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TWO-PARAMETER CHAOS

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Abstract. Let I be a real compact interval and $0 \leq \alpha \leq \beta$ be real numbers smaller than the length of I. A continuous function f from I into itself is said to be generically or densely (α, β)-chaotic if the set of all points $[x, y]$, for which $\liminf_{n\to\infty} |f^n(x) - f^n(y)| \leq \alpha$ and $\limsup_{n\to\infty} |f^n(x) - f^n(y)| > \beta$, is residual or dense in $I \times I$, respectively. In the paper such functions are characterized in terms of behaviour of subintervals of I under iterates of f provided $\alpha > 0$ (see [2] and [3] for $\alpha = 0$).

In the paper a function will always be a function belonging to the space $C^o(I, I)$ of all continuous maps of a real compact interval I into itself, endowed with the topology of uniform convergence. An interval will always be a nondegenerate interval lying in I. If J is an interval then diam J is its length. If $A, B \subset I$ then $\text{dist}(A, B) = \inf \{|x - y| : x \in A, y \in B\}$.

The k-th iterate of a function f is denoted by f^k. For a function f and α, β with $0 \leq \alpha \leq \beta <$diam I define the following planar sets:

$$C_1(f, \alpha) = \{[x, y] \in I^2 : \liminf_{n\to\infty} |f^n(x) - f^n(y)| \leq \alpha\},$$
$$C_2(f, \beta) = \{[x, y] \in I^2 : \limsup_{n\to\infty} |f^n(x) - f^n(y)| > \beta\},$$
$$C(f, \alpha, \beta) = C_1(f, \alpha) \cap C_2(f, \beta).$$

Due to A. Lasota, a function f is called generically chaotic if the set $C(f, 0, 0)$ is residual in $I \times I$ (cf. [1]).

Suppose we are studying some physical or biological system on which we make measurements at regular intervals. If we are just measuring a single quantity then the n-th measurement can be represented by a real number

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x_n. A very simple mathematical model of such a system is obtained by assuming that x_{n+1} is only a function of x_n, and that this function does not depend on n. That is, we assume there is a function f so that $x_{n+1} = f(x_n)$ for all $n \geq 0$. In connection with the definition of generic chaos we must realize that from the physical point of view we are not able to check for example whether $\liminf_{n\to\infty} |f^n(x) - f^n(y)|$ is zero or not. In fact, even if we admit that we are able to make infinitely many measurements, we are restricted by the accuracy of our measuring apparatus. So it seems that the following notion of (α, β)-chaos could have a physical sense.

Definition 1. A function $f \in C^o(I, I)$ is said to be generically or densely (α, β)-chaotic if the set $C(f, \alpha, \beta)$ is residual or dense in $I \times I$, respectively.

So the generic $(0, 0)$-chaos is the same as the generic chaos in the sense of A. Lasota. The generically (α, β)-chaotic functions and the densely (α, β)-chaotic functions were characterized in [2] and [3] provided $\alpha = 0$. In the present paper we show that an analogous characterization holds if $\alpha > 0$ (see Theorem 4). Further, we show that the set of all such maps is nowhere dense in the space $C^o(I, I)$.

The following lemma is a generalization of Lemma 4.3 from [2].

Lemma 2. *Let $f \in C^o(I, I)$ and $0 \leq \alpha \leq$diam I. Then the following three conditions are equivalent:*

(i) *$C_1(f, \alpha)$ is residual,*
(ii) *$C_1(f, \alpha)$ is dense,*
(iii) *for every two intervals J_1, J_2, $\liminf_{n\to\infty}$ dist $(f^n(J_1), f^n(J_2)) \leq \alpha$.*

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) are obvious. We are going to prove (iii) $\Longrightarrow$ (i). So let (iii) be fulfilled. Since $C_1(f, \alpha) = \bigcap_{n=1}^{\infty} L(n, \alpha + 1/n)$ where $L(n, \alpha + 1/n) = \{[x, y] \in I^2 : \inf_{k\geqslant n} |f^k(x) - f^k(y)| < \alpha + 1/n\}$ are open sets, it suffices to prove that for every n, $L(n, \alpha + 1/n)$ is dense in I^2. So take any positive integer n and intervals J_1, J_2. We prove that $L(n, \alpha + 1/n) \cap (J_1 \times J_2) \neq \emptyset$. From (iii) it follows that there exists $k \geq n$ with dist $(f^k(J_1), f^k(J_2)) < \alpha + 1/n$. This implies the existence of points $x \in J_1$, $y \in J_2$ such that $|f^k(x) - f^k(y)| < \alpha + 1/n$. Hence $[x, y] \in L(n, \alpha + 1/n)$ and the proof is complete.

The following lemma is a part of Lemma 4.16 in [2].

Lemma 3. *Let $f \in C^o(I, I)$ and $0 < \beta \leq$diam I. Then the following three conditions are equivalent:*

(i) *$C_2(f, \beta)$ is residual,*

(ii) $C_2(f,\beta)$ *is dense,*
(iii) *for every interval* J, $\limsup_{n\to\infty} \text{diam } f^n(J) > \beta$.

Since the intersection of two residual sets is a residual set, from Lemma 2 and Lemma 3 we immediately get

Theorem 4. *Let* $f \in C^o(I,I)$ *and* $0 < \alpha \le \beta <$diam I. *Then the following three conditions are equivalent:*

(i) f *is generically* (α,β)*-chaotic,*
(ii) f *is densely* (α,β)*-chaotic,*
(iii) *for every two intervals* J_1, J_2, $\liminf_{n\to\infty} \text{dist } (f^n(J_1), f^n(J_2)) \le \alpha$ *and for every interval* J, $\limsup_{n\to\infty} \text{diam } f^n(J) > \beta$.

From [2] it is known that if f is generically chaotic then it is generically $(0,\varepsilon)$-chaotic for some $\varepsilon > 0$ and so it is generically (α,β)-chaotic for any α, β with $0 \le \alpha \le \beta \le \varepsilon$. On the other hand, the following example shows that there are functions which are generically (α,β) -chaotic for some $0 < \alpha \le \beta <$ diam I without being generically chaotic.

Example 5. Take $I = [0,1]$ and numbers $0 < \alpha \le \beta < L$ and R with $L+\alpha+R = 1$. Let $f(0) = L+\alpha$, $f(L) = 1$, $f(L+\alpha) = 0$, $f(1-R/2) = L$, $f(1) = 0$ and let f be linear on each of the intervals $[0,L]$, $[L, L+\alpha]$, $[L+\alpha, 1-R/2]$ and $[1-R/2, 1]$. Then, using Theorem 4 it is easy to see that f is generically (α,β)-chaotic although it is not generically chaotic.

Now denote the set of all densely or generically (α,β)-chaotic maps from $C^o(I,I)$ by $D(\alpha,\beta)$ or $G(\alpha,\beta)$, respectively. Further denote $D = \bigcup\{D(\alpha,\beta) : 0 \le \alpha \le \beta <$diam $I\}$ and $G = \bigcup\{G(\alpha,\beta) : 0 \le \alpha \le \beta <$diam $I\}$. Clearly, $G \subset D$. We have (cf. Theorem 1.5 in [2])

Theorem 6. *The set* D *is nowhere dense in* $C^o(I,I)$.

Proof. Let $B(f,\varepsilon)$ be an open ball in $C^o(I,I)$. Since f has at least one fixed point x_o, it is possible to define a function $g \in B(f,\varepsilon)$ such that for some $x_1 < x_2$ very close to x_o and for some small $\eta > 0$ the intervals $J_i = [x_i - \eta, x_i + \eta]$, i=1,2 are disjoint and $g(J_i) = \{x_i\}$, i=1,2. Denote $M = [x_1 + \eta, x_2 - \eta]$. We may assume that diam $M >$ diam $J_i = 2\eta$, otherwise we can take smaller η. Take $\delta > 0$ such that simultaneously $B(g,\delta) \subset B(f,\varepsilon)$ and for every $h \in B(g,\delta)$, $h(J_i) \subset J_i$,i=1,2. Now suppose that there is a map $h \in B(g,\delta)$ and α, β with $0 \le \alpha \le \beta <$diam I such that h is densely (α,β)-chaotic. Since $\limsup_{n\to\infty} |f^n(x) - f^n(y)| \le$ diam J_1 whenever

$[x,y] \in J_1 \times J_1$, we have diam $J_1 > \beta$. Since $\liminf_{n\to\infty} |f^n(x) - f^n(y)| \geq$ diam M whenever $[x,y] \in J_1 \times J_2$, we have diam $M \leq \alpha$. The inequality diam M $>$ diam J_1 gives $\alpha > \beta$ which is a contradiction with the fact that $\alpha \leq \beta$. So $B(g,\delta) \cap D = \emptyset$. The proof is finished.

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