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Riemann-Stieltjes composition operators between weighted Banach spaces of holomorphic functions

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Abstract
We characterize boundedness and compactness of Riemann-Stieltjes composition operators acting between weighted Banach spaces of holomorphic functions.

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1 Introduction

Let $\mathbb{D}$ denote the open unit disk in the complex plane and $H(\mathbb{D})$ the set of all analytic functions on $\mathbb{D}$. Moreover, we consider an analytic self-map φ of $\mathbb{D}$ as well as an analytic map $g : \mathbb{D} \to \mathbb{C}$. Such maps induce the following *Riemann-Stieltjes composition operator*

$$I_{g,\varphi} : H(\mathbb{D}) \to H(\mathbb{D}),\ [I_{g,\varphi} f](z) = \int_0^1 f(\varphi(tz))g'(tz)z\, dt.$$

Recently, this type of operator has been of great interest, see e.g. [1], [2], [3], [6], [9], [10].

In this article we study operators of the above type acting in the following setting: Let v be a strictly positive, bounded and continuous function (*weight*) on $\mathbb{D}$. Then the *weighted Banach space of holomorphic functions* is defined by

$$H_v^\infty := \left\{ f \in H(\mathbb{D});\ \|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| < \infty \right\}.$$

Endowed with the weighted sup-norm $\|.\|_v$ this is a Banach space. Such spaces occur naturally in a variety of problems. For more information on that topic we refer the reader to the articles [4] and [7] and the references therein.

In [6] Li characterized boundedness and compactness of operators $I_{g,\varphi}$ acting between weighted Bergman spaces and weighed Bloch spaces, both generated by standard weights. In [8] we generalized his results to a more general setting. In this article we continue this branch of research by considering operators $I_{g,\varphi}$ acting between different weighted Banach spaces of holomorphic functions. We give a characterization of the boundedness and compactness of such operators that only involve the given weights as well as the holomorphic map g as well as the symbol φ.

2 Basics

Let ν be a holomorphic function on $\mathbb{D}$ that is additionally non-vanishing and strictly positive on $[0,1[$ and satisfies $\lim_{r\to 1}\nu(r)=0$. Then we define the corresponding weight by

$$v(z)=\nu(|z|) \text{ for every } z\in\mathbb{D}.$$

Moreover, we assume that $|\nu(z)|\geq \nu(|z|)$ for every $z\in\mathbb{D}$. The most relevant weights, such as the standard weights, the logarithmic weights and the exponential weights satisfy these conditions.

Such weights may be written as

$$v(z)=\min\{|\nu(\lambda z)|,\ |\lambda|=1\}.$$

For a better understanding we will give the proof. First, we use polar coordinates

$$\min\{|\nu(\lambda z)|,\ |\lambda|=1\}=\min\{|\nu(\lambda re^{i\Theta})|,\ |\lambda|=1\}\leq|\nu(e^{-i\Theta}re^{i\Theta})|=|\nu(r)|=\nu(|z|)=v(z).$$

On the other hand, for every $\lambda\in\partial\mathbb{D}$ we obtain for every $z\in\mathbb{D}$

$$|\nu(\lambda z)|\geq\nu(|\lambda z|)=\nu(|z|)=v(z).$$

We close this section with stating a very useful lemma, which can be easily derived from [5] Proposition 3.11.

Lemma 1. *Let v and w be weights. Then the operator $I_{g,\varphi}:H_v^\infty\to H_w^\infty$ is compact if and only if it is bounded and for every bounded sequence $(f_n)_n$ in H_v^∞ which converges to zero uniformly on the compact subsets of $\mathbb{D}$, $I_{g,\varphi}f_n$ tends to zero in H_w^∞ if $n\to\infty$.*

3 Results

Proposition 2. *The operator $I_{g,\varphi}:H_v^\infty\to H_w^\infty$ is bounded if and only if* $\sup_{z\in\mathbb{D}}w(z)\left|\int_0^1\frac{g'(tz)}{v(\varphi(tz))}z\,dt\right|<\infty$.

Proof. First, we assume that the operator $I_{g,\varphi}:H_v^\infty\to H_w^\infty$ is bounded. As we know, the weight v may be presented as

$$v(z)=\min\left\{|\nu(\lambda z)|,\ |\lambda|=1\right\},$$

where ν is a holomorphic function. Now, for fixed $\lambda\in\partial\mathbb{D}$ let

$$h_\lambda(z)=\frac{1}{\nu(\lambda z)}$$

for every $z\in\mathbb{D}$. Then $\|h_\lambda\|_v=\sup_{z\in\mathbb{D}}\frac{v(z)}{|\nu(\lambda z)|}\leq\sup_{z\in\mathbb{D}}\frac{v(z)}{\min_{|\lambda|=1}|\nu(\lambda z)|}=\sup_{z\in\mathbb{D}}\frac{v(z)}{v(z)}=1$ for every $\lambda\in\partial\mathbb{D}$. Now, we arrive at

$$\sup_{z\in\mathbb{D}}w(z)\left|\int_0^1\frac{g'(tz)z}{\nu(\lambda\varphi(tz))}\,dt\right|=\sup_{z\in\mathbb{D}}w(z)\left|\int_0^1 h_\lambda(\varphi(tz))g'(tz)z\,dt\right|\leq\|I_{g,\varphi}\|\|h_\lambda\|_v$$
$$\leq\|I_{g,\varphi}\|<\infty$$

for every $\lambda\in\partial\mathbb{D}$. Hence, since $\lambda\in\partial\mathbb{D}$ is arbitrary, we obtain

$$\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\,dt\right| < \infty$$

as desired.

Conversely, we assume that $\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\,dt\right| < \infty$. For every $f \in H_v^\infty$ we have

$$\begin{aligned}\|I_{g,\varphi}f\|_w &= \sup_{z\in\mathbb{D}}\left|\int_0^1 f(\varphi(tz))g'(tz)z\,dt\right| = \sup_{z\in\mathbb{D}}\left|\int_0^1 \frac{f(\varphi(tz))}{v(\varphi(tz))}v(\varphi(tz))g'(tz)z\,dt\right| \\ &\le \sup_{z\in\mathbb{D}} w(z) \sup_{t\in[0,1]} v(\varphi(tz))|f(\varphi(tz))| \left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))}z\,dt\right| \\ &\le \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))}z\,dt\right| \|f\|_v.\end{aligned}$$

Hence the operator $I_{g,\varphi}$ must be bounded. □

Remark 3. Let us assume that $I_{g,\varphi} : H_v^\infty \to H_w^\infty$ is bounded. Then

$$\|I_{g,\varphi}\| = \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))}z\,dt\right|.$$

First, for fixed $\lambda \in \partial\mathbb{D}$ we consider

$$h_\lambda(z) = \frac{1}{\nu(\lambda z)}$$

for every $z \in \mathbb{D}$. We have seen that $\|h_\lambda\|_v \le 1$. Moreover, the proof of Proposition 4 shows that

$$\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)z}{\nu(\lambda\varphi(tz))}\,dt\right| \le \|I_{g,\varphi}\|.$$

Since $\lambda \in \partial\mathbb{D}$ was arbitrary, we also obtain

$$\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)z}{\min_{|\lambda|=1}|\nu(\lambda\varphi(tz))|}\,dt\right| = \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)z}{v(\varphi(tz))}\,dt\right| \le \|I_{g,\varphi}\|.$$

On the other hand the proof of Proposition 4 yields for every $f \in H_v^\infty$

$$\|I_{g,\varphi}f\|_w \le \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))}z\,dt\right| \|f\|_v.$$

but by definition we have $\|I_{g,\varphi}\| = \inf\{M \ge 0,\ \|I_{g,\varphi}f\|_w \le M\|f\|_v \text{ for every } f \in H_v^\infty\}$. Hence $\|I_{g,\varphi}\| \le \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))}z\,dt\right|$ and the claim follows.

Proposition 4. *The operator $I_{g,\varphi} : H_v^\infty \to H_w^\infty$ is compact if and only if the following conditions are satisfied:*

(a) $\limsup_{|z|\to 1} w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))}z\,dt\right| = 0,$

(b) $\sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 g'(tz)z\,dt\right| < \infty.$

Proof. First, we assume that the conditions (a) and (b) are fulfilled. Let $(f_n)_n$ be a bounded sequence in H_v^∞ that converges to 0 uniformly on the compact subsets of $\mathbb{D}$ such that $\|f_n\|_v \leq M$ for every $n \in \mathbb{N}$. By hypothesis, for every $\varepsilon > 0$, there is $r > 0$ such that if $|z| > r$, then

$$w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| < \varepsilon.$$

Hence

$$w(z)|I_{g,\varphi} f_n(z)| \leq w(z)\left|\int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z\, dt\right| \|f_n\|_v < \varepsilon M$$

for every $n \in \mathbb{N}$ and every $z \in \mathbb{D}$ with $|z| > r$.

On the other hand, if $|z| \leq r$ there must be $0 < R < 1$ such that $|\varphi(z)| \leq R$. Since $f_n \to 0$ uniformly on $\{u;\ |u| \leq r\}$, we can find $n_0 \in \mathbb{N}$ such that if $|\varphi(z)| \leq R$ and $n \geq n_0$ then $|f_n(\varphi(z))| < \varepsilon$. Hence, we arrive at

$$w(z)|I_{g,\varphi} f_n(z)| = w(z)\left|\int_0^1 g'(tz) f_n(\varphi(tz)) z\, dt\right| < \varepsilon w(z)\left|\int_0^1 g'(tz) z\, dt\right| \leq \varepsilon N,$$

where $N = \sup_{z\in\mathbb{D}} w(z)\left|\int_0^1 g'(tz) z\, dt\right| < \infty$.

Conversely, we assume to the contrary that the condition (a) does not hold. Then there is a sequence $(z_n)_n \subset \mathbb{D}$ with $|z_n| \to 1$ as $n \to \infty$ such that

$$w(z_n)\left|\int_0^1 \frac{g'(tz_n)}{v(\varphi(tz_n))} z_n\, dt\right| \geq \alpha > 0$$

for every $n \in \mathbb{N}$. Next, we can choose an increasing sequence $(k(n))_n$ of natural numbers with $k(n) \to \infty$ such that $z_n^{k(n)} \geq \frac{1}{2}$ for every $n \in \mathbb{N}$ and for every $n \in \mathbb{N}$ we select $\lambda_n \in \partial\mathbb{D}$ such that $\frac{1}{\nu(\lambda_n \varphi(tz_n))} = \frac{1}{v(\varphi(tz_n))}$. Next, we consider the functions

$$h_{n,\lambda_n}(z) := \frac{z^{k(n)}}{\nu(\lambda_n z)}$$

for every $z \in \mathbb{D}$ and every $n \in \mathbb{N}$. Then obviously, $(h_{n,\lambda_n})_n \subset H_v^\infty$ is bounded, since $\|h_{n,\lambda_n}\|_v = \sup_{z\in\mathbb{D}} v(z)\frac{|z|^{k(n)}}{|\nu(\lambda_n z)|} \leq \sup_{z\in\mathbb{D}} |z|^{k(n)} \leq 1$. Moreover, $h_{n,\lambda_n} \to 0$ pointwise because of the factor $z^{k(n)}$. Finally,

$$\begin{aligned} \|I_{g,\varphi} h_{n,\lambda_n}\|_v &\geq w(z_n)\left|\int_0^1 g'(tz_n) h_{n,\lambda_n}(z_n) z_n\, dt\right| \geq w(z_n)\left|\int_0^1 g'(tz_n)\frac{z_n^{k(n)}}{\nu(\lambda_n\varphi(tz_n))} z_n\, dt\right| \\ &\geq \frac{1}{2} w(z_n)\left|\int_0^1 \frac{g'(tz_n)}{v(\varphi(tz_n))} z_n\, dt\right| \geq \frac{\alpha}{2} \end{aligned}$$

for every $n \in \mathbb{N}$ which is a contradiction.

It remains to show that condition (b) is satisfied. Since the operator $I_{g,\varphi}: H_v^\infty \to H_w^\infty$ is compact it also must be bounded. Now, take $f(z) = 1$ for every $z \in \mathbb{D}$. Then we have

$$\|f\|_v = \sup_{z\in\mathbb{D}} v(z)|f(z)| = \sup_{z\in\mathbb{D}} v(z) < \infty$$

since the weight is bounded by hypothesis. Thus, $f \in H_v^\infty$. Finally, we arrive at

$$\|I_{g,\varphi} f\|_w = \sup_{z\in\mathbb{D}} w(z) \left| \int_0^1 f(\varphi(tz)) g'(tz) z \, dt \right| = \sup_{z\in\mathbb{D}} w(z) \left| \int_0^1 g'(tz) z \, dt \right| \leq \|I_{g,\varphi}\| \|f\|_v.$$

□

Example 5. (a) We select $v(z) = w(z) = 1 - |z|$ as well as $\varphi(z) = g(z) = z$ for every $z \in \mathbb{D}$. Thus, we obtain

$$\sup_{z\in\mathbb{D}} w(z) \left| \int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z \, dt \right| = \sup_{z\in\mathbb{D}} (1 - |z|) \left| \int_0^1 \frac{z}{1 - t|z|} \, dt \right|$$
$$= \sup_{z\in\mathbb{D}} (1 - |z|) \left| [\ln(1 - t|z|)]_0^1 \right| = \sup_{z\in\mathbb{D}} (1 - |z|) \left| \ln(1 - |z|) \right| < \infty.$$

Hence the corresponding operator $I_{g,\varphi}$ must be bounded. Moreover, obviously, $\limsup_{|z|\to 1} w(z) \left| \int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z \, dt \right| = \limsup_{|z|\to 1} (1 - |z|) \left| \ln(1 - |z|) \right| = 0$ and $\sup_{z\in\mathbb{D}} w(z) \left| \int_0^1 g'(tz) z \, dt \right| = \sup_{z\in\mathbb{D}} (1 - |z|) \left| \int_0^1 z \, dt \right| = \sup_{z\in\mathbb{D}} (1 - |z|)|z| \leq 1$. Thus, the operator must be compact.

(b) We choose $w(z) = 1 - |z|, v(z) = (1 - |z|)^2$ and $\varphi(z) = g(z) = z$ for every $z \in \mathbb{D}$. Then, easy calculations show that

$$\sup_{z\in\mathbb{D}} w(z) \left| \int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z \, dt \right| = \sup_{z\in\mathbb{D}} (1 - |z|) \left| \int_0^1 \frac{z}{(1 - t|z|)^2} \, dt \right|$$
$$= \sup_{z\in\mathbb{D}} (1 - |z|) \left[\frac{1}{1 - |z|} - 1 \right] \leq 1.$$

But obviously $\limsup_{|z|\to 1} w(z) \left| \int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z \, dt \right| = \limsup_{|z|\to 1} (1 - |z|) \left[\frac{1}{1-|z|} - 1 \right] = \limsup_{|z|\to 1} [1 - 1 + |z|] = 1$. Hence the operator ist not compact.

(c) We consider $w(z) = 1 - |z|$, $v(z) = (1 - |z|)^3$ and $\varphi(z) = g(z) = z$ for every $z \in \mathbb{D}$. Then

$$\sup_{z\in\mathbb{D}} w(z) \left| \int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z \, dt \right| = \sup_{z\in\mathbb{D}} (1 - |z|) \left| \int_0^1 \frac{z}{(1 - t|z|)^3} \, dt \right|$$
$$= \sup_{z\in\mathbb{D}} \frac{1}{2} \left[\frac{1}{1 - |z|} - 1 + |z| \right] = \infty.$$

Hence, the corresponding operator is not bounded.

(d) Select $w(z) = v(z) = 1 - |z|$ as well as $\varphi(z) = z$ and $g(z) = \frac{1}{(1-z)^2}$ for every $z \in \mathbb{D}$. Then obviously $g'(z) = \frac{2}{(1-z)^3}$ for every $z \in \mathbb{D}$. Moroever,

$$\sup_{z\in\mathbb{D}} w(z) \left| \int_0^1 \frac{g'(tz)}{v(\varphi(tz))} z \, dt \right| = \sup_{z\in\mathbb{D}} (1 - |z|) \left| \int_0^1 \frac{2z}{(1 - tz)^3 (1 - t|z|)} \, dt \right|$$
$$\geq \sup_{z\in\mathbb{D}} 2(1 - |z|) \left| \int_0^1 \frac{z}{(1 - tz)^3} \, dt \right| = \sup_{z\in\mathbb{D}} (1 - |z|) \left| \frac{1}{(1 - z)^2} - 1 \right| = \infty.$$

Hence the operator is not bounded.

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