed., Chelsea Publishing Co., New York (NY), 1984.

[5] A. Janteng, S. A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, *J. Inequal. Pure Appl. Math.*,**7**(2), 1–5 (2006).

[6] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, *Proc. Amer. Math. Soc.*, **87**(2) 251–257 (1983).

[7] T. H. MacGregor, Functions whose derivative has a positive real part, *Trans. Amer. Math. Soc.*, **104**(3) , 532 – 537 (1962).

[8] Ch. Pommerenke, *Univalent functions*, Vandenhoeck and Ruprecht, Gottingen, 1975.

[9] Ch. Pommerenke, On the coefficients and Hankel determinants of univalent functions, *J. Lond. Math. Soc.*, **41**(1) , 111–122 (1966).

[10] B. Simon, "Orthogonal polynomials on the unit circle", part 1. Classical theory, vol. 54, American mathematical society colloquium publications. Providence (RI): American Mathematical Society; 2005.

[11] D. Vamshee Krishna and T. RamReddy, Coefficient inequality for multivalent bounded turning functions of order alpha, *Probl. Anal. Issues Anal.*, **23**(1), 45–54 (2016).

[12] D. Vamshee Krishna and T. RamReddy, Coefficient inequality for parabolic star like functions of order alpha, *Afr. Mat.*, **27**(1-2), 121–132 (2016).

[13] D. Vamshee Krishna and T. RamReddy, An upper bound to the second Hankel determinant for functions in Mocanu class, *Viet. J. Math.*, **43**(3), 541–549 (2015).

[14] D. Vamshee Krishna and T. RamReddy,Coefficient inequality for certain p- valent analytic functions, *Rocky Mountain J. Math.*, **44**(6), 1941–1959 (2014).

[15] D. Vamshee Krishna and T. RamReddy, An upper bound to the second Hankel determinant for certain subclass of analytic functions, *Proc. Janjgeon Math. Soc.*, **16**(4), 559–568 (2013).