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TWO-PARAMETER CHAOS

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Abstract. Let I be a real compact interval and $0 \leq \alpha \leq \beta$ be real numbers smaller than the length of I. A continuous function f from I into itself is said to be generically or densely (α, β)-chaotic if the set of all points $[x, y]$, for which $\liminf_{n\to\infty} |f^n(x) - f^n(y)| \leq \alpha$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \beta$, is residual or dense in $I \times I$, respectively. In the paper such functions are characterized in terms of behaviour of subintervals of I under iterates of f provided $\alpha > 0$ (see [2] and [3] for $\alpha = 0$).

In the paper a function will always be a function belonging to the space $C^o(I, I)$ of all continuous maps of a real compact interval I into itself, endowed with the topology of uniform convergence. An interval will always be a nondegenerate interval lying in I. If J is an interval then diam J is its length. If $A, B \subset I$ then dist(A, B) =inf $\{|x - y| : x \in A, y \in B\}$.

The k-th iterate of a function f is denoted by f^k. For a function f and α, β with $0 \leq \alpha \leq \beta <$diam I define the following planar sets:

$$C_1(f, \alpha) = \{[x, y] \in I^2 : \liminf_{n\to\infty} |f^n(x) - f^n(y)| \leq \alpha\},$$
$$C_2(f, \beta) = \{[x, y] \in I^2 : \limsup_{n\to\infty} |f^n(x) - f^n(y)| > \beta\},$$
$$C(f, \alpha, \beta) = C_1(f, \alpha) \cap C_2(f, \beta).$$

Due to A. Lasota, a function f is called generically chaotic if the set $C(f, 0, 0)$ is residual in $I \times I$ (cf. [1]).

Suppose we are studying some physical or biological system on which we make measurements at regular intervals. If we are just measuring a single quantity then the n-th measurement can be represented by a real number

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x_n. A very simple mathematical model of such a system is obtained by assuming that x_{n+1} is only a function of x_n, and that this function does not depend on n. That is, we assume there is a function f so that $x_{n+1} = f(x_n)$ for all $n \geq 0$. In connection with the definition of generic chaos we must realize that from the physical point of view we are not able to check for example whether $\liminf_{n\to\infty} |f^n(x) - f^n(y)|$ is zero or not. In fact, even if we admit that we are able to make infinitely many measurements, we are restricted by the accuracy of our measuring apparatus. So it seems that the following notion of (α, β)-chaos could have a physical sense.

Definition 1. A function $f \in C^o(I, I)$ is said to be generically or densely (α, β)-chaotic if the set $C(f, \alpha, \beta)$ is residual or dense in $I \times I$, respectively.

So the generic $(0, 0)$-chaos is the same as the generic chaos in the sense of A. Lasota. The generically (α, β)-chaotic functions and the densely (α, β)-chaotic functions were characterized in [2] and [3] provided $\alpha = 0$. In the present paper we show that an analogous characterization holds if $\alpha > 0$ (see Theorem 4). Further, we show that the set of all such maps is nowhere dense in the space $C^o(I, I)$.

The following lemma is a generalization of Lemma 4.3 from [2].

Lemma 2. *Let $f \in C^o(I, I)$ and $0 \leq \alpha \leq$diam I. Then the following three conditions are equivalent:*

(i) *$C_1(f, \alpha)$ is residual,*
(ii) *$C_1(f, \alpha)$ is dense,*
(iii) *for every two intervals J_1, J_2, $\liminf\limits_{n\to\infty}$ dist $(f^n(J_1), f^n(J_2)) \leq \alpha$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are obvious. We are going to prove (iii) $\Longrightarrow$ (i). So let (iii) be fulfilled. Since $C_1(f, \alpha) = = \bigcap_{n=1}^{\infty} L(n, \alpha + 1/n)$ where $L(n, \alpha + 1/n) = \{[x, y] \in I^2 : \inf_{k\geqslant n}|f^k(x) - f^k(y)| < \alpha + 1/n\}$ are open sets, it suffices to prove that for every n, $L(n, \alpha + 1/n)$ is dense in I^2. So take any positive integer n and intervals J_1, J_2. We prove that $L(n, \alpha + 1/n) \cap (J_1 \times J_2) \neq \emptyset$. From (iii) it follows that there exists $k \geq n$ with dist $(f^k(J_1), f^k(J_2)) < \alpha + 1/n$. This implies the existence of points $x \in J_1$, $y \in J_2$ such that $|f^k(x) - f^k(y)| < \alpha + 1/n$. Hence $[x, y] \in L(n, \alpha + 1/n)$ and the proof is complete.

The following lemma is a part of Lemma 4.16 in [2].

Lemma 3. *Let $f \in C^o(I, I)$ and $0 < \beta \leq$diam I. Then the following three conditions are equivalent:*

(i) *$C_2(f, \beta)$ is residual,*

(ii) $C_2(f,\beta)$ *is dense,*

(iii) *for every interval* J, $\limsup_{n\to\infty} \text{diam } f^n(J) > \beta$.

Since the intersection of two residual sets is a residual set, from Lemma 2 and Lemma 3 we immediately get

Theorem 4. *Let* $f \in C^o(I,I)$ *and* $0 < \alpha \le \beta <$diam I. *Then the following three conditions are equivalent:*

(i) f *is generically* (α,β)*-chaotic,*

(ii) f *is densely* (α,β)*-chaotic,*

(iii) *for every two intervals* J_1, J_2, $\liminf_{n\to\infty} \text{ dist } (f^n(J_1), f^n(J_2)) \le \alpha$ *and for every interval* J, $\limsup_{n\to\infty} \text{diam } f^n(J) > \beta$.

From [2] it is known that if f is generically chaotic then it is generically $(0,\varepsilon)$-chaotic for some $\varepsilon > 0$ and so it is generically (α,β)-chaotic for any α, β with $0 \le \alpha \le \beta \le \varepsilon$. On the other hand, the following example shows that there are functions which are generically (α,β) -chaotic for some $0 < \alpha \le \beta <$ diam I without being generically chaotic.

Example 5. Take $I = [0,1]$ and numbers $0 < \alpha \le \beta < L$ and R with $L+\alpha+R = 1$. Let $f(0) = L+\alpha$, $f(L) = 1$, $f(L+\alpha) = 0$, $f(1-R/2) = L$, $f(1) = 0$ and let f be linear on each of the intervals $[0,L]$, $[L,L+\alpha]$, $[L+\alpha, 1-R/2]$ and $[1-R/2, 1]$. Then, using Theorem 4 it is easy to see that f is generically (α,β)-chaotic although it is not generically chaotic.

Now denote the set of all densely or generically (α,β)-chaotic maps from $C^o(I,I)$ by $D(\alpha,\beta)$ or $G(\alpha,\beta)$, respectively. Further denote $D = \bigcup\{D(\alpha,\beta) : 0 \le \alpha \le \beta <\text{diam } I\}$ and $G = \bigcup\{G(\alpha,\beta) : 0 \le \alpha \le \beta <\text{diam } I\}$. Clearly, $G \subset D$. We have (cf. Theorem 1.5 in [2])

Theorem 6. *The set* D *is nowhere dense in* $C^o(I,I)$.

Proof. Let $B(f,\varepsilon)$ be an open ball in $C^o(I,I)$. Since f has at least one fixed point x_o, it is possible to define a function $g \in B(f,\varepsilon)$ such that for some $x_1 < x_2$ very close to x_o and for some small $\eta > 0$ the intervals $J_i = [x_i - \eta, x_i + \eta]$, i=1,2 are disjoint and $g(J_i) = \{x_i\}$, i=1,2. Denote $M = [x_1 + \eta, x_2 - \eta]$. We may assume that diam $M >$ diam $J_i = 2\eta$, otherwise we can take smaller η. Take $\delta > 0$ such that simultaneously $B(g,\delta) \subset B(f,\varepsilon)$ and for every $h \in B(g,\delta)$, $h(J_i) \subset J_i$,i=1,2. Now suppose that there is a map $h \in B(g,\delta)$ and α, β with $0 \le \alpha \le \beta <$diam I such that h is densely (α,β)-chaotic. Since $\limsup_{n\to\infty} |f^n(x) - f^n(y)| \le$ diam J_1 whenever

$[x, y] \in J_1 \times J_1$, we have diam $J_1 > \beta$. Since $\liminf_{n\to\infty} |f^n(x) - f^n(y)| \geq$ diam M whenever $[x, y] \in J_1 \times J_2$, we have diam $M \leq \alpha$. The inequality diam M $>$ diam J_1 gives $\alpha > \beta$ which is a contradiction with the fact that $\alpha \leq \beta$. So $B(g, \delta) \cap D = \emptyset$. The proof is finished.

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NOTES ON THE CONGRUENCE LATTICES OF ALGEBRAS

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Abstract. From [1] and [3] it follows that for any algebra $\mathcal{A}$ there exists a groupoid $\mathcal{G}$ for which $Con\mathcal{A} + 1 \simeq Con\mathcal{G}$ (where $Con\mathcal{A} + 1$ is the ordinal sum). In the paper we directly define the operation of groupoid $\mathcal{G}$ by using the operations of algebra $\mathcal{A}$. From this construction it follows that for any binary countable algebra $\mathcal{A}$ there exists a groupoid $\mathcal{G}$ for which $Con\mathcal{A} + 1 \simeq Con\mathcal{G}$ and moreover, $\mathcal{G}$ has no nontrivial subgroupoid and no nontrivial automorphism. We also present (in Theorem 3) some results related to the lattice of subuniverses and to the automorphism group of $\mathcal{A}$.

Throughout this paper Ord denotes the class of all ordinal numbers and N denotes the set of all natural numbers. An algebra (A, F) will be often denoted by $\mathcal{A}$. Further, $Con\mathcal{A}$, $Sub\mathcal{A}$ and $Aut\mathcal{A}$ denote the congruence lattice, the lattice of subuniverses and the automorphism group for the algebra $\mathcal{A}$, respectively.

Let (A, F) be a binary algebra

$$A = \{a_k; k < \alpha, k \in Ord\}, \quad \alpha \in Ord,$$

$$F = \{f_k; k < \beta, k \in Ord\}, \quad \beta \in Ord,$$

let γ be a limit ordinal such that $\gamma \geq max\{\alpha, \beta\}$ and let

$$M = \{k \in Ord; k < \gamma\}, \quad S = \{-3, -2, -1\} \cup M.$$

We consider the usual ordering

$$-3 < -2 < -1 < 0 < 1 < 2 < 3 < ...,$$

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on the set S. Now we put

$$G = \{(a_r, s); r < \alpha, r \in M, s \in S\}.$$

To define a groupoid operation o on G we consider the following cases for an element

$(a_i, r)o(a_j, s), \quad i, j < \alpha, \quad i, j \in M, \quad r, s \in S.$

I. $r = s$. a) $0 \leq r < \beta$

$$(a_i, r)o(a_j, s) = (f_r(a_i, a_j), r + 1), \tag{1}$$

b) either $r \in \{-3, -2, -1\}$ or $r \geq \beta$

$$(a_i, r)o(a_j, s) = (a_i, r + 1). \tag{2}$$

II. $r < s$. a) $r + 1 = s$

$$(a_i, r)o(a_j, r + 1) = (a_i, -2), \tag{3}$$

b) $r + 1 < s$

$$(a_i, r)o(a_j, s) = (a_i, -3). \tag{4}$$

III. $r > s$. a) $s = -3$

$$(a_i, r)o(a_j, -3) = (a_j, r), \tag{5}$$

b) $s = -2$

$$(a_i, r)o(a_j, -2) = (a_j, -3), \tag{6}$$

c) $s = -1, \quad 0 \leq r < \alpha$

$$(a_i, r)o(a_j, -1) = (a_r, -3), \tag{7}$$

d) otherwise

$$(a_i, r)o(a_j, s) = (a_i, r + 1). \tag{8}$$

Lemma 1. *Let (A, F) be a binary algebra and (G, o) be the groupoid whose operation is defined by (1) - (8). For any congruence relation Φ of (G, o) the following properties are satisfied:*

(i) $$(x,k)\Phi(y,k) \Longleftrightarrow (x,s)\Phi(y,s)$$

for all $k, s \in S$,

(ii) $$(x,k)\Phi(y,s), \quad k \neq s \quad \Rightarrow \quad \Phi = G^2.$$

Proof. (i). From $(x,k)\Phi(y,k)$ it follows $(x,k)o(x,k+2)\Phi(y,k)o(x,k+2)$, i.e., by (4) $(x,-3)\Phi(y,-3)$. Thus, $(x,s)o(x,-3)\Phi(x,s)o(y,-3)$ and so by (5) $(x,s)\Phi(y,s)$ holds for any $s \neq -3$.

(ii). Let $(x,k)\Phi(y,s)$, $k \neq s$. We may assume that $k < s$. By hypothesis $(x,k)o(x,s+1)\Phi(y,s)o(x,s+1)$, i.e., by (4) and (3),

(a) $$(x,-3)\Phi(y,-2).$$

From (a) it follows that for all $s \in S$, $s > -2$ $(x,s)o(x,-3)\Phi(x,s)o(y,-2)$, i.e., by (5) and (6),

(b) $$(x,s)\Phi(y,-3).$$

Furthermore, (b) implies $(x,s)o(x,-1)\Phi(y,-3)o(x,-1)$; thus, by (7) and (4),

(c) $$(a_s,-3)\Phi(y,-3)$$

for all $s \in M$, $s < \alpha$. Then $(c), (i), (b)$ and (a) yield $\Phi = G^2$.

Theorem 1. *Let (A, F) be an algebra. There exists a groupoid (G, o) such that*

$$Con\mathcal{A} + 1 \simeq Con\mathcal{G}$$

.

Proof. We can suppose that (A, F) is an algebra with binary operations. Let (G, o) be the groupoid whose operation is defined by $(1) - (8)$. Define a mapping $F : Con\mathcal{A} \to Con\mathcal{G}$ as follows:

(9) $$(x,k)F(\Theta)(y,s) \quad iff \quad k = s \quad and \quad x\Theta y, \quad \Theta \in Con\mathcal{A}.$$

1. First, we prove that the mapping F is well-defined, i.e., $F(\Theta) \in Con\mathcal{G}$. Obviously, $F(\Theta)$ is an equivalence relation on G. It suffices to prove that $(x,k)F(\Theta)(y,k)$ implies

$$(x,k)o(z,r)F(\Theta)(y,k)o(z,r) \tag{d}$$

and

$$(z,r)o(x,k)F(\Theta)(z,r)o(y,k) \tag{e}$$

for every $(z,r) \in G$. We only prove (d), since (e) can be proved in a similar way.

If $k = r$ and $0 \leq k < \beta$, then from $x\Theta y$ it follows $f_k(x,z)\Theta f_k(y,z)$ and so $(f_k(x,z), k+1)F(\Theta)(f_k(y,z), k+1)$ holds. Therefore we get (d) by (1).

In the other cases (d) is of the form

$$(x,m)F(\Theta)(y,m) \quad or \quad (t,m)F(\Theta)(t,m)$$

for some t, m. Hence, (d) obviously holds.

2. Let $\Phi \in Con\mathcal{G}$, $\Phi \neq G^2$. We define a relation Θ on A as follows:

$$x\Theta y \quad iff \quad (x,-3)\Phi(y,-3). \tag{10}$$

Obviously, Θ is an equivalence relation on A. Let $x\Theta y$ and let $k \in M, k < \beta$. Then by (i) (Lemma 1) we have $(x,k)\Phi(y,k)$. Therefore, we get $(x,k)o(z,k)\Phi(y,k)o(z,k)$, i.e. $(f_k(x,z), k+1)\Phi(f_k(y,z), k+1)$. Then again by (10) and (i) we have $f_k(x,z)\Theta f_k(y,z)$. Analogously, we get $f_k(z,x)\Theta f_k(z,y)$. Thus, $\Theta \in Con\mathcal{A}$ and obviously $F(\Theta) = \Phi$.

3. It is easy to check that

$$\Theta_1 \leq \Theta_2 \quad iff \quad F(\Theta_1) \leq F(\Theta_2)$$

for all $\Theta_1, \Theta_2 \in Con\mathcal{A}$. If we denote $F(A^2)$ by Ω, then we conclude that Ω is a unique dual atom of the lattice $Con\mathcal{G}$ and the mappping F is an isomorphism between $Con\mathcal{A}$ and the ideal $(\Omega]$ of the lattice $Con\mathcal{G}$. The proof is complete.

Corollary. *Let L be an algebraic lattice. There exists a groupoid $\mathcal{G}$ such that* $L + 1 \simeq Con\mathcal{G}$.

Theorem 2. *Let (A, F) be an algebra with binary operations such that A, F are countable. Then there exists a groupoid $\mathcal{G}$ having no proper subgroupoids and no nontrivial automorphisms such that*

$$Con\mathcal{A} + 1 \simeq Con\mathcal{G}.$$

Proof. Let (G, o) be the groupoid whose operation is defined by $(1) - (8)$, with $\gamma = \omega$. Then $Con\mathcal{A} + 1 \simeq Con\mathcal{G}$ by Theorem 1.

a) Now we shall prove that $\mathcal{G}$ has no proper subgroupoids. Let H be the subuniverse of the groupoid (G, o) generated by the element (a, p). Let $0 \leq p < \beta$. Then we successively get that H also contains the elements
$(a,p)o(a,p) = (f_p(a,a), p+1) = (b, p+1)$, $(b, p+1)o(b, p+1) = (f_{p+1}(b,b), p+2) = (c, p+2)$, whenever $p + 1 < \beta$
$(b, p+1)o(b, p+1) = (b, p+2) = (c, p+2)$, if $p + 1 = \beta$,
$(a,p)o(c, p+2) = (a, -3)$, $(a, -3)o(a, -3) = (a, -2)$, $(a, -2)o(a, -2) = (a, -1)$, etc.
For every $m \in S$ there exists an element $d \in A$ such that $(d, m) \in H$. Then for every m, $0 \leq m < \alpha$ we get
$(d, m)o(a, -1) = (a_m, -3) \in H$.
We also get
$(d, n)o(a_m, -3) = (a_m, n) \in H$, for every $n > -3$.
Hence, $H = G$ holds.

If either $p \in \{-3, -2, -1\}$ or $p \geq \beta$ we obtain $H = G$ in a similar way.

b) It remains to prove that $\mathcal{G}$ has no nontrivial automorphisms. Let g be an automorphism of $\mathcal{G}$. If $g(a, -3) = (b, -2)$ for some elements $a, b \in A$, then

$$g(a, -2) = g((a, -3)o(a, -3)) = g(a, -3)og(a, -3) = (b, -2)o(b, -2) = (b, -1)$$

but this contradicts the fact that

$$g(a, -2) = g((a, -3)o(a, -2)) = (b, -2)o(b, -1) = (b, -2).$$

We analogously check that $g(a, -3) = (b, p)$ where $p = -1$ or $O \leq p < \beta$ or $\beta \geq p$ is impossible, too. Hence, for every element $(a, -3) \in G$ there exists an element $(b, -3) \in G$ such that

$$\text{(f)} \qquad g(a, -3) = (b, -3).$$

But (f) implies

$$g(a,-2)=g((a,-3)o(a,-3))=(b,-3)o(b,-3)=(b,-2).$$

Similarly, $g(a,-1)=(b,-1)$, $g(a,0)=(b,0)$
$g(f_0(a,a),1)=g((a,0)o(a,0))=(b,0)o(b,0)=(f_0(b,b),1)$, etc.
One can prove by induction that for every $m \in S$ there exist elements $c,d \in A$ such that

$$g(c,m)=(d,m). \tag{g}$$

For any n, $O \leq n < \alpha$, (g) yields

$$g(a_n,-3)=g((c,n)o(a,-1))=(d,n)o(b,-1)=(a_n,-3).$$

Now, for any $m \neq -3$ we get

$$g(a_n,m)=g((c,m)o(a_n,-3))=(d,m)o(a_n,-3)=(a_n,m).$$

Therefore, g is the identity on G, and the proof is complete.

Theorem 3. *Let (A,F) be an algebra with binary operations such that A, F are countable. There exists a groupoid (G,o) such that*

$$Sub\mathcal{A} \simeq Sub\mathcal{G}, \quad Aut\mathcal{A} \simeq Aut\mathcal{G} \quad \textit{and} \quad Con\mathcal{A} \simeq (\Omega],$$

where $(\Omega]$ is an ideal of $Con\mathcal{G}$ generated by some element $\Omega \in Con\mathcal{G}$.

Proof. Let (G,o) be the groupoid constructed in the same way as in the proof of Theorem 1 where

$$(a_i,r)o(a_j,-1)=(a_i,r+1) \tag{7'}$$

holds instead of (7) (i.e., (8) also holds in the case $s=-1$).

1. We shall prove that $Sub\mathcal{A} \simeq Sub\mathcal{G}$. Let $\phi : Sub\mathcal{A} \to Sub\mathcal{G}$ be the mapping defined as follows:

$$\phi(A_1)=\{(a,n); \quad a \in A_1, \quad n \in S\}. \tag{11}$$

First, we shall show that ϕ is well-defined, i.e., that $\phi(A_1)$ is a subuniversum of the groupoid $\mathcal{G}$. Let $(a,n),(b,m) \in \phi(A_1)$.
If $0 \leq n = m < \beta$, then $(a,n)o(b,m)=(f_n(a,b),n) \in \phi(A_1)$. In the

other cases we have $(a,n)o(b,m) = (a,k)$ or $(a,n)o(b,m) = (b,k)$ for suitable $k \in S$, so again $(a,n)o(b,m) \in \phi(A_1)$.

In order to show that ϕ is surjective, let H be a subuniversum of $\mathcal{G}$, and let

$$A_1 = \{a \in A; \quad \exists n \quad (a,n) \in H\}.$$

Further, let $a, b \in A_1$ and let $f_p \in F$. Then there exist m,n such that $(a,m),(b,n) \in H$. In a similar way as in the proof of Theorem 2 one can prove that $(a,-3),(b,-3) \in H$, and that for every n there exists d such that $(d,n) \in H$. This implies
$(d,p)o(a,-3) = (a,p) \in H, \quad (d,p)o(b,-3) = (b,p) \in H,$
$(a,p)o(b,p) = (f_p(a,b), p+1) \in H, \quad$ i.e.,$f_p(a,b) \in A_1$.
Therefore, A_1 is the subuniversum of $\mathcal{A}$, and clearly $\phi(A_1) = H$.

Obviously, ϕ is one-to-one and

$$A_1 \subseteq A_2 \qquad iff \qquad \phi(A_1) \subseteq \phi(A_2).$$

Hence, ϕ is an isomorphism.

2.Now we shall prove that $Aut\mathcal{A} \simeq Aut\mathcal{G}$. Let $\phi : Aut\mathcal{A} \to Aut\mathcal{G}$ be the mapping defined as follows:

$$\phi(f)(a,n) = (f(a),n), \quad f \in Aut\mathcal{A}. \tag{12}$$

First, we shall again show that ϕ is well-defined, i.e., that $\phi(f)$ is an automorphism of the groupoid $\mathcal{G}$. Obviously, $\phi(f)$ is a bijection on G since f is a bijection on A. Let $(a,n),(b,m) \in G$. If $0 \leq n = m < \beta$, then
$\phi(f)((a,n)o(b,n)) = \phi(f)(f_n(a,b), n+1) = (f(f_n(a,b)), n+1) =$
$= (f_n(f(a), f(b)), n+1) = (f(a),n)o(f(b),n) = \phi(f)(a,n)o\phi(f)(b,n).$

If $n \neq m, \quad n+1 = m,$ then we have
$\phi(f)((a,n)o(b,m)) = \phi(f)(a,-2) = (f(a),-2) =$
$= (f(a),n)o(f(b),m) = \phi(f)(a,n)o\phi(f)(b,m).$

We analogously get
$\phi(f)((a,n)o(b,m)) = \phi(f)(a,n)o\phi(f)(b,m)$ in every other case (i.e., if $n+1 < m$, or $n > m$).

In order to show that ϕ is surjective, suppose that g is an automorphism of $\mathcal{G}$. In a similar way as in the proof of Theorem 2 one can prove that for every element $(a,-3) \in G$ there exists an element $(b,-3) \in G$ such that $g(a,-3) = (b,-3)$, and that for every $m \in S$ there exist elements $c, d \in G$ such that $g(c,m) = (d,m)$. This yields that for any $m \geq -3$

$$g(a,m) = g((c,m)o(a,-3)) = (d,m)o(b,-3) = (b,m).$$

On the other hand

$$g(a,-3) = g((a,n)o(c,n+2)) = (b,n)o(d,n+2) = (b,-3),$$

whenever $g(a,n) = (b,n)$ for some n. Therefore, for any automorphism $g \in Aut\mathcal{G}$ we can define a mapping $f : A \to A$ as follows:
$f(a) = b \iff g(a,n) = (b,n)$ for every $n \in S$.
We shall prove that the mapping f is an endomorphism on A. Let $a, c \in A$, $0 \le n < \beta$. Then

$$(f(f_n(a,c)), n+1) = g(f_n(a,c), n+1) = g((a,n)o(c,n)) =$$
$$= (f(a), n)o(f(c), n) = (f_n(f(a), f(c)), n+1)$$

which implies $f(f_n(a,c)) = f_n(f(a), f(c))$.

Since g is the automorphism on G, we conclude that f is the automorphism on A, and obviously $\phi(f) = g$.

One can easily check that ϕ is one-to-one, and that for any automorphisms $f, g \in Aut\mathcal{A}$

$$\phi(fog)(a,n) = ((fog)(a), n) = (f(g(a)), n) = (\phi(f)o\phi(g))(a,n).$$

Hence, ϕ is an isomorphism.

3. We define a relation Ω on G as follows:

$$(a,m)\Omega(b,n) \quad iff \quad m = n. \tag{13}$$

Obviously, Ω is the congruence of groupoid (G, o). Let
$F : Con\mathcal{A} \to Con\mathcal{G}$ be the mapping defined by (9) (in the proof of Theorem 1). In the same way as in the proof of Theorem 1, one can prove that F is an isomorphism between $Con\mathcal{A}$ and the ideal $(\Omega]$ of the lattice $Con\mathcal{G}$. The proof is complete.

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THE LATTICE OF ORDER VARIETIES

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Abstract. The list of all varieties of posets that cover the variety C ∨ A is given (C is the variety of all complete lattices and A is the variety of all antichains). Some results on varieties of posets containing the variety of antichains are established.

In the paper we obtain some new results on the lattice of varieties of posets which was introduced in [1]. Notations and terminology correspond to those from [1]. Throughout the paper the basic set of the poset is always assumed to be nonempty.

A poset Q is *a retract* of poset P (written $Q \triangleleft P$) if there are order-preserving maps $f : Q \to P$ (which is called a coretraction map) and $g : P \to Q$ (called a retraction map) such that $g \circ f$ is the identity map of Q to itself.

There is another characterisation of a retract: a subposet Q of a poset P is a retract of P if there is an order-preserving map $g : P \to Q$ which is identical on Q.

Let K be a class of partially ordered sets. The class of all retracts of posets from K will be denoted by **R**(K) and the class of all direct products of nonvoid families of posets from K will be denoted by **P**(K).

An order variety is a class V of ordered sets which contains all retracts of members of V and all direct product of nonvoid families of members of V (i.e. V is order variety iff $\mathbf{R}(\mathrm{V}) \subseteq \mathrm{V}$ and $\mathbf{P}(\mathrm{V}) \subseteq \mathrm{V}$).

For a class K of posets let K^τ denote the smallest order variety containing each member of K. A variety K^τ is called the order variety generated by K. In [1] it is proved that

$$\mathrm{K}^\tau = \mathbf{RP}(\mathrm{K}).$$

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The collection of all order varieties is a complete lattice ordered by inclusion.

There are only two atoms in this lattice:

C - the variety of all complete lattices (C = $\{\underset{\sim}{2}\}^\tau$ where $\underset{\sim}{2}$ denotes the two-element chain)

A - the variety of all antichains (A = $\{2\}^\tau$ where 2 denotes the two-element antichain).

Any order relation on P induces the relation of comparability on P defined by $a \sim b$ if $a \leq b$ or $b \leq a$. The relation of comparability is reflexive and symmetric. We note that the transitive closure of a reflexive and symmetric relation is an equivalence relation. The blocks of this equivalence relation are called *connected components* of the order relation. A poset P is called *a connected poset* if it has just one connected component. K is a class of connected posets if any member of K is connected.

Let $\{P_i : i \in I\}$ be a family of mutually disjoint posets. The *cardinal sum* of the family $\{ P_i : i \in I\}$ (written $\sum_{i\in I} P_i$) is the poset with the universe $P = \bigcup_{i\in I} P_i$ and the order relation $\leq$ on P defined as follows: $a \leq b$ iff there exists an index $i_o \in I$ such that $a, b \in P_{i_o}$ and $a \leq b$ holds in P_{i_o} .

Let $\sum$K denote the class of all isomorphic images of the cardinal sums of members of K.

It is easy to see that any poset is the cardinal sum of its connected components.

Lemma 1. *Let* $A = \sum_{i\in I} A_i$ *and* $B = \sum_{j\in J} B_j$ *be cardinal sums of nonvoid families* $\{ A_i : i \in I\}$ *and* $\{ B_j : j \in J\}$ *respectively. If for every* $i \in I$ *there exists* $j_i \in J$ *such that* $A_i \triangleleft B_{j_i}$ *and* $j_{i_1} = j_{i_2}$ *implies* $i_1 = i_2$*, then* $A \triangleleft B$.

Proof. Let $\varphi_i : B_{j_i} \to A_i$ be a retraction and

$\phi_i : A_i \to B_{j_i}$ be a coretraction for every $i \in I$.

We define the maps $g : \sum_{j\in J} B_j \to \sum_{i\in I} A_i$

and $f : \sum_{i\in I} A_i \to \sum_{j\in J} B_j$ as follows:

$$g(a) = \begin{cases} \varphi_i(a) \text{ if } a \in B_{j_i} \\ d \text{ otherwise} \end{cases}$$

where d is any fixed element of $\sum_{i\in I} A_i$ and

$$f(b) = \phi_i(b) \text{ if } b \in A_i.$$

The reader can straightforwardly verify that for every $x \in \sum_{i\in I} A_i$ there are indices i, j_i such that $x \in A_i$ and $f(A_i) \subseteq B_{j_i}$ and $((g \circ f)(x) = (\varphi_i \circ \phi_i)(x) = x$ and f and g are order-preserving maps. This completes the proof of the lemma.

Theorem 1. *The variety A of all antichains is contained in a variety V iff V= $\sum$V.*

Proof. a) If the poset P contains at least two connected components P_1, P_2 then choose two elements $x \in P_1$ and $y \in P_2$. The map $f : P \to \{x, y\}$ which maps each element $u \in P_1$ to x and each element $v \in P - P_1$ to y is a retraction map with fixed points x and y. Thus
$2 \lhd P$.This implies that if V contains at least one disconnected poset then V contains A.If V=$\sum$V then V contains a disconnected poset,so A$\subseteq$V.

b) Let A$\subseteq$V and $P \in \sum$V.Then $P = \sum_{i\in I} P_i$, where $P_i \in$ V for every $i \in I$. This implies that $\prod_{i\in I} P_i \in$ V. Let $\widetilde{I}$ denote the antichain with the universe I and let $Q = (\prod_{i\in I} P_i)\times\widetilde{I}$. Obviously Q is a member of V.(Note that $\widetilde{I} \in A \subseteq$ V.) We denote

$$Q_j = \{x \in Q : x = (f, j) \text{ where} f \in \prod_{i\in I} P_i\}.$$

It is easy to see that $Q = \sum_{i\in I} Q_i$. It is known that $U_{l_o} \lhd \prod_{l\in L} U_l$ for any family of posets $\{ U_l : l \in L\}$ and for any $l_o \in L$. The posets $P = \sum_{i\in I} P_i$ and $Q = \sum_{i\in I} Q_i$ satisfy the conditions from Lemma 1 and so $P \lhd Q$. This implies that $P \in$ V. The converse inclusion is obvious.

Theorem 2. *If V is an order variety then $\sum$ V is also an order variety.*

Proof. If there is a disconnected poset in V then the assertion follows from Theorem 1. So assume that V contains no disconnected poset.

a) Let $P \lhd \sum_{i\in I} P_i$ and $P_i \in$ V for any $i \in I$. Denote by f a retraction and by g a coretraction corresponding to f. Let S_i denote the set $\{ x \in P : g(x) \in P_i\}$ .Obviously, $P = \sum_{i\in I} S_i$. Let the subset $J \subseteq I$ be given by : $j \in J$ iff $S_j \neq \emptyset$. Then $P = \sum_{j\in J} S_j$. For every $j \in J$ the poset P_j is connected and so is the poset $g(f(P_j))$. Hence $g(f(P_j)) \subseteq P_j$. Thus we get

$f(P_j) \subseteq S_j$ for any $j \in J$. On the other hand $S_j = f(g(S_j)) \subseteq f(P_j)$. It implies that $f(P_j) = S_j$ for any $j \in J$. We have $S_j \lhd P_j$ with $f \restriction P_j$ as a retraction and $g \restriction S_j$ as a coretraction. This implies that $S_j \in$ V and $P \in \sum$V. Therefore $\mathbf{R}(\sum$ V$)\subseteq \sum$V.

b) Now we prove that $\mathbf{P}(\sum$ V$)\subseteq \sum$ V. The proof is based on fact that

$$\prod_{i\in I}(\sum_{j\in J_i} P_i{}^j)$$

and

$$\sum_{f\in \prod\limits_{i\in I} J_i}(\prod_{i\in I} P_i{}^{f(i)})$$

are isomorphic posets. The poset $\prod\limits_{i\in I} P_i{}^{f(i)}$ is the direct product of a family of connected sets from V and so $\prod\limits_{i\in I} P_i{}^{f(i)} \in$ V. Hence $\sum\limits_{f\in \prod\limits_{i\in I} J_i}(\prod\limits_{i\in I} P_i{}^{f(i)}) \in \sum$ V. This completes the proof.

Corollary 1. *In the lattice of order varieties $\sum V = A \vee V$ for any variety V.*

Theorem 3. *If V is a variety which contains no disconnected poset, then there is no variety W such that V$\subset$W $\subset \sum$V(i.e.,the variety V is covered by $\sum$V).*

Proof. Suppose V$\subseteq$W $\subseteq \sum$V. If every poset in W is connected then W=V. Otherwise A$\subseteq$W,thus W=$\sum$W and $\sum$V $\subseteq \sum$W=W. Hence W=$\sum$V.

Corollary 2. *If the system V_c of all connected posets of the variety V is a variety, then V_c is covered by V or $V_c = V$.*

Proof. This follows from the fact that V=$\sum$ V$_c$ or V=V$_c$.

Theorem 4. *If V and W are varieties of connected posets and V is covered by W then $\sum$V is covered by $\sum$W.*

Proof. Obviously, V$\subseteq$W implies $\sum$ V $\subseteq \sum$W. Let V$_1$ be a variety such that $\sum$ V $\subseteq$ V$_1$ $\subseteq \sum$ W. Denote by K the system of all connected posets of V$_1$.Then V$\subseteq$ K $\subseteq$W. It is easy to see that $\mathbf{R}$(K)$\subseteq$W, $\mathbf{P}$(K)$\subseteq$W, $\mathbf{R}$(K)$\subseteq$ V$_1$, $\mathbf{P}$(K)$\subseteq$ V$_1$. This implies $\mathbf{RP}$(K)$\subseteq$ V$_1$ and $\mathbf{RP}$(K)$\subseteq$W. So, for the variety $\mathbf{RP}$(K) generated by K we have V$\subseteq\mathbf{RP}$(K)$\subseteq$W. If $\mathbf{RP}$(K)=V then V$_1$ = $\sum$V. If $\mathbf{RP}$(K)=W then V$_1$ = $\sum$W.

Corollary 3. *Let V and W be varieties which contain A and let V be covered by W. Let V_c and W_c denote the systems of all connected posets of V and W, respectively.If V_c and W_c are varieties, then V_c is covered by W_c.*

Proof. If K is variety such that $\mathrm{V}_c \subseteq \mathrm{K} \subseteq \mathrm{W}_c$,$\mathrm{V}_c \neq \mathrm{K} \neq \mathrm{W}_c$ then $\sum \mathrm{V}_c \subseteq \sum \mathrm{K} \subseteq \sum \mathrm{W}_c$ and $\sum \mathrm{V}_c \neq \sum \mathrm{K} \neq \sum \mathrm{W}_c$, a contradiction.

Let P,Q be ordered sets.We will denote by $P \oplus Q$ the ordinal sum of P and Q. Further, let P^d denote the dual poset of P.Let F_3 be the poset with the universe { 1,2,3} and the order relation given by : 2≤1, 3≤1 and 2,3 are noncomparable elements (i.e.,F_3 is a fence).

Theorem 5 ([1]). *In the lattice of order varieties each of the following order varieties cover the variety C of all complete lattices*

$$\text{(L)}\quad \begin{array}{l} \{\alpha\}^{\tau}, \quad \{\beta^d\}^{\tau}, \quad \{\alpha \oplus \beta^d\}^{\tau}, \quad \{F_3\}^{\tau}, \quad \{{F_3}^d\}^{\tau}, \quad \{{F_3}^d \oplus F_3\}^{\tau}, \\ \{\alpha \oplus F_3\}^{\tau}, \quad \{{F_3}^d \oplus \beta^d\}^{\tau}, \quad \sum C, \end{array}$$

where α and β are any regular ordinals. Moreover, the system of all varieties which cover C consists just of the varieties in (L).

Theorem 6. *In the lattice of order varieties the variety $\sum C(= A \vee C)$ is covered only by the varieties of the form $\sum$ V for all V from the list (L) except of V= $\sum C$.*

Proof. Theorem 4 implies that these varieties cover $\sum$C. Let V be variety with $\sum \mathrm{C} \subseteq \mathrm{V}$ and $\sum \mathrm{C} \neq \mathrm{V}$. Then there exists a poset P such that P is not a complete lattice, P is connected, $P \in$V and $\{P\}^{\tau} \subseteq$V. In [1] it is shown (see proof of Theorem 7.1) that there is a poset Q generating a variety from the list (L) and Q is a retract of P. Thus, we get $\sum \{Q\}^{\tau} \subseteq \sum \{P\}^{\tau} \subseteq \sum$ V = V. This completes the proof.

Theorem 7. *The system of all varieties of connected posets is the ideal of the lattice of order varieties.*

Proof. Let V and W be varieties of connected posets. Then V∨W = **RP**(V∪W) (cf. [1]). If $P \in$ **RP**(V∪W) then $P \triangleleft Q_1 \times Q_2$, where $Q_1 \in$ **P**(V) and $Q_2 \in$ **P**(W). This implies that Q_1 and Q_2 are connected and $Q_1 \times Q_2$ is also connected.Every retract of connected poset is also connected.Let V and K be any varieties of posets.If V contains no disconnected poset and K⊆V ,then also K contains no disconnected poset.

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ON DECOMPOSITIONS OF COMPLETE GRAPHS INTO FACTORS WITH GIVEN DIAMETERS

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ABSTRACT. Let $F_m(d_1, d_2, \ldots, d_m)$ be the least positive integer n such that the complete graph K_n can be decomposed into m factors with the diameters $d_1, d_2, \ldots, d_m$. The estimations for $F_m(d_1, d_2, \ldots, d_m)$ are found.

By a factor of a graph G we mean a subgraph of G containing all the vertices of G. A system of factors of G such that every edge of G belongs to exactly one of them is called a decomposition of G. The symbol K_n denotes the complete graph with n vertices.

Let $m, d_1, d_2, \ldots, d_m$ be natural numbers. The symbol (see [1]) $F_m(d_1, d_2, \ldots, d_m)$ denotes the smallest natural number n such that the complete graph K_n can be decomposed into m factors with the diameters $d_1, d_2, \ldots, d_m$; if such a natural number does not exist then put $F_m(d_1, d_2, \ldots, d_m) = \infty$. In the case $d_1 = d_2 = \cdots = d_m = d$ we shall write $F_m(d, d, \ldots, d) = f_m(d)$. The significance of the function $F_m(d_1, d_2, \ldots, d_m)$ resides in the validity of the following assertion (proved in [1]): K_n is decomposable into m factors with diameters $d_1, d_2, \ldots, d_m$ if and only if $n \geq F_m(d_1, d_2, \ldots, d_m)$.

J. Bosák, A. Rosa and Š. Znám ([1]) initiated the studies of decompositions of complete graphs into factors with given diameters. Many papers deal with the problem of [1] or with its various modifications.

The following result is proved in [1]: Let $m, d_1, d_2, \ldots, d_m$ be natural numbers ≥ 3, then

$$F_m(d_1, d_2, \ldots, d_m) \leq d_1 + d_2 + \cdots + d_m - m.$$

This result can be strengthened. For $m = 3$ this was done in [1] (if $min\{d_1, d_2, d_3\} \geq 5$, then $F_3(d_1, d_2, d_3) \leq d_1 + d_2 + d_3 - 8$) and in [2]

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(if $min\{d_1, d_2, d_3\} > 65$, then $F_3(d_1, d_2, d_3) = d_1 + d_2 + d_3 - 8$). For $m > 3$ the following theorem gives a better result than that mentioned above.

Theorem 1. *Let $m \geq 3$, $d_1 \geq d_2 \geq \cdots \geq d_m \geq 6$, $d_1 \geq 2m - 1$ and $d_3 \geq m - 1$. Then*

$$F_m(d_1, d_2, \ldots, d_m) \leq d_1 + d_{k-1} + d_k + 3,$$

where k is the maximum natural number such that $3 \leq k \leq m$ and $d_k \geq m - 1$.

Proof. It is sufficient to show that the complete graph with $d_1 + d_{k-1} + d_k + 3$ vertices is decomposable into m factors $F_1, F_2, \ldots, F_m$ with the diameters $d_2, d_3, \ldots, d_{k-2}, d_{k+1}, d_{k+2}, \ldots, d_m, d_1, d_{k-1}, d_k$, respectively. Denote the vertices of the above mentioned graph by $u_1, u_2, \ldots, u_{d_1+1}, v_1, v_2, \ldots, v_{d_{k-1}+1}, w_1, w_2, \ldots, w_{d_k+1}$. Let $t = \left[\frac{d_1+1}{2}\right]$.
We shall consider the path P_i' of the length d_1 for $i = 1, 2, \ldots, m-2$

$$u_{i+1}u_iu_{i+2}u_{i-1}u_{i+3}u_{i-2}\ldots u_{i-t+2}u_{i+t}u_{i-t+1}$$

in the case that d_1 is an odd number or

$$u_{i+1}u_iu_{i+2}u_{i-1}u_{i+3}u_{i-2}\ldots u_{i+t}u_{i-t+1}u_{i+t+1}$$

in the case that d_1 is an even number. The subscripts j of u_j are taken as the integers $1, 2, \ldots, d_1 + 1 \bmod (d_1 + 1)$. Now we are going to construct the factors F_i for $i = 1, 2, \ldots, m$.

a) The factor F_i for $i = 1, 2, \ldots, m - 3$ consists of
  1) a path P_i with the following four properties
    (i) the length of P_i is equal to the diameter of the factor F_i,
    (ii) we get P_i from P_i' by deleting (if necessary) some vertices at the beginning and the end of the path P_i',
    (iii) P_i contains the vertices u_{2i+1} and u_{2i+2},
    (iv) neither u_{2i+1} nor u_{2i+2} is one of the first two vertices or one of the last two vertices of P_i,
  2) the edges

$$u_{2i+1}v_j, j = 1, 2, \ldots, d_{k-1} + 1,$$
$$u_{2i+2}w_j, j = 1, 2, \ldots, d_k + 1,$$

  3) for any vertex u_j which does not belong to P_i
    (i) the edge v_iu_j if j is an even number or
    (ii) the edge w_iu_j if j is an odd number.

b) The factor F_{m-2} will contain the path P'_{m-2} and the edges

$$u_{2m-3}v_j, j = 1, 2, \ldots, d_{k-1} + 1,$$
$$u_{2m-2}w_j, j = 1, 2, \ldots, d_k + 1.$$

c) The factor F_{m-1} will contain the path

$$v_1 v_2 v_{d_{k-1}+1} v_3 v_4 \ldots v_{d_{k-1}-1} v_{d_{k-1}}$$

and the edges

$$v_{d_{k-1}+1}u_{2j}, j = 1, 2, \ldots, t,$$
$$v_{d_{k-1}+1}w_j, j = 1, 2, \ldots, d_k + 1,$$
$$w_{d_k}u_{2j-1}, j = 1, 2, \ldots, s, \quad s = \left[\tfrac{d_1+2}{2}\right].$$

d) The factor F_m will contain the path

$$w_1 w_2 w_{d_k+1} w_3 w_4 \ldots w_{d_k-1} w_{d_k}$$

and the edges

$$w_{d_k+1}u_{2j-1}, j = 1, 2, \ldots, s \quad s = \left[\tfrac{d_1+2}{2}\right],$$
$$w_{d_k+1}v_j, j = 1, 2, \ldots, d_{k-1},$$
$$v_{d_{k-1}}u_{2j}, j = 1, 2, \ldots, t,$$
$$v_1 v_{d_{k-1}+1}.$$

Now consider all the edges which so far we have not included into any of the factors $F_1, F_2, \ldots, F_m$. Let those of them which are of the types $v_i w_j, v_i v_j, w_i w_j$ or $u_i u_j, u_i w_j$ or $u_i v_j$ belong to the factors F_{m-2} or F_{m-1} or F_m, respectively. It is easy to verify that the factors $F_1, F_2, \ldots, F_m$ have the desired diameters and they form a decomposition of the complete graph with $d_1 + d_{k-1} + d_k + 3$ vertices. Q.E.D.

In [1] a lower bound for $f_3(d)$ was found :

$$f_3(d) > \frac{3+\sqrt{3}}{2}d - \frac{5+4\sqrt{3}}{2}.$$

For $m > 3$ the following theorem holds.

Theorem 2. *If $m > 3$ and $d \geq 2m - 1$, then*

$$f_m(d) \geq \frac{m+\sqrt{m}}{m-1}d - \frac{m+\sqrt{m}(2m-1)}{m-1}.$$

Proof. The maximum number of edges in a graph with n vertices and with the diameter d is ([1], Lemma 1)

$$d + 3(n-d-1) + \frac{(n-d-1).(n-d-2)}{2}.$$

The necessary condition for the existence of a decomposition of the complete graph K_n into m factors with diameter d is the inequality ([1], Theorem 2)

$$m[d + 3(n - d - 1) + \frac{(n - d - 1).(n - d - 2)}{2}] \geq \frac{n(n - 1)}{2}$$

or, equivalently

$$(1) \qquad (m - 1)n^2 + (3m - 2md + 1)n + m(d^2 - d - 4) \geq 0.$$

In the following we shall use the idea from the proof of Lemma 6 in [1]. The quadratic function of the variable n defined by the left hand side of (1) takes negative values for $n_1 = d$ and for $n_2 = \frac{m+\sqrt{m}}{m-1}d - \frac{m+\sqrt{m}(2m-1)}{m-1}$ (with the exception of the case $d = 2m - 1$, when it takes the value 0). Since this function is convex and a graph with the diameter d has at least $d + 1$ vertices, the theorem is proved.

Remark. For $d = 2m - 1$ the estimation in Theorem 2 is the best possible. In fact, in this case the right hand side of the inequality of Theorem 2 gives the value 2m and and by [3] (Theorem 3) $f_m(d) = 2m$ for $m \geq 3$ and $3 \leq d \leq 2m - 1$.

Theorem 3. *If $d \geq 2m$ and $m > 3$ then*

$$f_m(d) \leq \frac{5}{2}d + 3.$$

Proof. We shall confine ourselves to the case when d is an odd number (in the case when d is an even number we can proceed in a similar way). It is sufficient to show that the complete graph with $\frac{5d+5}{2}$ vertices is decomposable into m factors with the diameter d. Denote the vertices of this complete graph by symbols $u_1, u_2, \ldots, u_{2k}, v_1, v_2, \ldots, v_k, w_1, w_2, \ldots, w_k, t_1, t_2, \ldots, t_k$, where $k = \frac{d+1}{2}$.

a) The factor F_i for $i = 1, 2, \ldots, m - 3$ will contain the path

$$u_{i+1}u_iu_{i+2}u_{i-1}u_{i+3}u_{i-2} \ldots u_{i+k}u_{i-k+1},$$

where the subscripts j of u_j are taken as the integers $1, 2, \ldots, 2k(mod 2k)$ and the edges

$$u_{2i+1}v_j, u_{2i+1}w_j, j = 1, 2, \ldots, k,$$
$$u_{2i+2}t_j, j = 1, 2, \ldots, k.$$

b) The factor F_{m-2} will contain the path

$$v_1v_2\dots v_kw_1w_2\dots w_k$$

and the edges

$$v_3u_{2j}, v_3t_j, t_ku_{2j-1}, \quad j = 1, 2, \dots, k.$$

c) The factor F_{m-1} will contain

1) the edges
t_3u_{2j-1}, v_1u_{2j}, $j = 1, 2, \dots, k$,
t_3v_j, $j = 1, 2, 4, 5, \dots, k\,(j \neq 3)$,
v_1v_3

2) and the path
(i) $t_1t_2\dots t_kw_1w_3\dots w_{k-1}w_2w_4\dots w_k$ if k is even or
(ii) $t_1t_2\dots t_kw_1w_3\dots w_kw_2w_4\dots w_{k-1}$ if k is odd.

d) The factor F_m will contain

1) the edges
$t_1u_{2j-1}, t_1w_j, w_1u_{2j}$, $j = 1, 2, \dots, k$

2) and the path
(i) $v_3v_5\dots v_{k-1}v_2v_4\dots v_kv_1t_1t_3\dots t_{k-1}t_2t_4\dots t_k$ if k is even or
(ii) $v_3v_5\dots v_kv_2v_4\dots v_{k-1}v_1t_1t_3\dots t_kt_2t_4\dots t_{k-1}$ if k is odd.

Now consider the edges which so far we have not included into any of the factors $F_1, F_2, \dots, F_m$. Let those of them which are of the types v_iw_j, v_it_j, w_it_j, v_iv_j, w_iw_j, t_it_j or u_it_j, u_iu_j or u_iv_j or u_iw_j belong to the factor F_1 or F_{m-2} or F_{m-1} or F_m, respectively. It is easy to check that the factors F_1, $F_2, \dots, F_m$ have the diameter d and they form a decomposition of the complete graph with $\frac{5d+5}{2}$ vertices.

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EDGE-LOCALLY HOMOGENEOUS GRAPHS

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Abstract. In the paper we shall investigate the relationship between locally homogenerous graphs and edge-locally homogeneous graphs. A local version of the well-known theorem establishing that an edge-transitive graph is either vertex-transitive, or bipartite is proved. Further we aply the theory of covering spaces to derive some general results on the family of edge-locally G_0 graphs for a fixed graph G_0.

Introduction

In 1986 Zelinka [11] introduced the concept of edge-locally homogeneous graphs. It can be understood as an edge version of the concept of locally homogeneous graphs (or graphs with a constant link, see [1,4]). Let G be a graph and x be either a vertex, or an edge of G. Denote the subgraph of G induced on the set of vertices at distance 1 from x by $link(x, G)$. The graph G is called locally homogeneous, or locally G , if there exists a finite graph G_0 such that for each vertex v of G $link(v, G) \cong G_0$. Similarly, the graph G is called edge-locally homogeneous, or edge-locally G_0 , if there exists a finite graph G_0 such that for each edge e of G $link(e, G) \cong G_0$. Two main problems for edge-locally homogeneous graphs can be considered:

(a) For which finite graphs G_0 does there exist an edge-locally G_0 graph?,

(b) For a fixed finite graph G_0 what can be said about the set of all connected edge-locally G_0 graphs?

Zelinka in [11] showed some examples of edge-locally G_0 graphs. A lot of examples of such graphs can be obtained using the concept of edge-transitive graphs. Further, it was proved in [11] that there is no edge-locally C_5 graph. This result was generalized by Fronček [2] by proving that for n odd, $n \neq 3$, there is no edge-locally C_n graph. In contrast, it is proved in [5] that a finite edge-locally C_n graph exists for all the remaining

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values of n. Another result contained in [2] reads as follows: if G is a complete multipartite graph then an edge-locally G graph exists if and only if all parts of G contain the same number of vertices. In this paper we shall investigate a connection between the locally homogeneous graphs and edge-locally homogeneous graphs. Further we apply the concept of covering spaces to derive some results analogous to those given in [7].

Strongly edge-locally homogeneous graphs

For a given graph G and its edge e denote by $Link(e,G)$ the subgraph of G induced on the set of vertices at distance ≤ 1 from e. That means $e \in Link(e,G)$. Then the graph G will be called *strongly edge-locally homogeneous* if for any two edges e, f in G there is an isomorphism $\varphi Link(e,G) \rightarrow Link(f,G)$ mapping e onto f. The following observation is clear.

Proposition 1. *If a graph G is strongly edge-locally homogeneous then G is edge-locally homogeneous .*

In fact we know no edge-locally homogeneous graph which is not strongly edge-locally homogeneous as well. Thus the question, whether the opposite implication in Proposition 1 holds true, is open. The following theorem can be considered as a local version of the well-known theorem (see [3]) establishing that an edge-transitive graph is either vertex-transitive or bipartite.

Theorem 2. *Let G be a strongly edge-locally homogeneous graph. Then either G is locally homogeneous or bipartite.*

Proof. Let $e = uv$ be a fixed edge of G. Let $f = xy$ be an arbitrary edge of G. Since G is strongly edge-locally homogeneous, there is an isomorphism $\varphi Link(f,G) \rightarrow Link(e,G)$ mapping f onto e. Then either there is an automorphism ψ of $Link(e,G)$ mapping u to v, or there is no such an automorphism. In the first case either φ, or $\psi\varphi$ maps $link(x,G)$ onto $link(u,G)$. Since f, and consequently x, is chosen arbitrarily, G is locally homogeneous in this case. In the second case the set of vertices of G splits into two subsets U, V. A vertex x is in U (V) if and only if there is an isomorphism mapping $link(x,G)$ onto $link(u,G)$ (or onto $link(v,G)$, respectively). Since there is no automorphism of $Link(e,G)$ mapping u onto v, $U \cap V = \varnothing$. Clearly, each edge of G joins a vertex from U to a vertex in V, otherwise it would be an automorphism of $Link(e,G)$ mapping u to v. Thus G is bipartite. $\square$

It was noted by Zelinka that if G is bipartite then the edge-local homogeneity of G implies the strong edge-local homogeneity of G. The following proposition shows further properties of strongly edge-locally homogeneous graphs. A bipartite graph G is called *biregular* if the vertices in one part of G have degree p while the vertices in the second part of G are of degree q, for some integers p, q. An r-regular graph in which each edge lies in t triangles and q induced quadrangles will be called an (r,t,q)-graph.

Proposition 3. *Let G be a strongly edge-locally homogeneous graph. Then either G is an (r,t,q)-graph for some integers r, t, q, or it is a bipartite biregular graph.*

Proof. It follows directly from the definition of strong local homogeneity that each edge of G lies in the same number of triangles and in the same number of induced quadrangles. According to Theorem 2 G is either locally homogeneous, and consequently regular, or it is bipartite, and therefore biregular. □

It follows from Proposition 3 that strongly edge-locally homogeneous graph G containing at least one triangle is an (r,t,q) graph. Thus for each vertex u of G $link(u,G)$ is a t-regular graph on r vertices. Regular graphs with regular links of vertices were investigated by Šoltés in [9]. He proved there that an r-regular graph G with t-regular links of vertices is the complete $(1+r/(r-t))$-partite graph, whose each part contains $r-t$ vertices, if $t < r < t + r\sqrt{\frac{8}{9}(t-1)+\frac{4}{3}}$.

Covering spaces of edge-locally homogeneous graphs

Let G be a graph. Denote by ΔG the simplicial complex, whose 0-simplexes are vertices of G, 1-simplexes are edges of G, 2-simplexes are bounded by triangles and induced quadrilaterals of G, and the incidence relation is given by the subgraph inclusion. That means that ΔG arises from G by gluing a 2-cell to each triangle and to each induced quadrangle of G. The following three propositions are analogous to Propositions 3, 4 and 5 in [7]. They follow from Theorems 1.2 and 1.5 in [6].

Proposition 4. *Let G be an edge-locally G graph, for some finite graph G . Let (X,p) be a connected covering space of ΔG. Then there is an edge-locally G graph H such that $p^{-1}(G) = H$ and $X = \Delta H$.*

Proposition 5. *Let $(G,\alpha)_n$ be a permutation voltage graph and G be a connected edge-locally G_0 graph. Then G_n^α is edge-locally G_0 if and only if the product of voltages in each triangle and quadrangle of G is 1.*

Proposition 6. *Let G be a connected edge-locally G_0 graph. Let (G, Γ, α) be an ordinary voltage graph. Then the derived graph G^α is edge-locally G_0 if and only if the product of voltages in each triangle and quadrangle of G is 1.*

The following proposition was motivated by the similar results of Vince [10] for locally homogeneous graphs. Call a subgroup B of the automorphism group $Aut\, G$ of a graph G strongly discontinous if for each $\varphi \in B$ and each vertex v of G the distance $\rho(v, \varphi(v)) \geq 5$.

Proposition 7. *Let G be edge-locally G_0 , for some finite graph G_0 . Let $\Gamma \subseteq Aut\, G$ be a strongly discontinous subgroup of $Aut\, G$. Then the regular quotient G/Γ is edge-locally G_0 .*

Proof. Consider $link([e], G/\Gamma)$ for some edge e in G. We show that the restriction $p' = p/link(e, G)$ of the covering projection mapping a vertex v onto [v] is an isomorphism mapping $link(e, G)$ onto $link([e], G/\Gamma)$. By its definition p' is onto. Since $Link(e, G)$ is a graph of diameter at most 3, by the assumption we have that p' is a bijection on the set of vertices of $link(e, G)$. Clearly, if $e = uv$ is an edge in $link(e, G)$ then $[u][v]$ is an edge in $link([e], G/\Gamma)$. On the other hand, let $[f] = [u][v]$ be an edge in $link([e], G/\Gamma)$, where $f = vw$ is the edge of G incident with v and mapped by p onto $[u][v]$. Suppose, on the contrary, that an edge uv is not in $link(e, G)$. Then $w \neq u$, $w \in [u]$ and the distance $\rho_G(u, w) \leq 4$, a contradiction with the assumption. Thus p' is an isomorphism of the graphs $link(e, G)$ and $link([e], G/\Gamma)$, and G/Γ is edge-locally G_0 . □

Note that if the graph G in Proposition 7 is strongly edge-locally G_0 then the graph G/Γ is strongly edge-locally G_0 as well. The following corollary allows us to build edge-locally homogeneous graphs from groups.

Corollary 8. *Let G be an edge-transitive graph. Let $\Gamma \subseteq Aut\, G$ be a strongly discontinous subgroup of the automorphism group $Aut\, G$. Then the regular quotient G/Γ is strongly edge-locally homogeneous.*

The following two theorems can be considered as edge variants of results in [7].

Theorem 9. *Let G be a finite graph and let the Euler characteristic $\chi(\Delta\, G) \leq 0$. Then for each $n \geq 1$ there exists an n-fold cover of $\Delta\, G$.*

Proof. Denote by v, e, f_3 and f_4 the number of vertices, edges, triangles and induced quadrangles in G, respectively. The fundamental group $\pi(\Delta\, G)$ is generated by $gen = e - v + 1$ generators satisfying $rel = f_3 + f_4$ relations.

By the assumption we have $rel = f_3 + f_4 < e - v + 1 = gen$. Thus $\pi(\Delta G)$ contains a subgroup of index n for each $n > 1$. It follows from the well-known correspondence betweeen the covers of a topological space and subgroups of its fundamental group that for each $n > 1$ there is an n-fold cover of ΔG. □

Theorem 10. *Let G_0 be a finite graph. Let G be a finite connected edge-locally G_0 graph and let $\chi(\Delta G) \leq 0$. Then*

(a) for each $n > 1$ there exists a connected edge-locally G_0 graph with $n.v(G)$ vertices,

(b) there exists an infinite connected edge-locally G graph.

Proof. Theorem 9 implies that there is an n-fold cover of ΔG for each $n > 1$. The statement (a) now follows from Proposition 4. Consider the universal cover $\tilde{X}$ of ΔG. By Proposition 4 $\tilde{X} \cong \Delta \tilde{G}$, where $\tilde{G}$ is edge-locally G_0 , and moreover, $\tilde{G}$ covers each the edge-locally G_0 graph constructed in the proof of part (a) of the theorem. Thus the number of vertices of $\tilde{G}$ is infinite. □

If G is strongly edge-locally homogeneous then the following proposition enumerates the cases for which $\chi(\Delta G) \leq 0$.

Proposition 11. *Let G be strongly edge-locally G_0 . Then $\chi(\Delta G) \leq 0$ if and only if either G is an (r, t, s)-graph, where $0 \leq 4t + 3s \leq 11$ and $r \geq 24/(12 - 4t - 3s)$, or G is bipartite biregular, $e(G_0) = s \leq 3$ and $e(G) \geq v(G)/(1 - s/4)$.*

Proof. Let v and e be the numbers of vertices and edges in G, respectively. Denote by f_3 and f_4 the numbers of triangles and induced quadrangles of G, respectively. Then $\chi(\Delta G) = v - e + f_3 + f_4$. According to Proposition 3 G is either an (r, s, t)-graph, or it is bipartite biregular. In the first case we have $e = vr/2$, $f_3 = et/3 = vrt/3$, and $f_4 = es/4 = vrs/8$. Thus the inequality $\chi(\Delta G) \leq 0$ is equivalent to the inequality $24 + r(4t + 3s - 12) \leq 0$ implying the first part of the statement. In the second case $f_3 = 0$ and the inequality $\chi(\Delta G) \leq 0$ is equivalent to the inequality $v \leq e(1 - s/4)$, where $s = f_4 = e(G)$. □

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A NOTE ON REGULAR LANGUAGES

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Abstract. In this paper there are constructed two non-regular languages satisfying the well-known necessary condition for regular languages and there is modified this necessary condition. It is not known if the new modification of the necessary condition is sufficient.

1. Introduction.

We are going to show that the wel-known necessary condition for regular languages (here: Theorem 2) can be replaced (without any change in the method of the proof) by certain more strong necessary condition (here: Theorem 3).

We shall use the following notations:

N ...the set of all non-negative integers

T^* ...the set of all strings of the form $x_1x_2\dots x_k$ with $k \in N,\ x_i \in T$, including the empty string $\mathbf{e}$.
We shall assume that the set T is finite. The subsets of T^* are called **languages**.

w_1w_2 ...(the concatenation) if $w_1 = x_1\dots x_k$ and $w_2 = y_1\dots y_m$, then $w_1w_2 = x_1\dots x_k y_1\dots y_m$. (Special cases: $\mathbf{ee} = \mathbf{e}$, $\mathbf{e}w = w\mathbf{e} = w$.)

$|w|$ ...the length of the string w ($|\mathbf{e}| = 0$.)

a^n ...the string $aa\dots a$ (n-times, $a^o = \mathbf{e}$).

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Remark. The notion of regular language can be defined by many ways. More precisely, there are many possible (pairwise non-equivalent) definitions of regular grammars. However, all these definitions yield the same class of languages.

2. Two conditions for regular languages.

Theorem 1. (An easy consequence of Theorem 3.1 in [4], resp. Theorem 1.3.3 in [5]. See also [1].) *Let T be a finite set and let L be a subset of the set T^*.*

a) *Put*
$R_L = \{[x, y] \mid L_x = L_y\}$, *where*
$L_z = \{t \mid zt \in L\}$.

(Here $x, y, z, t \in T^*$.) If the language L is regular then R_L is a right congruence on T^* such that L can be written in the form of the union of some classes of the equivalence relation R_L and this equivalence relation has a finite index.

b) *If R is a right congruence on T^* of a finite index and if the language L can be written in the form of the union of some classes of R, then L is regular.*

Remark. For each finite set T, the set T^* with the operation of concatenation is a free monoid over the set T. The "right congruence" is an equivalence relation R on the set T^* such that

$$xRy \quad \text{implies} \quad xyRyz \quad (x, y, z \in T^*).$$

The index of any equivalence relation R is the number of R-blocks.

Example 1. (by [5], Example 1.3.6.) Put $T = \{a, b\}$, $L = \{a^n b^n | n > 0\}$. Applying Theorem 1, we shall prove that the language L is not regular.

If L is an union of certain classes of a right congruence R on $\{a, b\}^*$ of a finite index p, then at least two of the strings

$$a, a^2, a^3, \ldots, a^{p+1}.$$

are in the same class of R, say

$$a^i R a^j, 1 \leq i < j \leq p + 1.$$

Then (R is a right congruence!) it holds:

$$a^i b^i R a^j b^i,$$

a contradiction ($a^i b^i \in L$ but not $a^j b^i \in L$).

Lemma 1. *Put $T = \{a, b\}$ and put*
$L_1 = \{a^n b^n | n \geq 1\}$,
$L_2 = \{w | w \in T^*, \#_a(w) = \#_b(w) > 0\}$,
where $\#_t(w)$ denotes the number of occurences of the symbol t in the string w. Then it holds:

$$L_1 \subseteq L \subseteq L_2 \Longrightarrow L \text{ is not regular.}$$

Proof. Similarly as in Example 1, at least two of the strings $a, a_2, a_3, \ldots, a_{p+1}$ are in the same block of the right congruence R, say

$$a^i R a^j, 1 \leq i < j \leq p + 1.$$

Then it holds

$$w_1 = a^i b^i R a^j b^i = w_2,$$

a contradiction. (Here $w_1 \in L_1$ but not $w_2 \in L_2$.)

Theorem 2. (See [6], Theorem 3.6 with a fault in the last row.) *Let L be a regular language. Then there exists a constant $p > 0$ such that for every $w \in L$, $|w| \geq p$, the string w can be written in the form*

$$w = w_1 w_2 w_3,$$

where $0 < |W_2| \leq p$ and for all $i \in N$, $w_1 {w_2}^i w_3 \in L$.

Remark. In Theorem 2, the case $i = 0$ is included. In the proof of this Theorem there are used finite automata. The fundamental properties of regular languages and finite automata can be found, for instance, in [1], [2], [3].

Example 2. Let L_2 be a language from Lemma 1. We know that L_2 is not regular. However, the non regularity of this language can not be proved by a direct application of Theorem 2. In fact, the necessary condition is satisfied for $p = 3$. (Each string $w \in L_2$, $|w| \geq 3$, contains a substring identical to "ab" or "ba" and this substring can be used in the role of w_2 in Theorem 2.)

Remark. (See [5], Example 2.2.13.) It is known that the class of all regular languages is closed to the operation of the intersection. Using this fact and Theorem 2 we can easily prove the non-regularity of L_2. In fact, if we put

$$L_3 = \{a^i b^j | i \geq 1\},$$

then $L_2 \cap L_3 = L_1$ and the non-regularity of L_1 can be proved by Theorem 2.

3. A more strong condition of regular languages.

Theorem 3. *Let L be a regular language. Then there exists a constant $p > 0$ such that for every $w \in L$, if $|w| \geq p$ and w is written in the form*

$$w = x_1 u x_2, |u| = p,$$

then the string u can be wrtitten in the form

$$u = y_1 v y_2, |v| > 0,$$

in such a way that for each $i \in N$, $x_1 y_1 v^i y_2 x_2 \in L$.

Remark. Theorem 3 can be proved by the same method as Theorem 2. We are going to show that the necessary condition in Theorem 3 is more strong than the necessary condition in Theorem 2.

Example 3. Let us continue the Example 2. The language L_2 is non-regular but this fact can not be proved by a direct application of Theorem 2. On the other hand, it is possible to apply Theorem 3: it suffices to put

$$w = a^p b^p, x_1 = a^p, u = b^p, x_2 = \mathbf{e}.$$

Remark. By the same argument it can be proved the non-regularity of the language L_1 (see Lemma 1).

Example 4. Put $T = \{a, b\}$ and put

$$L = \{a^m (ab)^j b^k : m \geq k \geq 0, j > 0\}.$$

First we shall try to apply Theorem 2. However, the necessary condition is satisfied for $p = 3$. In fact, assume that

$$w = a^a (ab)^j b^k, m \geq k \geq 0, j > 0, m + 2j + k \geq 3.$$

There are 3 possibilities:

1) $j > 1$. Then it suffices to put

$$w_1 = a^m, w_2 = ab, w_3 = (ab)^{j-1} b^k.$$

2) $j = 1, k > 0$. Then $w = a^m (ab) b^k, m \geq k \geq 1$ and it suffices to put

$$w_1 = a^m, w_2 = ab, w_3 = b^k.$$

3) $j = 1, k = 0, m > 0$. Then $w = a^{m+1} b, m \geq 1$ and it suffices to put

$$w_1 = \mathbf{e}, w_2 = a, w_3 = a^m b.$$

Therefore, it is impossible to prove the non-regularity of the language L by a direct application of Theorem 2, but it suffices to apply Theorem 3 to the strings

$$x_1 = a^p(ab), u = b^p, x_2 = \mathbf{e}, w = ap(ab)b^p.$$

Remark. The language L from Example 4 is not regular but it is context-free. In fact, it is generated by the following context-free grammar:

$$S \longrightarrow aSb \,|\, aS \,|\, R,$$
$$R \longrightarrow Rab \,|\, ab.$$

(Here S is the starting non-terminal symbol.) The author does not know if the non-regularity of this language can be proved by a similar method as in the remark after the Example 2.

4. Two open problems.

In [4] many unsolvable problems concerning context-free grammars and languages can be found. For instance, it is unsolvable to decide if the language generated by arbitrary context-free grammar is regular. Therefore, at least one of the following two problems has the negative answer.

Problem 1. *Is it possible (for an arbitary context-free grammar G) to decide if the language generated by G satisfies the necessary condition from the Theorem 3?*

Problem 2. *Is it true that each context-free language satisfying the necessary condition from Theorem 3 is regular?*

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SUBDIRECT DECOMPOSITIONS OF DIGRAPHS

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Abstract. Direct product decompositions of the covering graph $C(\overline{\mathcal{G}})$ of a digraph $\overline{\mathcal{G}}$ and direct product decompositions of $\overline{\mathcal{G}}$ were studied in [1]. The relations between a certain type of subdirect decompositions of $C(\overline{\mathcal{G}})$ and subdirect decompositions of $\overline{\mathcal{G}}$ will be studied in the present paper.

A *graph* $\mathcal{G} = (V, E)$ consists of a nonempty set V of vertices together with a prescribed set E of unordered pairs of distinct vertices of V. Each pair $\{x, y\} \in E$ is an *(undirected) edge* of the graph $\mathcal{G}$ and shall be denoted by xy.

A *digraph* $\overline{\mathcal{G}} = (V, \overline{E})$ consists of a nonempty set V of vertices together with a prescribed set $\overline{E}$ of ordered pairs of distinct vertices. Each ordered pair $(x, y) \in \overline{E}$ is a *(directed) edge* of the digraph $\overline{G}$ and shall be denoted by $\overline{xy}$.

Let I be a nonempty set and $\mathcal{G}_i = (V_i, E_i)$, $i \in I$ be graphs. Let V be the cartesian product of the sets V_i ($V = \prod_{i\in I} V_i$). The elements of V will be denoted $a = (a_i)$, $i \in I$, where $a_i = a(i) \in V_i$. Let $\mathcal{G}$ be a graph whose set of vertices is V and whose set of edges consists of those pairs $\{x, y\}$, $x, y \in V$ which satisfy the following condition: there is $i \in I$ such that $x_i y_i \in E_i$ and $x_j = y_j$ for each $j \in I \setminus \{i\}$. Then $\mathcal{G}$ is said to be the *direct product of the graphs* $\mathcal{G}_i$, $i \in I$ and we write $\mathcal{G} = \prod_{i\in I} \mathcal{G}_i$.

The direct product of digraphs is defined similarly.

For all further notions concerning digraphs and graphs we refer the reader to [2].

Let $\prod_{i\in I} \mathcal{G}_i = (V, E)$. If $W \subseteq V$, then we denote $O_i(W) = \{a_i a \in W\}$.

Let $\prod_{i\in I} \mathcal{G}_i = (V, E)$ be the direct product of graphs $\mathcal{G}_i = (V_i, E_i)$ ($i \in I$). If $W \subseteq V$ and $O_i(W) = V_i$ for each $i \in I$, then a graph $\mathcal{G} = (W, F)$, where

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$F = \{ab \in Ea, b \in W\}$, will be called a *subdirect product* of the graphs $\mathcal{G}_i$. If $\mathcal{G}$ is a subdirect product of graphs $\mathcal{G}_i$ we write $\mathcal{G} = (\text{sub}) \prod_{i \in I} \mathcal{G}_i$.

Subdirect products of digraphs are defined similarly.

Remark. If $W = V$, then $(\text{sub}) \prod_{i \in I} \mathcal{G}_i = \prod_{i \in I} \mathcal{G}_i$.

The subgraph of a graph $\mathcal{G} = (V, E)$ induced by a set $W \subseteq V$ will be denoted by $\mathcal{G}\langle W \rangle$.

Remark. Since a graph $(\text{sub}) \prod_{i \in I} \mathcal{G}_i$ is in fact a subgraph of the graph $\prod_{i \in I} \mathcal{G}_i$ induced by a suitable set W with $O_i(W) = V_i$ for each $i \in I$, then $(\text{sub}) \prod_{i \in I} \mathcal{G}_i = (\prod_{i \in I} \mathcal{G}_i)\langle W \rangle$.

If a mapping $fV_1 \to V_2$ is an isomorphism of a graph $\mathcal{G}_1 = (V_1, E_1)$ onto a graph $\mathcal{G}_2 = (V_2, E_2)$, then we shall write $\mathcal{G}_1 \overset{\text{f}}{\simeq} \mathcal{G}_2$ or shortly $\mathcal{G}_1 \simeq \mathcal{G}_2$.

If $\mathcal{G} \overset{\text{f}}{\simeq} (\text{sub}) \prod_{i \in I} \mathcal{G}_i$ then we shall say that $(\text{sub}) \prod_{i \in I} \mathcal{G}_i$ is a *subdirect decomposition* of the graph $\mathcal{G}$ (with respect to the mapping f).

In the present paper every subdirect decomposition $(\text{sub}) \prod_{i \in I} \mathcal{G}_i$, where $\mathcal{G}_i = (V_i, E_i)$, is supposed to be nontrivial (i. e. $|V_i| > 1$ for each $i \in I$).

Analogous terminology and notation are used for digraphs.

Let $\overline{\mathcal{G}} = (V, \overline{E})$ be a digraph. By the *covering graph* of $\overline{\mathcal{G}}$ we mean the graph $C(\overline{\mathcal{G}}) = (V, E)$ where $ab \in E$ iff $\overline{ab} \in \overline{E}$.

The following two lemmas are easy to verify.

Lemma 1. *Let $\overline{\mathcal{G}}_1 = (V_1, \overline{E}_1)$, $\overline{\mathcal{G}}_2 = (V_2, \overline{E}_2)$ be digraphs. If $\overline{\mathcal{G}}_1 \overset{\text{f}}{\simeq} \overline{\mathcal{G}}_2$ then $C(\overline{\mathcal{G}}_1) \overset{\text{f}}{\simeq} C(\overline{\mathcal{G}}_2)$.*

Lemma 2. *Let $\prod_{i \in I} \overline{\mathcal{G}}_i = (V, \overline{E})$ be the direct product of digraphs $\overline{\mathcal{G}}_i$, $i \in I$ and let $W \subseteq V$. Then $C((\prod_{i \in I} \overline{\mathcal{G}}_i)\langle W \rangle) = (\prod_{i \in I} C(\overline{\mathcal{G}}_i))\langle W \rangle$.*

Lemma 1 and Lemma 2 imply the following

Theorem 1. *Let $\overline{\mathcal{G}}$, $\overline{\mathcal{G}}_i$, $i \in I$ be digraphs and $\overline{\mathcal{G}} \overset{\text{f}}{\simeq} (\text{sub}) \prod_{i \in I} G_i$. Then $C(\overline{\mathcal{G}}) \overset{\text{f}}{\simeq} (\text{sub}) \prod_{i \in I} C(\overline{\mathcal{G}}_i)$.*

Definition. Let $\overline{\mathcal{G}} = (V, \overline{E})$ be a digraph and let $C(\overline{\mathcal{G}}) \overset{\text{f}}{\simeq} (\text{sub}) \prod_{i \in I} \mathcal{G}_i$, where $\mathcal{G}_i = (V_i, E_i)$, $i \in I$. We shall say that the subdirect decomposition $(\text{sub}) \prod_{i \in I} \mathcal{G}_i$ of the graph $C(\overline{\mathcal{G}})$ induces a subdirect decomposition of the digraph $\overline{\mathcal{G}}$ if there exist such digraphs $\overline{\mathcal{G}}_i = (V_i, \overline{E}_i)$ that $C(\overline{\mathcal{G}}_i) = \mathcal{G}_i$ for each $i \in I$ and $\overline{\mathcal{G}} \overset{\text{f}}{\simeq} (\text{sub}) \prod_{i \in I} \overline{\mathcal{G}}_i$.

A subdirect decomposition of $C(\overline{\mathcal{G}})$ does not induce a decomposition of $\overline{\mathcal{G}}$ in general. The digraph $\overline{\mathcal{G}} = (\{a,b,c,d\},\{\overline{ab},\overline{bc},\overline{cd},\overline{da}\})$ is not isomorphic to the subdirect product of any two digraphs but its covering graph is isomorphic to the subdirect (direct) product of two complete graphs K_2.

We are going to investigate when a subdirect decomposition of $C(\overline{\mathcal{G}})$ induces a subdirect decomposition of $\overline{\mathcal{G}}$.

Let $\mathcal{G} = (V,E)$ be a graph. If there exists a four-element set $W = \{a,b,c,d\} \subseteq V$ such that $\mathcal{G}\langle W\rangle = (W,\{ab,bc,cd,ad\})$, then we say that the graph $\mathcal{G}\langle W\rangle$ is a *square* (in $\mathcal{G}$) and we denote it by $\mathcal{S}(a,b,c,d)$. If $\overline{\mathcal{G}}$ is a digraph and $C(\overline{\mathcal{G}}\langle W\rangle) = \mathcal{S}(a,b,c,d)$, then the digraph $\overline{\mathcal{G}}\langle W\rangle$ is called a *square* (in $\overline{\mathcal{G}}$) and will be denoted by $\overline{\mathcal{S}}(a,b,c,d)$.

An edge ab of a graph $\prod_{i\in I}\mathcal{G}_i$ ((sub)$\prod_{i\in I}\mathcal{G}_i$) will be called a *k-edge* whenever $a_j = b_j$ for each $j \in I\setminus\{k\}$.

We say that ordered pairs (a,b) and (c,d) of vertices of a direct product $\prod_{i\in I}\mathcal{G}_i$ (subdirect product (sub)$\prod_{i\in I}\mathcal{G}_i$) are *r-equivalent* and write $(a,b) \overset{\mathrm{r}}{\sim} (c,d)$ if ab and cd are r-edges and $a_r = c_r$, $b_r = d_r$.

It is easy to see that if $(a,b) \overset{\mathrm{r}}{\sim} (c,d)$ then $(b,a) \overset{\mathrm{r}}{\sim} (d,c)$.

A square $\mathcal{S}(a,b,c,d)$ in $\prod_{i\in I}\mathcal{G}_i$ ((sub)$\prod_{i\in I}\mathcal{G}_i$) will be called an *r-square* whenever all its edges are r-edges for some $r\in I$. If such $r\in I$ does not exist, it will be called a *mixed square*.

Let $C(\overline{\mathcal{G}}) \overset{\mathrm{f}}{\simeq} \prod_{i\in I}\mathcal{G}_i$. We shall say that the edge $\overline{ab}$ of the digraph $\overline{\mathcal{G}}$ and the edge ab of the covering graph $C(\overline{\mathcal{G}})$ are k-edges (with respect to the isomorphism f) if $f(a)f(b)$ is a k-edge of the graph $\prod_{i\in I}\mathcal{G}_i$. In an analogous way the other notions concerning the direct product $\prod_{i\in I}\mathcal{G}_i$ can be introduced for the digraph $\overline{\mathcal{G}}$ and the covering graph $C(\overline{\mathcal{G}})$.

In [1] it was proved that if $\mathcal{S}(a,b,c,d)$ is a mixed square, then there exist $r,s\in I$, $r\neq s$ such that ab, cd are r-edges and bc, ad are s-edges (cf. Lemmas 2, 3, 4 in [1]).

Lemma 3 [1]. *Let $\mathcal{S}(a,b,c,d)$ be a mixed square in $\prod_{i\in I}\mathcal{G}_i$, where ab is an r-edge and bc is an s-edge. Then $(a,b) \overset{\mathrm{r}}{\sim} (d,c)$, $(b,c) \overset{\mathrm{s}}{\sim} (a,d)$.*

Since (sub)$\prod_{i\in I}\mathcal{G}_i = (\prod_{i\in I}\mathcal{G}_i)\langle W\rangle$, the above mentioned facts hold also for the subdirect products.

Let (sub)$\prod_{i\in I}\mathcal{G}_i = (V,E)$ be a subdirect product of graphs $\mathcal{G}_i = (V_i,E_i)$ and let $(a_i), (b_i) \in V$, $i\in I$. We shall say that the subdirect product (sub)$\prod_{i\in I}\mathcal{G}_i$ is *orientable* if the following condition is fulfilled:

If $a_kb_k \in E_k$ then there exists a k-edge $(a_i)(b_i)\in E$, $i\in I$.

Example. Let $\mathcal{G} = (\{a,b,c\},\{ab,bc\})$, $\mathcal{G}' = (\{1,2,3,4\},\{12,23,34\})$ be graphs. Let $W = \{(a,a),(a,b),(a,c),(b,a),(c,a),(c,c)\}$ and $W' =$

$= \{(1,1),(1,2),(1,3),(2,1),(3,1),(4,4)\}$. Then the subdirect product $(\mathrm{sub})\prod_{i\in\{1,2\}}\mathcal{G}_i = (\prod_{i\in\{1,2\}}\mathcal{G}_i)\langle W\rangle$, where $\mathcal{G}_i = \mathcal{G}$, $i \in \{1,2\}$, is orientable and the subdirect product $(\mathrm{sub})\prod_{i\in\{1,2\}}\mathcal{G}'_i = (\prod_{i\in\{1,2\}}\mathcal{G}'_i)\langle W'\rangle$, where $\mathcal{G}'_i = \mathcal{G}'$, $i \in \{1,2\}$, is not orientable. Let us notice that $(\mathrm{sub})\prod_{i\in\{1,2\}}\mathcal{G}_i \simeq (\mathrm{sub})\prod_{i\in\{1,2\}}\mathcal{G}'_i$.

All subdirect products considered in the next are assumed to be orientable.

Lemma 4. *Let $C(\overline{\mathcal{G}}) \overset{\mathrm{f}}{\simeq} (\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$, where $\overline{\mathcal{G}} = (V,\overline{E})$ and $\mathcal{G}_i = (V_i,E_i)$. The subdirect decomposition $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$ of $C(\overline{\mathcal{G}})$ induces a subdirect decomposition of $\overline{\mathcal{G}}$ if and only if for any two r-equivalent ordered pairs (a,b), (c,d) of vertices of $\overline{\mathcal{G}}$ the following condition is fulfilled:*

$$(1) \qquad \overline{ab} \in \overline{E} \quad \text{if and only if} \quad \overline{cd} \in \overline{E}.$$

Proof. It suffices to define $\overline{\mathcal{G}}_i$ for each $i \in I$ by $\overline{\mathcal{G}}_i = (V_i,\overline{E}_i)$, where $\overline{f(a)_i f(b)_i} \in \overline{E}_i$ if and only if there exists an i-edge $\overline{ab} \in \overline{E}$.

A subdirect product $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i = (W,E) = \mathcal{G}$ is said an *l-product* if the following condition is fulfilled:

> If $a,b,c,d \in W$ and $(a,b) \overset{\mathrm{r}}{\sim} (c,d)$, then there exist a nonnegative integer n and vertices $x^0 = a, x^1, \dots, x^n = c$, $y^0 = b, y^1, \dots, y^n = d \in W$ such that $\mathcal{G}\langle x^j, x^{j+1}, y^{j+1}, y^j\rangle$ is a mixed square $\mathcal{S}(x^j, x^{j+1}, y^{j+1}, y^j)$ for each $j \in \{0,1,\dots,n-1\}$.

Remark. If $\mathcal{G} = \prod_{i\in I}\mathcal{G}_i$ is a connected graph, then the direct product $\prod_{i\in I}\mathcal{G}_i$ is an l-product (cf. Lemma 6 in [1]).

The following theorem is a generalization of a result from [1].

Theorem 2. *Let $C(\overline{\mathcal{G}}) \overset{\mathrm{f}}{\simeq} (\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$, where $\overline{\mathcal{G}} = (V,\overline{E})$ is a digraph and $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$ is an l-product. The subdirect decomposition $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$ of $C(\overline{\mathcal{G}})$ induces a subdirect decomposition of $\overline{\mathcal{G}}$ if and only if the following condition is fulfilled:*

(2) If $\overline{\mathcal{S}}(a,b,c,d)$ is a mixed square in $\overline{\mathcal{G}}$, then there exists $i \in \{1,2,3\}$ with $\overline{\mathcal{S}}(a,b,c,d) \simeq \overline{\mathcal{S}}_i$, where
$\overline{\mathcal{S}}_1 = (\{a,b,c,d\},\{\overline{ab},\overline{bc},\overline{dc},\overline{ad}\})$,
$\overline{\mathcal{S}}_2 = (\{a,b,c,d\},\{\overline{ab},\overline{ba},\overline{bc},\overline{cd},\overline{dc},\overline{ad}\})$,
$\overline{\mathcal{S}}_3 = (\{a,b,c,d\},\{\overline{ab},\overline{ba},\overline{bc},\overline{cb},\overline{cd},\overline{dc},\overline{da},\overline{ad}\})$.

Proof. Let the subdirect decomposition $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$ of $C(\overline{\mathcal{G}})$ induce a subdirect decomposition of $\overline{\mathcal{G}}$ and $\overline{\mathcal{S}}(a,b,c,d)$ be its mixed square. Then, by

Lemma 3, there exist $r, s \in I$, $r \neq s$, such that $(a,b) \overset{\text{r}}{\sim} (d,c)$, $(b,c) \overset{\text{s}}{\sim} (a,d)$ and $(b,a) \overset{\text{r}}{\sim} (c,d)$, $(c,b) \overset{\text{s}}{\sim} (d,a)$. From Lemma 4 it follows that $\overline{ab} \in \overline{E}$ iff $\overline{dc} \in \overline{E}$, $\overline{bc} \in \overline{E}$ iff $\overline{ad} \in \overline{E}$ and $\overline{ba} \in \overline{E}$ iff $\overline{cd} \in \overline{E}$, $\overline{cb} \in \overline{E}$ iff $\overline{da} \in \overline{E}$. Thus there exists $i \in \{1,2,3\}$ with $\overline{\mathcal{S}}(a,b,c,d) \simeq \overline{\mathcal{S}}_i$. To prove the converse implication, suppose that (2) is fulfilled. With respect to Lemma 4, it suffices to prove that if $(x,y) \overset{\text{r}}{\sim} (u,v)$, then (1) holds. Since (sub) $\prod_{i \in I} \mathcal{G}_i$ is an l-product, then there exist a nonnegative integer n and vertices $x^0 = x, x^1, \ldots, x^n = u$, $y^0 = y, y^1, \ldots, y^n = v \in V$ such that $\overline{\mathcal{G}}\langle x^j, x^{j+1}, y^{j+1}, y^j\rangle$ is a mixed square $\overline{\mathcal{S}}(x^j, x^{j+1}, y^{j+1}, y^j)$ in $\overline{\mathcal{G}}$ for each $j \in \{0, 1, \ldots, n-1\}$. If $n = 0$, then (1) holds, since $(x,y) = (u,v)$. If $n = 1$, then $\overline{\mathcal{S}}(x,u,v,y)$ is a mixed square and from (2) it follows (1). Now it is easy to complete the proof by induction on n.

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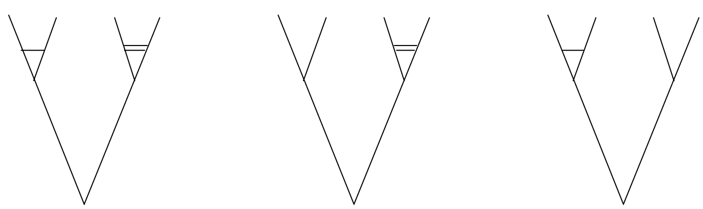
CHARACTERIZATION OF UNIVERSAL QUASIGROUP IDENTITIES OF CANONICAL TYPE

Gabriela Monoszová

Abstract. Quasigroup identities of canonical type are defined. Conditions which are necessary and sufficient for such identities to be universal are found.

In 1968 Belousov [1] posed the conjecture that the variety of quasigroups is invariant under the isotopies if and only if it can be characterized by equalities whose coresponding diagrams satisfy the following two conditions:

(1) they can contain only forks of the following three types



(2) if the numbers 1, 2 and 3 are assigned to the tops of the forks as indicated below

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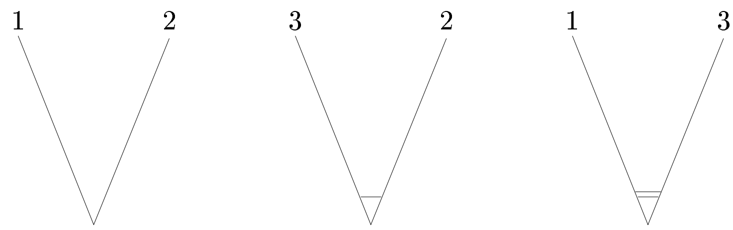


then whenever an element x has a position i, $i = 1, 2, 3$ in a fork then the element x has the same position i in any fork containing x.

In the present paper necessary and sufficient conditions for a class of quasigroup identities to be invariant under the isotopies are given.

Let $(Q; A)$ be a quasigroup. The right or the left inverse operation to the operation A will be denoted by ${}^rA$ or ${}^lA$, respectively. Using the unary functors r and l, we can assign the set $\Sigma_A = \{A, {}^rA, {}^lA, {}^{rl}A, {}^{lr}A, {}^{rlr}A\}$, to the quasigroup $(Q; A)$. Here ${}^{rl}A := {}^r({}^lA)$ and similarly for ${}^{rl}A$ and ${}^{rlr}A$. Denote by ι the identity map. Since $r^2 = l^2 = \iota$ and $rlr = lrl$, the set $\Sigma = \{\iota, r, l, rl, lr, rlr\}$ with the composition is a group isomorphic to the group of all permutations of a three-element set. Hence for every $\sigma \in \Sigma$, $A(x, y) = z$ if and only if ${}^\sigma A(\sigma x, \sigma y) = \sigma z$.

Throughout the paper X is a countable set of variables, $T(\Sigma_A)$ is the set of all terms of the language Σ_A over X and $R(\Sigma_A)$ is the set of all identities of the language Σ_A over X. For $w \in R(\Sigma_A)$ let W_w be the set of all variables contained in w and V_w the variety characterized by the identity w.

To every quasigroup identity $w \in R(\Sigma_A)$ a diagram can be assigned such that to every operation from Σ_A a vertex coresponds as shown in Fig.1.

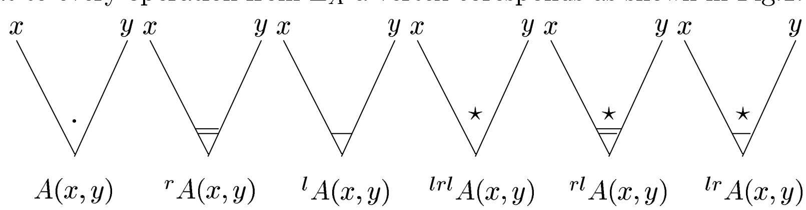


Fig.1

Moreover, to every variable occuring in the identity w we can assign one of the numbers 1, 2 and 3 according to Fig.2 (note that here the notation of vertices plays an important role).

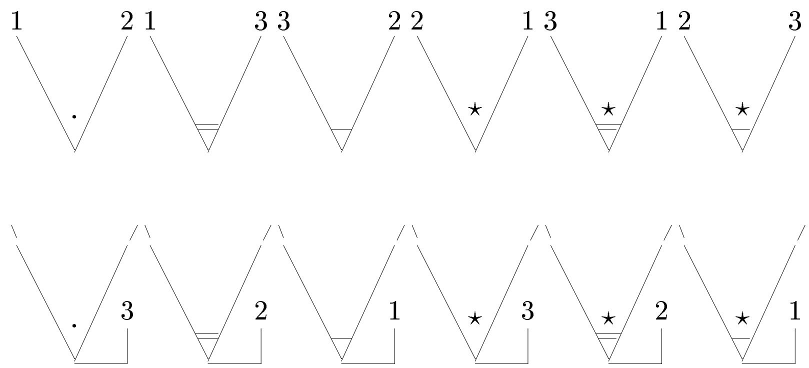


Fig.2

We give an example. Consider the identity

$$((zx) * y)\ {}^{rl}(.)z = (x\backslash y)(z/x)$$

where $(A) = (.)$, $({}^{r}A) = (\backslash)$, $({}^{l}A) = (/)$, $({}^{rlr}A) = (*)$ (i.e., the multiplicative notation is used). Then the diagram assigned to w is that in Fig.3.

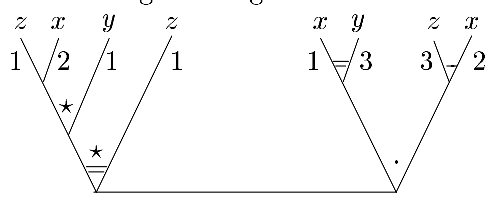


Fig.3

Definition 1. Every quasigroup identity $w \in R(\Sigma_A)$ of the form

$${}^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \ldots, {}^{\sigma_n}A(x_n, x_{n+1}))\ldots) = x_o$$

where $n \in N$, $x_o, x_1, \ldots, x_{n+1} \in X$, $\sigma_1, \sigma_2, \ldots, \sigma_n \in \Sigma$ will be called an identity of canonical type.

Definition 2. An identity $w \in R(\Sigma_A)$ of canonical type will be called optimal, if it satisfies the following three conditions

(i) w contains no terms of the forms

$$B(x, {}^{r}B(x, t)),$$

$${}^{l}B(x, {}^{r}B(y, x)),$$

where $B \in \Sigma_A$, $t \in T(\Sigma_A)$, $x, y \in W_w$;

(ii) for every i, j with $2 \le i \le j \le n$ there exists a quasigroup $Q \in V_w$ in which the identity

$$A_i(x_i, A_{i+1}(x_{i+1}, \dots, A_n(x_n, x_{n+1}))\dots) =$$

$$= A_j(x_j, A_{j+1}(x_{j+1}, \dots, A_n(x_n, x_{n+1}))\dots)$$

does not hold;

(iii) for every i with $2 \le i \le n$ there exists $Q \in V_w$ in which the identity

$$A_i(x_i, A_{i+1}, \dots, A_n(x_n, x_{n+1}))\dots) = x_{n+1}$$

does not hold.

An optimal quasigroup identity will be called nontrivial if it contains no isolated variable, i.e., if every variable $x \in W_w$ occurs at least twice in the identity w. In the sequel only nontrivial quasigroup identities will be considered.

Let (Q_1, B) and (Q_2, A) be quasigroups. An ordered triple $T = (\alpha, \beta, \gamma)$ of bijections of the set Q_1 onto the set Q_2 is called an isotopy of the quasigroup (Q_1, B) onto the quasigroup (Q_2, A) provided the following diagram commutes

$$\begin{array}{ccc} Q_1 \times Q_1 & \xrightarrow{B} & Q_1 \\ {\scriptstyle \alpha\times\beta}\downarrow & & {\scriptstyle \gamma}\downarrow \\ Q_2 \times Q_2 & \xrightarrow{A} & Q_2 \end{array}$$

The quasigroup $(Q_2; A)$ is also called an isotope of the quasigroup $(Q_1; B)$.

Let $T(\varphi_1, \varphi_2, \varphi_3)$ be an isotopy of a quasigroup $(Q; B)$ onto $(Q; A)$, (without loss of generality we can assume that $Q_1 = Q_2 =: Q$, since $Q_1 \cong Q_2$). An identity $w \in R(\Sigma_A)$ of the form

$$\text{(w)} \qquad {}^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \dots, {}^{\sigma_n}A(x_n, x_{n+1}))\dots) = x_o$$

is called universal [2], if it is invariant under every quasigroup isotopy. In other words, w is an universal identity if it holds in some quasigroup $(Q; A)$

if and only if for any isotopy $T = (\varphi_1, \varphi_2, \varphi_3)$ onto quasigroup (Q, A), the identity

(Tw)

$$^{\sigma_1}A(\varphi_{\sigma_1 1}x_1, \varphi_{\sigma_1 2}\varphi^{-1}_{\sigma_2 3}{}^{\sigma_2}A(\varphi_{\sigma_2 1}x_2, \dots, \varphi_{\sigma_{n-1}2}\varphi^{-1}_{\sigma_n 3}{}^{\sigma_n}A(\varphi_{\sigma_n 1}x_n, \varphi_{\sigma_n 2}x_{n+1}))\dots) =$$

$$= \varphi_{\sigma_1 3}x_o$$

also holds in $(Q; A)$.

Definition 3. We will say that the diagram of a quasigroup identity satisfies the condition $\boxed{B1}$ if all forks contained in the diagram are of the following form (see Fig. 4):

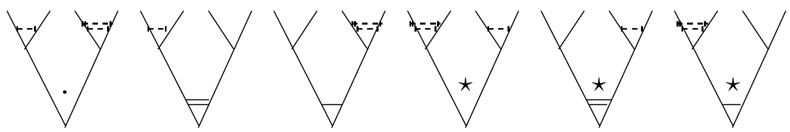
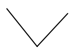
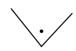
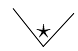
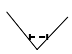
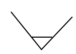
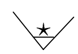
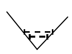
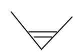


Fig. 4

Here  denotes any of the vertices  or 

Here  denotes any of the vertices  or 

Here  denotes any of the vertices  or 

Definition 4. We will say that the diagram of a quasigroup identity satisfies the condition $\boxed{B2}$ if every variable occuring at a top of the diagram has the same position in each of its occurences (see Fig.2).

Theorem 5. *Let an identity $w \in R(\Sigma_A)$ be nontrivial. If the diagram of the identity w satisfies the conditions $\boxed{B1}$ and $\boxed{B2}$ then w is invariant under the isotopies of quasigroups.*

Proof. Let $w \in R(\Sigma_A)$

(w) $$^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \ldots, {}^{\sigma_n}A(x_n, x_{n+1}))\ldots) = x_o$$

holds in a quasigroup $(Q;\Sigma_A)$ and let $T = (\varphi_1, \varphi_2, \varphi_3)$ be an isotopy onto the qausigroup $(Q;\Sigma_A)$. We are going to prove that also the identity

(Tw)

$$^{\sigma_1}A(\varphi_{\sigma_1 1}x_1, \varphi_{\sigma_1 2}\varphi^{-1}_{\sigma_2 3}{}^{\sigma_2}A(\varphi_{\sigma_2 1}x_2, \ldots, \varphi_{\sigma_{n-1}2}\varphi^{-1}_{\sigma_n 3}{}^{\sigma_n}A(\varphi_{\sigma_n 1}x_n, \varphi_{\sigma_n 2}x_{n+1}))\ldots) =$$
$$= \varphi_{\sigma_1 3}x_o$$

holds in the quasigroup $(Q;\Sigma_A)$.
Write the identity Tw in the following more convenient form

(Tw)
$$\begin{pmatrix} & \varphi_{\sigma_1 1} & & \varphi_{\sigma_2 1} & & & \varphi_{\sigma_n 1} & \varphi_{\sigma_n 2} & & \varphi_{\sigma_1 3} \\ & & \varphi_{\sigma_1 2}\varphi^{-1}_{\sigma_2 3} & & & \varphi_{\sigma_{n-1}2}\varphi^{-1}_{\sigma_n 3} & & & & \\ {}^{\sigma_1}A & (x_1, & {}^{\sigma_2}A & (x_2, & \ldots, & {}^{\sigma_n}A & (x_n, & x_{n+1})) & \ldots) = & x_o \end{pmatrix}$$

Here the coefficients (i.e., the bijections) from the isotopy $T(\varphi_1, \varphi_2, \varphi_3)$ corresponding to the variables occuring in the identity are written in the first row. The second row contains the coefficients (i.e. compositions of two bijections from the isotopy $T(\varphi_1, \varphi_2, \varphi_3)$) corresponding to the symbols of binary operations.

By the assumption the diagram of the identity w satisfies the condition $\boxed{B2}$. Thus every variable has the same position in each of its occurences. Therefore we can omit the first row and write the identity w using only two rows.

Since the diagram of the identity w satisfies also the condition $\boxed{B1}$, we have $\varphi_{\sigma_k 2} = \varphi_{\sigma_{k+1}3}$, i.e. $\varphi_{\sigma_k 2}\varphi^{-1}_{\sigma_{k+1}3} = \iota$ for any $k \in \{1, 2, \ldots, n-1\}$. Therefore also the second row can be omitted. Then the identities Tw and w have the same form and so it is obvious that Tw holds in a quasigroup if and only if w holds in it.

Theorem 6. *Let $w \in R(\Sigma_A)$ be a nontrivial identity. If the diagram of the identity w satisfies the condition $\boxed{B2}$ and does not satisfy the condition $\boxed{B1}$ then w is not a universal identity.*

Proof. Let $w \in R(\Sigma_A)$ be a nontrivial identity and let $T = (\varphi_1, \varphi_2, \varphi_3)$ be an isotopy onto a quasigroup $(Q;A)$. Write the identities w and Tw as follows:

$$\text{(w)} \qquad {}^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \ldots, {}^{\sigma_n}A(x_n, x_{n+1}))\ldots) = x_o$$

(Tw)

$$\left(\begin{matrix} & \varphi_{\sigma_1 1} & & \varphi_{\sigma_2 1} & & & \varphi_{\sigma_n 1} & \varphi_{\sigma_n 2} & & \varphi_{\sigma_1 3} \\ & & \varphi_{\sigma_1 2}\varphi_{\sigma_2 3}^{-1} & & & \varphi_{\sigma_{n-1} 2}\varphi_{\sigma_n 3}^{-1} & & & & \\ {}^{\sigma_1}A & (x_1, & {}^{\sigma_2}A & (x_2, & \ldots, & {}^{\sigma_n}A & (x_n, & x_{n+1})) & \ldots) = & x_o \end{matrix} \right)$$

Choose $T = (\varphi_1, \varphi_2, \varphi_3) = (\iota, \beta, \iota)$. The diagram of the identity w satisfies the condition $\boxed{B2}$. For the same reasons as in the proof of Theorem 1 we can omit the first row in Tw to get

$$\text{(Tw)} \qquad \left(\begin{matrix} \beta^{\varepsilon_1} & & \beta^{\varepsilon_2} & & \ldots & \beta^{\varepsilon_n} & \\ {}^{\sigma_1}A & (x_1, & {}^{\sigma_2}A & (x_2, & \ldots, & {}^{\sigma_n}A & (x_n, x_{n+1}))\ldots) = x_o \end{matrix} \right)$$

Here $\varepsilon_1 = 0$ and $\varepsilon_i \in \{-1, 0, 1\}$ for $i \in \{2, \ldots, n\}$.

The ordered n-tuple $< \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n >$ is said to be the signature of the identity Tw and every pair $< \varepsilon_j, \varepsilon_{j+m} >$ with $j, m \in \{1, 2, \ldots, n-1\}$, $j + m \leq n$ a sign change in the signature provided $\varepsilon_j . \varepsilon_{j+m} = -1$ and simultaneously $\varepsilon_{j+1} = \varepsilon_{j+2} = \cdots = \varepsilon_{j+m-1} = 0$.

We are going to show that there exist a permutation β and a quasigroup Q such that the identity w holds in Q but the identity Tw where $T = (\iota, \beta, \iota)$, does not hold in Q. Denote the number of sign changes in the signature $< \varepsilon_1, \ldots, \varepsilon_n >$ of w by d and the number of non-zero elements of this signature by p. Without loss of generality we can assume that, for example,

$$\underbrace{\varepsilon_n = \varepsilon_{k_p} = 1, \quad \varepsilon_{n-1} = 0, \quad \varepsilon_{n-2} = \varepsilon_{k_{p-1}} = -1, \ldots, \varepsilon_3 = 0, \quad \varepsilon_2 = \varepsilon_{k_1} = 1}_{\text{the signature has} \quad \text{p} \quad \text{non-zero elements}}$$

Then the following table can be assigned to the identity Tw

	${}^{\sigma_n}A(x_n, x_{n+1})$		$\dots$	${}^{\sigma_n}A(x_2, {}^{\sigma_3}A(x_3, {}^{p-1}y))$
		${}^{\sigma_{n-2}}A(x_{n-2}, {}^{\sigma_{n-1}}A(x_{n-1}, {}^{1}y))$	$\dots$	
βx	${}^{1}y$			${}^{p}y$
$\beta^{-1}x$		${}^{2}y$	$\dots$	

Tab.

Here in the first row we have the arguments of the permutation β and in the second row the arguments of the permutation β^{-1}. The values of β or β^{-1} at these arguments are written in the third or fourth row, respectively. Since we want β to be a permutation, from the table we can see that all the values in the first and fourth row must be mutually different. Hence we get $\binom{p}{2}$ conditions. Similarly we get other $\binom{p}{2}$ conditions by taking into consideration that all the values in the second and third row must be mutually different. We are going to define β so that it satisfies all the mentioned conditions and, moreover, the condition

$$^{\sigma_1}A(x_1, \beta^{\varepsilon_2}\ {}^{\sigma_2}A(x_2, \dots, \beta^{\varepsilon_n}\ {}^{\sigma_n}A(x_n, x_{n+1}))\dots) \neq x_o,$$

i.e.

$$^{\sigma_1}A(x_1, {}^{p}y) \neq x_o.$$

Exactly d conditions from all p(p-1)+1 ones have the following form

$$(q_k) \qquad {}^{\sigma_i}A(x_i, \dots, {}^{\sigma_j}A(x_j, {}^{k}y))\dots) \neq {}^{k}y$$

Since the identity w is optimal, for every condition (q_k), $k \in \{1, \dots, d\}$ there is a quasigroup $Q_k \in V_w$ and elements of this quasigroup for which (q_k) holds. By Birkhoff theorem, $Q := Q_1 \times Q_2 \times \cdots \times Q_d \in V_w$. Then it is possible to define (we do not go into details) n+2 elements a_1, a_2, ... ,a_{n+1}, a_o from the quasigroup Q and a permutation β on Q such that

$$\beta^{\varepsilon_1 \sigma_1}A(a_1, \beta^{\varepsilon_2 \sigma_2}A(a_2, \dots, \beta^{\varepsilon_n \sigma_n}A(a_n, a_{n+1}))\dots) \neq a_o.$$

So the identity w holds in the quasigroup $Q = Q_1 \times Q_2 \times \cdots \times Q_d$, but the identity Tw, $T = (\iota, \beta, \iota)$ does not hold in this quasigroup. Therefore w is not universal.

Theorem 7. *Let $w \in R(\{A, {}^rA, {}^lA\})$ be a nontrivial identity. If the diagram of the identity w satisfies the condition $\boxed{B1}$ and simultaneously does not satisfy the condition $\boxed{B2}$ then w is not universal.*

Proof. If the diagram of the identity

$$^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \dots, {}^{\sigma_n}A(x_n, x_{n+1}))\dots) = x_o$$

satisfies the condition $\boxed{B1}$, then for every $\sigma_i \in \{\iota, r, l,\}$ the following holds
- if $\sigma_1 \in \{\iota, r\}$, then $\sigma_{k+1} = r\sigma_k$, k=1,2, ... ,n-1;
- if $\sigma_1 = l$, then $\sigma_2 = r$ and $\sigma_{k+1} = r\sigma_k$, k=2,3, ... ,n-1.

Depending on σ_1 and parity of the length $l(w)$ of the identity w we obtain six possible forms of w (the multiplicative notation will be used):

1) if $\sigma_1 = \iota$ and $l(w)$ is an even number

$$(w_1) \qquad x_1(x_2\backslash(x_3(x_4\backslash\dots\backslash(x_{n-1}(x_n\backslash x_{n+1}))\dots) = x_o;$$

2) if $\sigma_1 = \iota$ and $l(w)$ is an odd number

$$(w_2) \qquad x_1(x_2\backslash(x_3(x_4\backslash\dots\backslash(x_{n-1}\backslash(x_n.x_{n+1}))\dots) = x_o;$$

3) if $\sigma_1 = r$ and $l(w)$ is an even number

$$(w_3) \qquad x_1\backslash(x_2(x_3\backslash\dots(x_{n-1}\backslash(x_n.x_{n+1}))\dots) = x_o;$$

4) if $\sigma_1 = r$ and $l(w)$ is an odd number

$$(w_4) \qquad x_1\backslash x_2(x_3\backslash\dots\backslash(x_{n-1}(x_n\backslash x_{n+1}))\dots) = x_o;$$

5) if $\sigma_1 = l$ and $l(w)$ is an even number

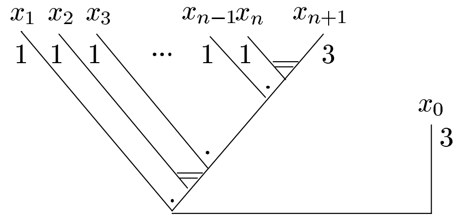
$$(w_5) \qquad x_1/(x_2\backslash(x_3(x_4\backslash\dots\backslash(x_{n-1}(x_n\backslash x_{n+1}))\dots) = x_o;$$

6) if $\sigma_1 = l$ and $l(w)$ is an odd number

$$(w_6) \qquad x_1/(x_2\backslash(x_3(x_4\backslash\dots(x_{n-1}\backslash(x_n.x_{n+1}))\dots) = x_o.$$

By using transformations which transform the identities fulfilled or not fulfilled in a quasigroup into the identities which are fulfilled or not fulfilled, respectively, in it, one can get the identity w_1 from w_5 and the identity w_2 from w_6. So it suffices to prove that the identities w_1, w_2, w_3 and w_4 are universal:

1a) If $x_{n+1} \neq x_o$, then the diagram of the identity w_1 has the following form:



Then there is at least one i and at least one j, $i, j \in \{1, 2, \ldots, n\}$ such that $x_i = x_{n+1}$ and $x_j = x_o$. Let, e.g., i=2 and j=3. Then

$$(Tw_1) \qquad \alpha x_1(\alpha x_2 \backslash (\alpha x_3(\alpha x_4 \ldots (\alpha x_{n-1} \backslash \gamma x_2)) \ldots) = \gamma x_3.$$

Choose a quasigroup Q from the variety V_{w_1} and elements $a_1, a_2, \ldots, a_n, b \in Q$ such that $b \neq a_2 \neq a_3 \neq b$. After substituting these elements into the identity w_1 we get

$$a_1(a_2 \backslash (a_3(a_4 \backslash \ldots (a_{n-1}(a_n \backslash a_2)) \ldots) = a_3.$$

Choose the isotopy $T(\iota, \iota, \gamma)$ onto the quasigroup Q with
$\gamma a_3 = b$, $\gamma b = a_3$, $\quad \gamma x = x$, for all $x \in Q - \{a_3, b\}$.
Substitute these elements into

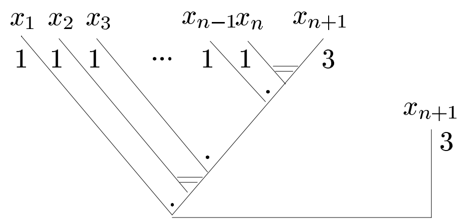
$$(Tw_1) \qquad x_1(x_2 \backslash (x_3(x_4 \backslash \ldots (x_{n-1}(x_n \backslash \gamma x_2)) \ldots) = \gamma x_3$$

to get

$$a_1(a_2 \backslash (a_3(a_4 \backslash \ldots (a_{n-1}(a_n \backslash a_2)) \ldots) = b.$$

Clearly, the identity Tw_1 does not hold in Q. Therefore w_1 is not universal.

1b) If $x_{n+1} = x_o$ then we have the following diagram of the identity w_1



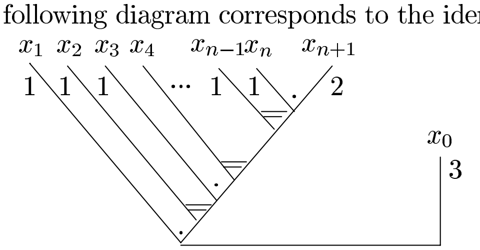
Since this diagram does not satisfy the condition $\boxed{B2}$, there is at least one $i \in \{1, 2, \ldots, n\}$ with $x_i = x_{n+1}$. Let, e.g. $x_2 = x_4 = x_{n+1}$. Then

$$(w_1) \qquad x_1(x_2\backslash(x_3\backslash(x_2(x_5\backslash \ldots (x_n\backslash x_2)) \ldots) = x_2$$

$$(Tw_1) \qquad x_1(x_2\backslash(x_3\backslash(x_2(x_5\backslash \ldots (x_n\backslash \gamma x_2)) \ldots) = \gamma x_2$$

where $\gamma x_2 =: y \notin W_{w_1}$.
Denote the left-hand side of the identity w_1 by f and consider the free quasigroup F whose free generators are the elements of the set $W_{w_1} \cup \{y\}$. In the quotient quasigroup $F\big|_{\Theta<f,x_2>}$ (where the relation $\Theta < f, x_2 >$ is the smallest congruence containing the element $[f, x_2]$) the identity w_1 holds whereas Tw_1 does not hold in it. Therefore w_1 is not universal.

2) The following diagram corresponds to the identity w_2



Consider quasigroup $(Q; ., \backslash, /)$ and take elements $a_o, a_1, \ldots, a_{n+1} \in Q$ in places of the variables $x_o, x_1, \ldots, x_{n+1}$. Choose an isotopy $T(\iota, \iota, \gamma)$ onto the quasigroup $(Q; ., \backslash, /)$ such that the permutation γ on Q has the property:

$\gamma a_o = b \neq a_o$. Then the identity Tw_2 does not hold in the quasigroup $(Q; ., \backslash, /)$ and so w_2 is not universal.

Analogously as in the case of the identity w_1 or w_2 one can prove that the identity w_3 or w_4, respectively, is not universal.

Theorem 8. *Let $w \in R(\Sigma_A)$ be a nontrivial identity. If the diagram of the identity w satisfies neither the condition $\boxed{B1}$ nor the condition $\boxed{B2}$ then w is not universal.*

Proof. Take $w \in R(\Sigma_A)$

$$\text{(w)} \qquad {}^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \ldots, {}^{\sigma_n}A(x_n, x_{n+1}))\ldots) = x_o$$

By the assumption the diagram of the identity w does not satisfy the condition $\boxed{B2}$. Then there is a variable $x_i \in W_w$ which occurs in the diagram in two different positions. Without loss of generality we may assume that one of them is the position 2. Consider the variety V_w and choose a suitable quasigroup $Q \in V_w$ (the choice will be specified later). Take an isotopy $T = (\iota, \beta, \iota)$ onto Q (β will be specified later)

(Tw)

$$\begin{pmatrix} & \beta^{\varepsilon'_1} & & \beta^{\varepsilon'_2} & \ldots & & \beta^{\varepsilon'_n} & \beta^{\varepsilon'_{n+1}} & & \beta^{\varepsilon'_o} \\ \beta^{\varepsilon_1} & & \beta^{\varepsilon_2} & & \ldots & \beta^{\varepsilon_n} & & & & \\ {}^{\sigma_1}A & (x_1, & {}^{\sigma_2}A & (x_2, & \ldots, & {}^{\sigma_n}A & (x_n, & x_{n+1})) & \ldots) = & x_o \end{pmatrix}$$

where $\varepsilon'_i \in \{0,1\}$ and $\varepsilon_j \in \{-1,0,1\}$, $i = 0,1,\ldots,n+1$, $j = 1,2,\ldots,n$. Then n-tuple $< \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n >$ can be regarded as the signature of Tw. Denote by p the number of non-zero elements of this signature and by d the number of its sign changes. Now analogously as in the proof of Theorem 6 consider p(p-1)+1 conditions and substitute suitable chosen elements a_1, a_2, ... , a_{n+1}, a_o from the quasigroup Q into the variables x_1, x_2, ... , x_{n+1}, x_o.

Further, we get p(n+2) new conditions by requiring

$$t_m \neq a_k, \qquad k = 0,1,\ldots,n+1, \quad m = 1,2,\ldots,p,$$

where t_1, t_2, ... ,t_p are terms from the first and from the fourth row in Tab. So together we have $\overline{p} := p(p-1) + 1 + p(n+2)$ conditions. Let $\overline{d}$ of them have the (q_k) form (see proof of Theorem 6). Then the wanted quasigroup $Q \in V_w$ will be of the form $Q := Q_1 \times \cdots \times Q_{\overline{d}}$, where Q_1, ... ,$Q_{\overline{d}}$ are suitable quasigroups chosen from V_w by using conditions (q_k),

$k = 1, 2, \dots, \overline{d}$. Now we are going to define a permutation β on Q. Take a map β satisfying the above mentioned $\overline{p}$ conditions as well as the conditions $\beta a_k = a_k$, k=0,1, ... ,n+1. Then extend β to be a permutation on Q. Then Tw is not fulfilled in the quasigroup Q though w is fulfilled in it. Therefore w is not universal, which finishes the proof.

From Theorem 5 and Theorem 8 it follows that $\boxed{B1}$ is a necessary condition for a quasigroup identity $w \in R(\Sigma_A)$ to be universal. Similarly, from Theorem 6 and Theorem 7 we get that $\boxed{B2}$ is a necessary condition for a quasigroup identity $w \in R(\{A, {}^rA, {}^lA\})$ to be universal.

Hence we have

Theorem 9. *Let $w \in R(\{A, {}^rA, {}^lA)\}$ be a nontrivial identity. Then the diagram of the identity w satisfies the conditions $\boxed{B1}$ and $\boxed{B2}$ if and only if the identity w is universal.*

Recall that every quasigroup identity such that its length is at most 7 and neither it nor the identities obtained from it by any transformations contain a square, can be transformed into an identity of canonical type by using only transformations which do not change the universality of the identities [3]. This fact enlarges the class of quasigroup identities for which Theorem 8 gives a necessary and sufficient condition for the universality of the quasigroup identities.

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