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Existence Result for an Impulsive Ordinary Differential Problem

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Abstract
In this paper, we treated the existence and uniqueness of solutions to a Cauchy problem for impulsive ordinary differential equations of first order on an unbounded interval $[0, \infty)$. We also treated in this paper some topological and geometric properties of the solutions set.

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1 Introduction

The dynamics of many evolving processes are subject to abrupt changes, such as shocks, harvesting and natural disasters. These phenomena involve short-term perturbations from continuous and smooth dynamics, whose duration is negligible in comparison with the duration of an entire evolution. In models involving such perturbations, it is natural to assume these perturbations act instantaneously or in the form of impulses. As a consequence, impulsive differential equations have been developed in modeling impulsive problems in physics, population dynamics, ecology, biological systems, biotechnology, industrial robotics, pharmcokinetics, optimal control, and electrical engineering. Important contributions to the study of the mathematical aspects of such equations have been undertaken in [2, 3, 9, 12].

In this work we consider the following problem

$$\begin{cases} \dot{y}(t) = f(t, y(t)), & t \in [0, +\infty)/\{t_1, \cdots\} \\ y(t_k^+) - y(t_k^-) = I_k(y(t_k^-)), & k = 1, 2, 3, \cdots \\ y(0) = a, & a \in \mathbb{R}^n \end{cases} \tag{1.1}$$

where $0 = t_0 < t_1 < \ldots t_k < t_{k+1}, \ldots, \lim_{k\to\infty} t_k = \infty,\ f : [0, +\infty) \times \mathbb{R}^n \to \mathbb{R}^n$ Carathéodory function, and $I_k \in C(\mathbb{R}^n, \mathbb{R}^n), y(t_k^+) = \lim_{h\to 0^+} y(t_k + h), y(t_k^-) = \lim_{h\to 0^-} y(t_k - h)$.

Our goal in this work is to give present some existence and solution sets of some class impulsive differential equations.

The paper is organized as follows: In Section 2 we introduce all the background material needed in this paper such as elements from functional analysis and also some results from homology and algebraic topology. Section 3 is devoted to establishing the existence and uniqueness of solution of the problem (1.1). In Section 4, we investigate the geometric structure of solution set (R_δ, acyclicity) of the problem (1.1). In Section 5, as an application, we present an example to illustrate our main result.

2 Preliminaries

In this section, we recall some notations, definitions, and auxiliary results which will be used throughout this paper.

Denote by $\mathcal{P}(X) = \{Y \subset X\colon\ Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X)\colon\ Y \text{ bounded}\}$. Let (X, d) and (Y, ρ) be two metric spaces and $F : X \to \mathcal{P}(Y)$ be a multi-valued mapping. The map F is called *upper semi-continuous* (*u.s.c.*) on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $F(M) \subseteq Y$. That is, if the set $F^{-1}(V)$ is closed for any closed set V in Y. Equivalently, F is *u.s.c.* if the set $F^{+1}(V)$ is open for any open set V in Y.

The mapping F is said to be *completely continuous* if it is *u.s.c.* and, for every bounded subset $A \subseteq X$, $F(A)$ is relatively compact, i.e., there exists a relatively compact set $K = K(A) \subset X$ such that

$$F(A) = \bigcup \{F(x) : x \in A\} \subset K.$$

Also, F is *compact* if $F(X)$ is relatively compact, and it is called *locally compact* if for each $x \in X$, there exists an open set U containing x such that $F(U)$ is relatively compact.

Theorem 1. *[6] Let $F : X \to \mathcal{P}_{cp}(Y)$ be a closed locally compact multifunction. Then F is u.s.c.*

Definition 2 (Homotopic)**.** Let $f, g : X \longrightarrow Y$ two continuous functions. We say that f is homotopic to g if there exists is a continuous function $h : X \times [0, 1] \longrightarrow Y$ such that:

(1) $h(x, 0) = f(x); \forall x \in X$,

(2) $h(x, 1) = g(x); \forall x \in X$.

Definition 3 (Contractible)**.** The set $A \in \mathcal{P}(X)$ is contractible, if there exist $x_0 \in A$ and homotopic continuous function $h : A \times [0, 1] \longrightarrow A$ such that

(i) $h(x, 0) = x$, $\forall x \in A$, et

(ii) $h(x, 1) = x_0$, $\forall x \in A$.

Definition 4 (Retract)**.** Let $A \in \mathcal{P}(X)$, we say that A is a retract set in X, if there exists a continuous map $r : X \longrightarrow A$, such that $r(x) = x; \quad \forall x \in A$.

Definition 5 (R_δ−set)**.** A space X is said R_δ−set, if there exists a sequence of non-empty compact contractible spaces $\{X_n\}_n$ such that

$$X_{n+1} \subset X_n; \forall n$$

and

$$X = \cap_{n=1}^{\infty} X_n.$$

Definition 6 (Acyclic)**.** A space A is closed acyclic if
(a) $H_0(A) = \mathbb{Q}$,
(b) $H_n(A) = 0$, for every $n > 0$, where $H_* = \{H_n\}_{n\geq 0}$ is the Čech-homology functor with compact carriers and coefficients in the field of rationals $\mathbb{Q}$. In other words, a space A is acyclic if the map $j : \{p\} \to X$, $j(p) = x_0 \in A$, induces an isomorphism $j_* : H_*(\{p\}) \to H_*(A)$.

Lemma 7 ([11])**.** *Let X be a compact metric space, if X is $R_\delta-$set, then X is an acyclic space.*

Theorem 8 ([10])**.** *Let E be a normal space, and X a metric space, and let $f : X \longrightarrow E$ a continuous map. Then for every $\varepsilon > 0$ there exists a locally Lipschitz function $f_\varepsilon : X \longrightarrow E$ such that*

$$\|f(x) - f_\varepsilon(x)\| \leq \varepsilon; \forall x \in X. \tag{2.1}$$

Definition 9. A function $f : X \to Y$ is said specific, if it is continuous and the inverse image of a compact set is compact.

Theorem 10 (Theorem of Browder and Gupta)**.** *[4] Let $(E, \|\cdot\|)$ a Banach space, and let $f : X \longrightarrow E$ a specific function, and for all $\varepsilon > 0$ we have a specific function $f_\varepsilon : X \longrightarrow E$, satisfying:*

(i) $\|f_\varepsilon(x) - f(x)\| < \varepsilon$ *for all* $x \in X$.

(ii) For all $u \in E$ such that $\|u\| \leq \varepsilon$, the equation $f_\varepsilon(x) = u$ has an unique solution. Then the set $S = f^{-1}(0)$ is R_δ.

3 Existence and Uniqueness

Let $J_k = (t_k, t_{k+1}]$, $k \in \mathbb{N}$, and let y_k be the restriction of a function y to J_k. In order to define solutions for problem (1.1), consider the space. Let $[\alpha, \beta]$ be a interval in $\mathbb{R}$ and $C([\alpha, \beta], \mathbb{R}^n)$ be the Banach space of all continuous functions from $[\alpha, \beta]$ into $\mathbb{R}^n$ with the norm

$$\|y\|_\infty = \sup\{|y(t)| : \ \alpha \leq t \leq \beta\}.$$

Set

$$AC([\alpha, \beta], \mathbb{R}^n) = \left\{ y \in C([\alpha, \beta], \mathbb{R}^n) : \quad y(t) = y(\alpha) + \int_\alpha^t y'(s)ds, \ y' \in L^1([\alpha, \beta], \mathbb{R}^n) \right\},$$

$$\begin{aligned} PC \quad = \quad & \{y : [0, +\infty) \longrightarrow \mathbb{R}^n : y_k \in C(J_k, \mathbb{R}^n), y(t_k^-) \text{ and } y(t_k^+) \text{ exist and satisfy} \\ & y(t_k^-) = y(t_k), \ k \in \mathbb{N}\}, \end{aligned}$$

and

$$PC_b = \{y \in PC : \|y\|_{PC_b} < \infty\}.$$

Endowed with the norm

$$\|y\|_{PC_b} = \sup_{t\in[0,\infty)} \|y(t)\|,$$

PC_b is a Banach space.

Definition 11. A function $y \in PC \cap \cup_{k=1}^{m} AC(J_k, \mathbb{R}^n)$ is a solution of the problem (1.1) if and only if

$$y(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad t \in [0, \infty).$$

Our first main result is the existence and uniqueness of the problem (1.1) in PC_b.

Theorem 12. *Suppose that there exist* $p \in L^1([0, +\infty[, \mathbb{R}_+)$ *such that:*

$$\|f(t, y) - f(t, x)\| \leq p(t) \|y - x\|, \ \forall x, y \in \mathbb{R}^n, \textit{ almost all elements } t \in [0, \infty),$$

and positives real numbers $c_k > 0; k \in \mathbb{N}$ *such that:*

$$\|I_k(y) - I_k(x)\| \leq c_k \|y - x\|, \textit{ for every } \quad x, y \in \mathbb{R}^n.$$

and

$$\sum_{k=1}^{+\infty} \|I_k(0)\| < +\infty, \quad \int_0^{+\infty} \|f(s, 0)\| ds < \infty.$$

If $\sum_{k=1}^{+\infty} c_k < 1$. *Then the problem (1.1) has a unique solution on* $[0, \infty)$.

Proof. Consider the application

$$\begin{aligned} N : PC_b &\longrightarrow PC_b \\ y &\longrightarrow Ny, \end{aligned}$$

defined by

$$Ny(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad t \in [0, \infty).$$

Step 1 N is well defined
Let $y \in PC_b$, we have

$$Ny(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad t \in [0, \infty).$$

we will prove that $Ny \in PC_b$,

$$\begin{aligned}
\|Ny(t)\| &\leq \|a\| + \int_0^t \|f(s,y(s))\|\,ds + \sum_{0<t_k<t} \|I_k(y(t_k))\| \\
&\leq \|a\| + \int_0^{+\infty} \|f(s,y(s)) - f(s,0)\|\,ds + \int_0^{+\infty} \|f(s,0)\|\,ds \\
&+ \sum_{k=1}^{+\infty} \|I_k(y(t_k)) - I_k(0)\| + \sum_{k=1}^{+\infty} \|I_k(0)\| \\
&\leq \|a\| + \int_0^{+\infty} p(s)\,\|y(s)\|\,ds + \int_0^{+\infty} \|f(s,0)\|\,ds \\
&+ \sum_{k=1}^{+\infty} c_k\,\|y(t_k)\| + \sum_{k=1}^{+\infty} \|I_k(0)\| \\
&\leq \|a\| + \int_0^{+\infty} p(s)\,\|y\|_{PC_b}\,ds + \int_0^{+\infty} \|f(s,0)\|\,ds \\
&+ \sum_{k=1}^{+\infty} c_k\,\|y\|_{PC_b} + \sum_{k=1}^{+\infty} \|I_k(0)\| < \infty.
\end{aligned}$$

Step 2 N is a contracting:
We let $y_1, y_2 \in PC_b$:

$$\begin{aligned}
\|Ny_1(t) - Ny_2(t)\| &\leq \int_0^t \|f(s,y_1(s)) - f(s,y_2(s))\|\,ds \\
&+ \sum_{0<t_k<t} \|I_k(y_1(t_k)) - I_k(y_2(t_k))\| \\
&\leq \int_0^t p(s)\,\|y_1(s) - y_2(s)\|\,ds \\
&+ \sum_{k=1}^{+\infty} c_k\,\|y_1(t_k) - y_2(t_k)\| \\
&\leq \int_0^t \tau\frac{1}{\tau}e^{\tau P(s)}e^{-\tau P(s)}p(s)\,\|y_1(s) - y_2(s)\|\,ds \\
&+ e^{\tau P(t)}e^{-\tau P(t)}\sum_{k=1}^{+\infty} c_k\,\|y_1(t_k) - y_2(t_k)\| \\
&\leq \frac{1}{\tau}\left(\int_0^t \tau p(s)e^{\tau P(s)}ds\right)\|y_1 - y_2\|_* + e^{\tau P(t)}\left(\sum_{k=1}^{+\infty} c_k\right)\|y_1 - y_2\|_* \\
&\leq \frac{1}{\tau}\left(e^{\tau P(t)}\right)\|y_1 - y_2\|_* + e^{\tau P(t)}\left(\sum_{k=1}^{+\infty} c_k\right)\|y_1 - y_2\|_*.
\end{aligned}$$

Then

$$\|Ny_1 - Ny_2\|_* \leq \left(\frac{1}{\tau} + \sum_{k=1}^{+\infty} c_k\right)\|y_1 - y_2\|_*.$$

where

$$\|x\|_* = \sup_{t\geq 0} e^{-\tau P(t)} \|x(t)\|, \quad P(t) = \int_0^t p(s)ds.$$

By assumption we have $\sum_{k=1}^{+\infty} c_k < 1$, so there exists $\epsilon \in (0,1)$ such that

$$\varepsilon + \sum_{k=1}^{+\infty} c_k < 1.$$

If we take $\tau = \frac{1}{\varepsilon}$, we obtain that N is contracting, hence the problem (1.1) has an unique solution. □

Next, we present an existence and uniqueness result of the problem (1.1) in the following Fréchet space $PC = \cap_{m\in\mathbb{N}} PC_m$ such that

$$PC_m = PC([0,t_m], \mathbb{R}^n)$$

- $(PC_m, \|\cdot\|_m)$ is a Banach space endowed with the norm $\|\cdot\|_m$ such that

  $$\|y\|_m = \sup_{t\in[0,t_m]} \|y(t)\|,$$

  and

  $$PC_1 \subset PC_2 \subset PC_3 \subset \cdots$$
  $$\|\cdot\|_1 \leq \|\cdot\|_2 \leq \|\cdot\|_3 \leq \cdots$$

- PC is a Fréchet space for the family of semi-normes $\{\|\cdot\|_m\}_{m\in\mathbb{N}}$.

Theorem 13 (Frigon et Granas, [8]). *Let E be a Fréchet space with a family of semi semi–normes $\{\|\cdot\|_n\}_{n\in N^*}$, $N : E \longrightarrow E$ a continuous operator, with*

$$E = \cap_{n\in N^*} E_n, \quad E_n \subset E_{n+1}, \quad \text{and} \quad \|\cdot\|_n \leq \|\cdot\|_{n+1}, \quad n \in \mathbb{N}.$$

assume that for all $n \in \mathbb{N}$, $\exists k_n \in (0,1)$ such that:

$$\|Ny - Nx\|_n \leq k_n \|y - x\|_n; \quad \forall n \in \mathbb{N}, \text{for all} x, y \in E_n,$$

and $N : \mathcal{O} \longrightarrow E$ contractant, $\mathcal{O}$ a closed part in E. Then either,

1) there exists $\lambda \in (0,1) : y = \lambda Ny, \quad \forall y \in \partial\mathcal{O}$; or

2) there exists unique $y \in \mathcal{O}$ such that $y = Ny$.

Now, we consider the following hypotheses:

$(\mathcal{A}_1)$ $\forall R > 0; \exists \ell_R \in L^1_{loc}([0,+\infty), \mathbb{R}_+)$ such that

$$\begin{array}{lcl} \|f(t,y) - f(t,x)\| & \leq & \ell_R(t)\|y - x\|; \text{ almost all elements } x, y \in \mathbb{R}^n \\ \|y\| \leq R, \|x\| \leq R & ; & \forall t \in [0,+\infty) \end{array}$$

$(\mathcal{A}_2)$ there exist $c_k > 0; k \in \mathbb{N}$ such that

$$\|I_k(y) - I_k(x)\| \leq c_k \|y - x\|, \forall x, y \in \mathbb{R}^n, \quad \forall t \in [0, \infty),$$

with

$$\sum_{k=1}^{+\infty} c_k < 1,$$

$(\mathcal{A}_3)$ there exist $p \in L^1_{loc}([0,\infty), \mathbb{R}^n)$ and $\psi \in C([0,\infty),(0,\infty))$ such that

$$\|f(t, y(t))\| \leq p(t)\psi(\|y(t)\|); \forall t \in [0,\infty), \forall y \in C([0,\infty), \mathbb{R}^n),$$

with

$$\int_{\|a\|}^{\infty} \frac{du}{\psi(u)} = \infty.$$

Theorem 14. *Assume $(\mathcal{A}_1) - (\mathcal{A}_3)$ are satisfied. Then the problem (1.1) has an unique solution on $[0, +\infty)$.*

Proof. Consider the operator $N : PC \longrightarrow PC$ defined by

$$(Ny)(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)); \quad t \in [0, \infty).$$

We let $y \in PC$, $\lambda \in]0, 1[$, such that $y = \lambda Ny$. Then

$$y(t) = \lambda \left(a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)) \right); \ t \in [0, t_m],$$

we can prove that there exists $M_m > 0$ such that

$$\|y\| \leq M_m,$$

Let

$$\mathcal{O} = \{y \in PC : \|y\|_m \leq M_m + 1\}.$$

• $\mathcal{O}$ is closed
• $N : PC_m \longrightarrow PC_m$ is contracting $\forall m \in \mathbb{N}$.
Let $y_1, y_2 \in PC_m$, we have

$$\|Ny_1 - Ny_2\|_{PC_m} \leq \left(\frac{1}{\tau} + \sum_{k=1}^{m} c_k \right) \|y_1 - y_2\|_{PC_m},$$

where

$$\|y\|_{PC_m} = \sup_{t\in[0,t_m]} e^{-\tau P(t)} \|y(t)\|, \quad P(t) = \int_0^t p(s)ds.$$

So, $N : \mathcal{O} \longrightarrow PC$ is contracting. Then by the alternative of Frigon and Granas, we have either,

1. there exists $\lambda \in (0, 1 =: y = \lambda Ny, \quad \forall y \in \partial\mathcal{O}$; or
2. there exists unique $y \in \mathcal{O}$ such that $y = Ny$.

Assume that $\exists \lambda \in (0, 1) : y = \lambda Ny, \quad \forall y \in \partial\mathcal{O}$,
if $y \in \partial\mathcal{O} \Longrightarrow \|y\|_m \leq M_m + 1$, we also have

$$\begin{aligned} y = \lambda Ny \quad &\Longrightarrow \quad \|y\|_m \leq \|\lambda Ny\|_m \\ &\Longrightarrow \quad M_m + 1 \leq M_m, \quad \text{contradiction.} \end{aligned}$$

Then there exist unique $y \in \mathcal{O}$ such that $y = Ny$. □

4 Solution sets

In this section, we present an existence result, compactness and R_δ solution sets of the problem (1.1).

The following compactness criterion on unbounded domains is a simple extension of a compactness criterion in $PC_b(\mathbb{R}_+, \mathbb{R})$ (see [1, 5]).

Lemma 15. *Let $C \subset PC_b$. Then C is relatively compact if it satisfies the following conditions:*
(a) C is uniformly bounded in $PC_\ell(\mathbb{R}_+, \mathbb{R}^n)$.
(b) The functions belonging to C are almost equicontinuous on $\mathbb{R}_+$, i.e. equicontinuous on every compact interval of $\mathbb{R}^+$.
(c) The functions from C are equiconvergent, that is, given $\varepsilon > 0$, there corresponds $T(\varepsilon) > 0$ such that $|x(\tau_1) - x(\tau_2)| < \varepsilon$ for any $\tau_1, \tau_2 \geq T(\varepsilon)$ and $x \in M$.

The following so-called nonlinear alternatives of Leray and Schauder will be needed in the proof (see [7, 10]).

Theorem 16. *Let X be a Banach space C a convex subset of X, U an open subset in C, and $N : U \to X$ is continuous and compact operator, then*

(a) either $\exists u \in \partial U; \exists \lambda \in]0, 1[$ such that $u \in \lambda F(u)$,

(b) or F has a fixed point in $\overline{U}$.

Theorem 17. *Let $f\colon [0, \infty) \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory function. Assume that the following conditions*

($\mathcal{H}_1$) There exist $c_k, d_k > 0$ such that

$$\sum_{k=1}^{+\infty} c_k < 1 \ \ et \ \sum_{k=1}^{\infty} d_k < \infty$$

with

$$\|I_k(x)\| \leq c_k\|x\| + d_k, \ \ \text{for all } x \in \mathbb{R}^n, \ k \in \mathbb{N}.$$

($\mathcal{H}_2$) There is a continuous increasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$ and $p \in L^1([0, \infty), \mathbb{R}_+)$ such that

$$\|f(t, x)\| \leq p(t)\psi(\|x\|), \ \ \text{for almost all elements } t \in [0, \infty) \ \text{and all } x \in \mathbb{R}^n,$$

with

$$\int_0^{+\infty} m(s)ds < \int_c^{+\infty} \frac{du}{\psi(u)},$$

where

$$m(s) = \frac{p(s)}{1 - \sum_{k=1}^{+\infty} c_k} \ \ \text{and} \ \ c = \frac{\|a\| + \sum_{k=1}^{\infty} d_k}{1 - \sum_{k=1}^{+\infty} c_k}.$$

hold. Then the problem (1.1) has at least one solution. Moreover the solution set is compact, R_δ, acyclic, and the solution operator $S : a \to S(a)$ is u.s.c.

Proof.

- Existence solutions:

Consider the operator $N\colon PC_b([0,\infty),\mathbb{R}^n) \to PC_b([0,\infty),\mathbb{R}^n)$ defined by

$$Ny(t) = a + \int_0^t f(s,y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)),\ t \in [0,\infty).$$

We show that N satisfies all the conditions of theorem 16 on PC_b.

Step 1 N is well defined.

Let $y \in PC_b$ then, we have

$$Ny(t) = a + \int_0^t f(s,y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)),\ t \in [0,\infty). \tag{4.1}$$

Then

$$\begin{aligned}\|Ny(t)\| &\leq \|a\| + \int_0^t \|f(s,y(s))\|ds + \sum_{0<t_k<t} \|I_k(y(t_k))\| \\ &\leq \|a\| + \int_0^t p(s)\psi(\|y(s)\|)ds + \sum_{0<t_k<t} (c_k\|y(t_k)\| + d_k).\end{aligned}$$

So

$$\|Ny\|_{PC_b} \leq \|a\| + \psi(\|y\|_{PC_b}) \int_0^\infty p(s)ds + \sum_{k=1}^{\infty}(c_k\|y\|_{PC_b} + d_k) < \infty.$$

Step 2 N is continuous

Let $(y_n)_n$ a sequence in $PC_b([0,\infty),\mathbb{R}^n)$ such that $y_n \longrightarrow y$ as $n \to \infty$, it suffices to prove that $Ny_n \longrightarrow Ny$ as $n \to \infty$. For all $t \in [0,\infty)$, we have

$$Ny_n(t) = a + \int_0^t f(s,y_n(s))ds + \sum_{0<t_k<t} I_k(y_n(t_k)),$$

then

$$\begin{aligned}\|Ny_n(t) - Ny(t)\| &\leq \int_0^t \|f(s,y_n(s)) - f(s,y(s))\|\, ds \\ &+ \sum_{k=1}^{m} \|I_k(y_n(t_k)) - I_k(y(t_k))\|\end{aligned}$$

Since, $I_k, k = 1,\cdots,m$ are continuous functions, and f is L^1-Carathéodory function. The Lebesgue dominated convergence theorem implies that

$$\begin{aligned}\|Ny_n - Ny\|_{PC_b} &\leq \int_0^{+\infty} \|f(s,y_n(s)) - f(s,y(s))\|\, ds \\ &+ \sum_{k=1}^{+\infty} \|I_k(y_n(t_k)) - I_k(y(t_k))\| \longrightarrow 0 \ \text{ when } \ n \longrightarrow \infty\end{aligned}$$

so, N is continuous.

Step 3 N is compact:
Let $r > 0$, $B_r := \{y \in PC : \|y\|_{PC_b} \leq r\}$, for to prove that $N(B_r)$ is relatively compact we use the lemma 15.

- $N(B_r)$ is uniformly bounded in PC_b. Let $y \in B_r$, then we have

$$Ny(t) = a + \int_0^t f(s, y(s)ds + \sum_{0<t_k<t} I_k(y(t_k)),\ t \in [0, +\infty). \tag{4.2}$$

Then

$$\begin{aligned}\|Ny(t)\| &\leq \|a\| + \int_0^t \|f(s, y(s))\|ds + \sum_{0<t_k<t} \|I_k(y(t_k))\| \\ &\leq \|a\| + \int_0^t p(s)\psi(\|y(s)\|)ds + \sum_{0<t_k<t} (c_k\|y(t_k)\| + d_k).\end{aligned}$$

So

$$|Ny|_{PC_b} \leq \|a\| + \psi(r)\int_0^\infty p(s)ds + \sum_{k=1}^{\infty}(c_k r + d_k).$$

- $N(B_r)$ is equicontinuous in PC_b. For each $\tau_1, \tau_2 \in [0, \infty)$, $\tau_1 < \tau_2$, and $y \in B_r$, we have

$$\begin{aligned}\|Ny(\tau_2) - Ny(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2} \|f(s, y(s))\|ds + \sum_{\tau_1<t_k<\tau_2} \|I_k(y(t_k))\| \\ &\leq \psi(r)\int_{\tau_1}^{\tau_2} p(s)ds \\ &+ \sum_{\tau_1<t_k<\tau_2} (c_k r + d_k) \to 0 \text{ when } \tau_2 \to \tau_1.\end{aligned}$$

Then, we proved the equicontinuity in the case where $\tau_1 \neq t_i$ and $\tau_2 \neq t_i,\ i = 1, \ldots$.

If $\tau_1 = t_i^-$, let $\varepsilon_0 > 0$ such that $\{t_j : j \neq i\} \cap [t_i - \varepsilon_0, t_i + \varepsilon_0] = \emptyset$. Then for all $0 < \varepsilon < \varepsilon_0$, we have

$$\|Ny(t_i) - Ny(t_i - \varepsilon)\| \leq \int_{t_i-\varepsilon}^{t_i} \|f(s, y(s))\|ds \leq \psi(r)\int_{t_i-\varepsilon}^{t_i} p(s)ds.$$

The terms in the right-hand side tend to zero as $\varepsilon \to 0$.
In the same way we have also the equicontinuity if $t_2 = t_i^+$ $(i = 1, \ldots)$.

- $N(B_r)$ is equiconvergent at ∞

We show that for all $\varepsilon > 0$, there exists $T_\varepsilon > 0$ such that

$$\|Ny(t) - Ny(\infty)\| \leq \varepsilon \text{ for all } t \geq T_\varepsilon \text{ and all } y \in B_r.$$

Let $y \in B_r$, then we have

$$\begin{aligned}\|Ny(t) - Ny(\infty)\| &\leq \int_t^\infty \|f(s, y(s))\|ds + \sum_{t\leq t_k<\infty} \|I_k(y(t_k))\| \\ &\leq \psi(r)\int_t^\infty p(s)ds + \sum_{t\leq t_k<\infty} (c_k r + d_k).\end{aligned}$$

As $\sum_{k=1}^{\infty} c_k < \infty$, $\sum_{k=1}^{\infty} d_k < \infty$ and $p \in L^1([0,\infty),\mathbb{R}_+)$, so there exist k_0 and $T_\varepsilon > 0$ such that

$$\sum_{k=k_0}^{\infty}(c_k r + d_k) \leq \frac{\varepsilon}{2}$$

and

$$\int_t^{\infty} p(s) < \frac{\varepsilon}{2\psi(r)}, \ \forall\, t \geq T_\varepsilon.$$

Then

$$\|Ny(t) - Ny(\infty)\| \leq \varepsilon, \ \forall\, t \geq \max(k_0, T_\varepsilon).$$

Then $N(B_r)$ is equiconvergent. Hence by Lemma 15, the operator N is compact.

Step 4 A priori estimates.

Let $y \in PC_b$ such that $y = \lambda N y$, et $0 < \lambda < 1$. Then

$$y(t) = \lambda\left(a + \int_0^t f(s,y(s))ds + \sum_{0<t_k<t} I_k(y(t_k^-))\right), \quad \text{for } \ t \in [0,\infty).$$

And

$$\|y(t)\| \leq \|a\| + \int_0^t p(s)\psi(\|y(s)\|)ds + \sum_{0<t_k<t}(c_k\|y(t_k)\| + d_k).$$

Let $\alpha(t) = \sup\{\|y(s)\| : s \in [0,t]\}$, we get

$$\alpha(t) \leq \|a\| + \int_0^t p(s)\psi(\alpha(s))ds + \sum_{0<t_k<t}(c_k\alpha(t) + d_k).$$

Then

$$\alpha(t) \leq \frac{1}{1-\sum_{k=1}^{\infty} c_k}\left(\|a\| + \int_0^t p(s)\psi(\alpha(s))ds + \sum_{k=1}^{\infty} d_k\right).$$

Thus

$$\|y(t)\| \leq \alpha(t) \leq \beta(t), \ t \in [0,+\infty[.$$

where

$$\beta(t) = \frac{1}{1-\sum_{k=1}^{\infty} c_k}\left(\|a\| + \int_0^t p(s)\psi(\alpha(s))ds + \sum_{k=1}^{\infty} d_k\right).$$

Hence

$$\beta(0) = \frac{\|a\| + \sum_{k=1}^{\infty} d_k}{1-\sum_{k=1}^{\infty} c_k} \ \text{ and } \ \beta'(t) = \frac{p(t)\psi(\alpha(t))}{1-\sum_{k=1}^{\infty} c_k} \leq \frac{p(t)\psi(\beta(t))}{1-\sum_{k=1}^{\infty} c_k}.$$

By $(\mathcal{H}_2)$, we have for all $t \in [0,\infty)$

$$\int_{\beta(0)}^{\beta(t)} \frac{ds}{\psi(s)} \leq \frac{1}{1-\sum_{k=1}^{\infty} c_k}\int_0^{\infty} p(s)ds < \int_{\beta(0)}^{\infty}\frac{ds}{\psi(s)}.$$

Then

$$\beta(t) \leq \Gamma^{-1}\left(\frac{\|p\|_{L^1}}{1-\sum_{k=1}^{\infty} c_k}\right), \ \text{ for all } \ t \in [0,\infty),$$

where $\Gamma(z) = \displaystyle\int_{\beta(0)}^{z} \frac{du}{\psi(u)}\cdot$

Consequently
$$\|y\|_{PC_b} \leq \Gamma^{-1}\left(\frac{\|p\|_{L^1}}{1-\sum_{k=1}^{\infty} c_k}\right) := \widetilde{M}.$$

Consider the set
$$U := \{y \in PC_b : \|y\|_{PC_b} < \widetilde{M}+1\}.$$

So, the operator $N : U \to PC_b$ is completely continuous, From theorem 16, we deduce that N has a fixed point which is a solution of problem (1.1).

- $S(a)$ is compact.

Let
$$S(a) = \{y \in PC_b : y \text{ solution of the problem (1.1) and } y(0) = a\}.$$

As in step 3, we can prove that, there exists $\widetilde{M} > 0$ such that, for all $y \in S(a)$, we have
$$\|y\|_{PC_b} \leq \widetilde{M}.$$

Since N is completely continuous, then $\overline{N(S(a))}$ is relatively compact in PC_b.
Let $y \in S(a)$; then $y = N(y)$ so $S(a) \subset \overline{N(S(a))}$.

Let $\{y_n : n \in N\} \subset S(a)$ such that $(y_n)_{n\in\mathbb{N}}$ converges to y. Then for all $n \in \mathbb{N}$, we have
$$y_n(t) = a + \int_0^t f(s, y_n(s))ds + \sum_{0<t_k<t} I_k(y_n(t_k)), \quad t \in [0,\infty).$$

Then
$$y_n(t) = (Ny_n)(t), \quad t \in [0,\infty).$$

By the continuity of N, we obtain
$$y_n(t) = N(y_n(t)) \to N(y(t)), \text{ as } n \to \infty,$$

then
$$y(t) = a + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \ t \in [0,\infty).$$

So, $y \in S(a)$. This implies that $S(a)$ is closed. Hence, we deduce that $S(a)$ is compact in PC_b.

- The solution set is R_δ.

Clair that, $FixN = S(a)$, and by the previous step 4, there exists $\widetilde{M} > 0$ such that for every $y \in S(a)$, we have
$$\|y\|_{PC_b} \leq \widetilde{M}.$$

Let $\widetilde{f} : [0,\infty) \times R^n \to R^n$ be a map defined by
$$\widetilde{f}(t,x) = \begin{cases} f(t,x) & \text{if } \|x\| \leq \widetilde{M}, \\ f(t, \frac{\widetilde{M}x}{\|x\|}) & \text{if } \|x\| \geq \widetilde{M}, \end{cases}$$

and the function $\widetilde{I}_k : \mathbb{R}^n \to \mathbb{R}^n$ defined by
$$\widetilde{I}_k(y(t)) = \begin{cases} I_k(x) & \text{if } \|x\| \leq \widetilde{M}, \\ I_k(\frac{\widetilde{M}x}{\|x\|}) & \text{if } \|x\| \geq \widetilde{M}. \end{cases}$$

f is a L^1-Carathéodory function, then $\widetilde{f}$ is also L^1-Carathéodory, and there exist $h \in L^1([0,\infty),\mathbb{R}_+)$ such that

$$\left\|\widetilde{f}(t,x)\right\| \leq h(t); \quad \text{a.e. } t \in [0,\infty); \quad \text{and} \quad x \in \mathbb{R}^n. \tag{4.3}$$

We consider the following modified problem

$$\dot{y}(t) = \widetilde{f}(t, y(t)), \quad t \in [0,\infty)/\{t_1, \cdots, t_m\},$$

$$y(t_k^+) - y(t_k^-) = \widetilde{I}_k(y(t_k^-)), k = 1, 2, \cdots$$

$$y(0) = a.$$

We can easily prove that $FixN = Fix\widetilde{N}$, where $\widetilde{N} : PC_b([0,\infty),\mathbb{R}^n) \longrightarrow PC_b([0,\infty),\mathbb{R}^n)$ defined by

$$\widetilde{N}(y)(t) = a + \int_0^t \widetilde{f}(s, y(s))ds + \sum_{0<t_k<t} \widetilde{I}_k(y(t_k)), \quad t \in [0,\infty). \tag{4.4}$$

By inequality (4.3) and continuity of I_k, we have

$$\begin{aligned} \left\|\widetilde{N}(y)\right\|_{PC} &\leq \|a\| + \|h\|_{L^1} + \sum_{k=1}^{+\infty} \left(c_k \|y\|_{PC_b} + d_k\right) \\ &\leq \|a\| + \|h\|_{L^1} + \sum_{k=1}^{+\infty} \left(c_k \widetilde{M} + d_k\right) := r, \end{aligned}$$

then $\widetilde{N}$ is uniformly bounded.

We can easily prove that the function $\mathcal{M}$ defined by $\mathcal{M}(y) = y - \widetilde{N}(y)$ is a specific function. Also we have $\widetilde{N}$ is compact, so by the Lasota Yorke theorem 8, we can easily prove that the conditions of Theorem 10 are satisfied, then the set $\mathcal{M}^{-1}(0) = Fix\widetilde{N} = S(a)$ is R_δ, and it is also acyclic and those by the lemma (7).

- The solution operator S is $u.s.c.$

The graph of S is closed.

First we show that S has a closed graph. Let G_S be a graph of S defined by

$$G_S = \{(x,y) \in \mathbb{R}^n \times PC | y \in S(x)\}.$$

Let $((x_q, y_q))_q$ is a sequence in G_S, and let $(x_q, y_q) \to (x, y)$ when $q \to \infty$.
As $y_q \in S(x_q)$, then we have

$$y_q(t) = x_q + \int_0^t f(s, y_q(s))ds + \sum_{0<t_k<t} I_k(y_q(t_k)), \ t \in [0,\infty).$$

Let

$$Z(t) = x + \int_0^t f(s, y(s))ds + \sum_{0<t_k<t} I_k(y(t_k)), \quad t \in [0,\infty).$$

Let $t \in [0,\infty)$, we have

$$\begin{aligned}\|y_q(t)-Z(t)\| &\leq \|x_n-x\|+\int_0^t \|f(s,y_q(s))-f(s,y(s))\|\,ds\\ &+ \sum_{0<t_k<t}\|I_k(y_q(t))-I_k(y(t))\|\\ &\leq \|x_n-x\|+\int_0^{+\infty} \|f(s,y_q(s))-f(s,y(s))\|\,ds\\ &+ \sum_{k=1}^{+\infty}\|I_k(y_q(t))-I_k(y(t))\|,\end{aligned}$$

by Lebesgue dominated convergence theorem, we have

$$\|y_q(t)-Z(t)\| \longrightarrow 0 \ \text{ when } \ q\longrightarrow\infty.$$

Hence, $\lim_{q\longrightarrow 0} y_q = y = Z \in S(x)$.

S transforms every bounded set in a relatively compact set
Let $r>0, B_r := \{y\in PC_b : \|y\|\leq r\}$.

(a) $S(B_r)$ is uniformly bounded.
Let $y \in S(B_r)$, then there exists $x\in B_r$ such that

$$y(t)=x+\int_0^t f(s,y(s))ds+\sum_{0<t_k<t} I_k(y(t_k)),\ t\in[0,\infty).$$

As in step 4, we can prove that there exists $\widetilde{M}>0$ such that

$$\|y\|_{PC_b}\leq \widetilde{M}.$$

(b) $S(B_r)$ is equicontinuous.
We let $\tau_1,\tau_2\in[0,+\infty), \tau_1<\tau_2$, and $y\in B_r$, then we have

$$\begin{aligned}\|Ny(\tau_2)-Ny(\tau_1)\| &\leq \int_{\tau_1}^{\tau_2}\|f(s,y(s))\|ds+\sum_{\tau_1<t_k<\tau_2}\|I_k(y(t_k))\|\\ &\leq \int_{\tau_1}^{\tau_2} p(s)\psi(\|y(s)\|)ds+\sum_{\tau_1<t_k<\tau_2}(c_k\|y(t_k)\|+d_k)\\ &\leq \psi(\widetilde{M})\int_{\tau_1}^{\tau_2}p(s)ds\\ &+\sum_{\tau_1<t_k<\tau_2}(c_k\widetilde{M}+d_k)\to 0 \text{ quand } \tau_1\to\tau_2.\end{aligned}$$

(c) $S(B_r)$ is equiconvergent at ∞.
i.e. for all $\varepsilon>0$, there exist $T_\varepsilon>0$ such that $\|y(t)-y(\infty)\|\leq\varepsilon$ for all $t\geq T_\varepsilon$ and all $y\in S(B)$.

We take $y \in S(B)$ then there exist $x \in B$, and we have

$$\begin{aligned}
\|y(t) - y(\infty)\| &\leq \int_t^\infty \|f(s, y(s))\| ds + \sum_{t \leq t_k < \infty} \|I_k(y(t_k))\| \\
&\leq \int_t^\infty p(s)\psi(\|y(s)\|) ds + \sum_{t \leq t_k < \infty} (c_k \|y(t_k)\| + d_k) \\
&\leq \psi(\widetilde{M}) \int_t^\infty p(s) ds \\
&+ \sum_{t \leq t_k < \infty} (c_k \widetilde{M} + d_k) \to 0 \text{ when } t \to \infty.
\end{aligned}$$

So, the set $\overline{S(B_r)}$ is compact. Hence we obtain that the operator S is locally compact, and S has a closed graph, then S is *u.s.c.* □

5 Example

Consider the problem:

$$\begin{cases} y' = \frac{1}{100}(1+y)^{\frac{2}{3}}, \quad t \in J = [0, \infty), \ t \neq k; \\ \Delta y(k) = \frac{1}{8^k}|y(k)|, \quad k \in \mathbb{N}, \\ y(0) = a \in \mathbb{R} \end{cases} \tag{5.1}$$

$$f(t, x) = \frac{1}{100}(1+x)^{\frac{2}{3}},$$

$$I_k(x) = \frac{1}{8^k}|x| \ k \in \mathbb{N}.$$

For every $x \in \mathbb{R}$, we have

$$|f(t, x)| \leq \frac{1}{100}\left(1 + \frac{2}{3}|x|\right),$$

and

$$\int_1^\infty \frac{1}{100(1 + \frac{2}{3}u)} du = \infty.$$

Hence the condition (H_2) holds

Also for all $u, \bar{u}, v, \bar{v} \in \mathbb{R}^+$, we have

$$|I_k(u) - I_k(\bar{u})| \leq \frac{1}{8^k}|u - \bar{u}|, \ k = 1, 2, \cdots,$$

and

$$\sum_{k=1}^\infty \frac{1}{8^k} < 1.$$

Thus (H_1) holds.

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