Acta Univ. M. Belii
Math. no.1(1993), pp.7-14

NOTES ON THE CONGRUENCE LATTICES OF ALGEBRAS

ALFONZ HAVIAR

ABSTRACT. From [1] and [3] it follows that for any algebra $\mathcal{A}$ there exists a groupoid $\mathcal{G}$ for which $Con\mathcal{A} + 1 \simeq Con\mathcal{G}$ (where $Con\mathcal{A} + 1$ is the ordinal sum). In the paper we directly define the operation of groupoid $\mathcal{G}$ by using the operations of algebra $\mathcal{A}$. From this construction it follows that for any binary countable algebra $\mathcal{A}$ there exists a groupoid $\mathcal{G}$ for which $Con\mathcal{A} + 1 \simeq Con\mathcal{G}$ and moreover, $\mathcal{G}$ has no nontrivial subgroupoid and no nontrivial automorphism. We also present (in Theorem 3) some results related to the lattice of subuniverses and to the automorphism group of $\mathcal{A}$.

Throughout this paper Ord denotes the class of all ordinal numbers and N denotes the set of all natural numbers. An algebra (A, F) will be often denoted by $\mathcal{A}$. Further, $Con\mathcal{A}$, $Sub\mathcal{A}$ and $Aut\mathcal{A}$ denote the congruence lattice, the lattice of subuniverses and the automorphism group for the algebra $\mathcal{A}$, respectively.

Let (A, F) be a binary algebra

$$A = \{a_k; k < \alpha, k \in Ord\}, \quad \alpha \in Ord,$$

$$F = \{f_k; k < \beta, k \in Ord\}, \quad \beta \in Ord,$$

let γ be a limit ordinal such that $\gamma \geq max\{\alpha, \beta\}$ and let

$$M = \{k \in Ord; k < \gamma\}, \quad S = \{-3, -2, -1\} \cup M.$$

We consider the usual ordering

$$-3 < -2 < -1 < 0 < 1 < 2 < 3 < ...,$$

1991 *Mathematics Subject Classification.* 08A30, 08A35.
Key words and phrases. congruence relation, subalgebra, automorphism.

on the set S. Now we put

$$G = \{(a_r, s); r < \alpha, r \in M, s \in S\}.$$

To define a groupoid operation o on G we consider the following cases for an element

$(a_i, r)o(a_j, s), \quad i, j < \alpha, \quad i, j \in M, \quad r, s \in S.$

I. $r = s$. a) $0 \leq r < \beta$

$$(a_i, r)o(a_j, s) = (f_r(a_i, a_j), r + 1), \tag{1}$$

b) either $r \in \{-3, -2, -1\}$ or $r \geq \beta$

$$(a_i, r)o(a_j, s) = (a_i, r + 1). \tag{2}$$

II. $r < s$. a) $r + 1 = s$

$$(a_i, r)o(a_j, r + 1) = (a_i, -2), \tag{3}$$

b) $r + 1 < s$

$$(a_i, r)o(a_j, s) = (a_i, -3). \tag{4}$$

III. $r > s$. a) $s = -3$

$$(a_i, r)o(a_j, -3) = (a_j, r), \tag{5}$$

b) $s = -2$

$$(a_i, r)o(a_j, -2) = (a_j, -3), \tag{6}$$

c) $s = -1, \quad 0 \leq r < \alpha$

$$(a_i, r)o(a_j, -1) = (a_r, -3), \tag{7}$$

d) otherwise

$$(a_i, r)o(a_j, s) = (a_i, r + 1). \tag{8}$$

Lemma 1. *Let (A,F) be a binary algebra and (G,o) be the groupoid whose operation is defined by (1) - (8). For any congruence relation Φ of (G,o) the following properties are satisfied:*

(i) $$(x,k)\Phi(y,k) \Longleftrightarrow (x,s)\Phi(y,s)$$

for all $k,s \in S$,

(ii) $$(x,k)\Phi(y,s), \quad k \neq s \quad \Rightarrow \quad \Phi = G^2.$$

Proof. (i). From $(x,k)\Phi(y,k)$ it follows $(x,k)o(x,k+2)\Phi(y,k)o(x,k+2)$, i.e., by (4) $(x,-3)\Phi(y,-3)$. Thus, $(x,s)o(x,-3)\Phi(x,s)o(y,-3)$ and so by (5) $(x,s)\Phi(y,s)$ holds for any $s \neq -3$.

(ii). Let $(x,k)\Phi(y,s)$, $k \neq s$. We may assume that $k < s$. By hypothesis $(x,k)o(x,s+1)\Phi(y,s)o(x,s+1)$, i.e., by (4) and (3),

(a) $$(x,-3)\Phi(y,-2).$$

From (a) it follows that for all $s \in S$, $s > -2$ $(x,s)o(x,-3)\Phi(x,s)o(y,-2)$, i.e., by (5) and (6),

(b) $$(x,s)\Phi(y,-3).$$

Furthermore, (b) implies $(x,s)o(x,-1)\Phi(y,-3)o(x,-1)$; thus, by (7) and (4),

(c) $$(a_s,-3)\Phi(y,-3)$$

for all $s \in M$, $s < \alpha$. Then $(c),(i),(b)$ and (a) yield $\Phi = G^2$.

Theorem 1. *Let (A,F) be an algebra. There exists a groupoid (G,o) such that*

$$Con\mathcal{A} + 1 \simeq Con\mathcal{G}$$

.

Proof. We can suppose that (A,F) is an algebra with binary operations. Let (G,o) be the groupoid whose operation is defined by $(1)-(8)$. Define a mapping $F: Con\mathcal{A} \to Con\mathcal{G}$ as follows:

(9) $$(x,k)F(\Theta)(y,s) \quad iff \quad k=s \quad and \quad x\Theta y, \quad \Theta \in Con\mathcal{A}.$$

1. First, we prove that the mapping F is well-defined, i.e., $F(\Theta) \in Con\mathcal{G}$. Obviously, $F(\Theta)$ is an equivalence relation on G. It suffices to prove that $(x,k)F(\Theta)(y,k)$ implies

$$\text{(d)} \qquad (x,k)o(z,r)F(\Theta)(y,k)o(z,r)$$

and

$$\text{(e)} \qquad (z,r)o(x,k)F(\Theta)(z,r)o(y,k)$$

for every $(z,r) \in G$. We only prove (d), since (e) can be proved in a similar way.

If $k = r$ and $0 \leq k < \beta$, then from $x\Theta y$ it follows $f_k(x,z)\Theta f_k(y,z)$ and so
$(f_k(x,z), k+1)F(\Theta)(f_k(y,z), k+1)$ holds. Therefore we get (d) by (1).

In the other cases (d) is of the form

$$(x,m)F(\Theta)(y,m) \quad or \quad (t,m)F(\Theta)(t,m)$$

for some t, m. Hence, (d) obviously holds.

2. Let $\Phi \in Con\mathcal{G}$, $\Phi \neq G^2$. We define a relation Θ on A as follows:

$$\text{(10)} \qquad x\Theta y \quad iff \quad (x,-3)\Phi(y,-3).$$

Obviously, Θ is an equivalence relation on A. Let $x\Theta y$ and let $k \in M, k < \beta$. Then by (i) (Lemma 1) we have $(x,k)\Phi(y,k)$. Therefore, we get $(x,k)o(z,k)\Phi(y,k)o(z,k)$, i.e. $(f_k(x,z), k+1)\Phi(f_k(y,z), k+1)$. Then again by (10) and (i) we have $f_k(x,z)\Theta f_k(y,z)$. Analogously, we get $f_k(z,x)\Theta f_k(z,y)$. Thus, $\Theta \in Con\mathcal{A}$ and obviously $F(\Theta) = \Phi$.

3. It is easy to check that

$$\Theta_1 \leq \Theta_2 \quad iff \quad F(\Theta_1) \leq F(\Theta_2)$$

for all $\Theta_1, \Theta_2 \in Con\mathcal{A}$. If we denote $F(A^2)$ by Ω, then we conclude that Ω is a unique dual atom of the lattice $Con\mathcal{G}$ and the mappping F is an isomorphism between $Con\mathcal{A}$ and the ideal $(\Omega]$ of the lattice $Con\mathcal{G}$. The proof is complete.

Corollary. *Let L be an algebraic lattice. There exists a groupoid $\mathcal{G}$ such that $L + 1 \simeq Con\mathcal{G}$.*

Theorem 2. *Let (A, F) be an algebra with binary operations such that A, F are countable. Then there exists a groupoid $\mathcal{G}$ having no proper subgroupoids and no nontrivial automorphisms such that*

$$Con\mathcal{A} + 1 \simeq Con\mathcal{G}.$$

Proof. Let (G, o) be the groupoid whose operation is defined by $(1) - (8)$, with $\gamma = \omega$. Then $Con\mathcal{A} + 1 \simeq Con\mathcal{G}$ by Theorem 1.

a) Now we shall prove that $\mathcal{G}$ has no proper subgroupoids. Let H be the subuniverse of the groupoid (G, o) generated by the element (a, p). Let $0 \leq p < \beta$. Then we successively get that H also contains the elements $(a,p)o(a,p) = (f_p(a,a), p+1) = (b, p+1)$, $(b, p+1)o(b, p+1) = (f_{p+1}(b,b), p+2) = (c, p+2)$, whenever $p+1 < \beta$
$(b, p+1)o(b, p+1) = (b, p+2) = (c, p+2)$, if $p+1 = \beta$,
$(a,p)o(c, p+2) = (a, -3)$, $(a,-3)o(a,-3) = (a,-2)$, $(a,-2)o(a,-2) = (a,-1)$, etc.
For every $m \in S$ there exists an element $d \in A$ such that $(d, m) \in H$. Then for every m, $0 \leq m < \alpha$ we get
$(d, m)o(a, -1) = (a_m, -3) \in H$.
We also get
$(d, n)o(a_m, -3) = (a_m, n) \in H$, for every $n > -3$.
Hence, $H = G$ holds.

If either $p \in \{-3, -2, -1\}$ or $p \geq \beta$ we obtain $H = G$ in a similar way.

b) It remains to prove that $\mathcal{G}$ has no nontrivial automorphisms. Let g be an automorphism of $\mathcal{G}$. If $g(a, -3) = (b, -2)$ for some elements $a, b \in A$, then

$$g(a,-2) = g((a,-3)o(a,-3)) = g(a,-3)og(a,-3) = (b,-2)o(b,-2) = (b,-1)$$

but this contradicts the fact that

$$g(a,-2) = g((a,-3)o(a,-2)) = (b,-2)o(b,-1) = (b,-2).$$

We analogously check that $g(a,-3) = (b,p)$ where $p = -1$ or $O \leq p < \beta$ or $\beta \geq p$ is impossible, too. Hence, for every element $(a,-3) \in G$ there exists an element $(b,-3) \in G$ such that

(f) $$g(a,-3) = (b,-3).$$

But (f) implies

$$g(a,-2)=g((a,-3)o(a,-3))=(b,-3)o(b,-3)=(b,-2).$$

Similarly, $g(a,-1)=(b,-1)$, $g(a,0)=(b,0)$
$g(f_0(a,a),1)=g((a,0)o(a,0))=(b,0)o(b,0)=(f_0(b,b),1)$, etc.
One can prove by induction that for every $m \in S$ there exist elements $c,d \in A$ such that

(g) $$g(c,m)=(d,m).$$

For any n, $O \leq n < \alpha$, (g) yields

$$g(a_n,-3)=g((c,n)o(a,-1))=(d,n)o(b,-1)=(a_n,-3).$$

Now, for any $m \neq -3$ we get

$$g(a_n,m)=g((c,m)o(a_n,-3))=(d,m)o(a_n,-3)=(a_n,m).$$

Therefore, g is the identity on G, and the proof is complete.

Theorem 3. *Let (A,F) be an algebra with binary operations such that A, F are countable. There exists a groupoid (G,o) such that*

$$Sub\mathcal{A} \simeq Sub\mathcal{G}, \quad Aut\mathcal{A} \simeq Aut\mathcal{G} \quad \text{and} \quad Con\mathcal{A} \simeq (\Omega],$$

where $(\Omega]$ is an ideal of $Con\mathcal{G}$ generated by some element $\Omega \in Con\mathcal{G}$.

Proof. Let (G,o) be the groupoid constructed in the same way as in the proof of Theorem 1 where

(7') $$(a_i,r)o(a_j,-1)=(a_i,r+1)$$

holds instead of (7) (i.e., (8) also holds in the case $s=-1$).

1. We shall prove that $Sub\mathcal{A} \simeq Sub\mathcal{G}$. Let $\phi : Sub\mathcal{A} \to Sub\mathcal{G}$ be the mapping defined as follows:

(11) $$\phi(A_1)=\{(a,n);\quad a \in A_1,\quad n \in S\}.$$

First, we shall show that ϕ is well-defined, i.e., that $\phi(A_1)$ is a subuniversum of the groupoid $\mathcal{G}$. Let $(a,n),(b,m) \in \phi(A_1)$.
If $0 \leq n = m < \beta$, then $(a,n)o(b,m)=(f_n(a,b),n) \in \phi(A_1)$. In the

other cases we have $(a,n)o(b,m) = (a,k)$ or $(a,n)o(b,m) = (b,k)$ for suitable $k \in S$, so again $(a,n)o(b,m) \in \phi(A_1)$.

In order to show that ϕ is surjective, let H be a subuniversum of $\mathcal{G}$, and let

$$A_1 = \{a \in A; \quad \exists n \quad (a,n) \in H\}.$$

Further, let $a,b \in A_1$ and let $f_p \in F$. Then there exist m,n such that $(a,m),(b,n) \in H$. In a similar way as in the proof of Theorem 2 one can prove that $(a,-3),(b,-3) \in H$, and that for every n there exists d such that $(d,n) \in H$. This implies
$(d,p)o(a,-3) = (a,p) \in H, \quad (d,p)o(b,-3) = (b,p) \in H,$
$(a,p)o(b,p) = (f_p(a,b),p+1) \in H, \quad$ i.e.,$f_p(a,b) \in A_1$.
Therefore, A_1 is the subuniversum of $\mathcal{A}$, and clearly $\phi(A_1) = H$.

Obviously, ϕ is one-to-one and

$$A_1 \subseteq A_2 \qquad iff \qquad \phi(A_1) \subseteq \phi(A_2).$$

Hence, ϕ is an isomorphism.

2.Now we shall prove that $Aut\mathcal{A} \simeq Aut\mathcal{G}$. Let $\phi : Aut\mathcal{A} \to Aut\mathcal{G}$ be the mapping defined as follows:

$$\phi(f)(a,n) = (f(a),n), \quad f \in Aut\mathcal{A}. \tag{12}$$

First, we shall again show that ϕ is well-defined, i.e., that $\phi(f)$ is an automorphism of the groupoid $\mathcal{G}$. Obviously, $\phi(f)$ is a bijection on G since f is a bijection on A. Let $(a,n),(b,m) \in G$. If $0 \le n = m < \beta$, then
$\phi(f)((a,n)o(b,n)) = \phi(f)(f_n(a,b),n+1) = (f(f_n(a,b)),n+1) =$
$= (f_n(f(a),f(b)),n+1) = (f(a),n)o(f(b),n) = \phi(f)(a,n)o\phi(f)(b,n).$

If $n \neq m, \quad n+1 = m,$ then we have
$\phi(f)((a,n)o(b,m)) = \phi(f)(a,-2) = (f(a),-2) =$
$= (f(a),n)o(f(b),m) = \phi(f)(a,n)o\phi(f)(b,m).$

We analogously get
$\phi(f)((a,n)o(b,m)) = \phi(f)(a,n)o\phi(f)(b,m)$ in every other case (i.e., if $n+1 < m$, or $n > m$).

In order to show that ϕ is surjective, suppose that g is an automorphism of $\mathcal{G}$. In a similar way as in the proof of Theorem 2 one can prove that for every element $(a,-3) \in G$ there exists an element $(b,-3) \in G$ such that $g(a,-3) = (b,-3)$, and that for every $m \in S$ there exist elements $c,d \in G$ such that $g(c,m) = (d,m)$. This yields that for any $m \ge -3$

$$g(a,m) = g((c,m)o(a,-3)) = (d,m)o(b,-3) = (b,m).$$

On the other hand

$$g(a,-3)=g((a,n)o(c,n+2))=(b,n)o(d,n+2)=(b,-3),$$

whenever $g(a,n)=(b,n)$ for some n. Therefore, for any automorphism $g\in Aut\mathcal{G}$ we can define a mapping $f:A\to A$ as follows:
$f(a)=b \iff g(a,n)=(b,n)$ for every $n\in S$.
We shall prove that the mapping f is an endomorphism on A. Let $a,c\in A$, $0\le n<\beta$. Then

$$(f(f_n(a,c)),n+1)=g(f_n(a,c),n+1)=g((a,n)o(c,n))=$$
$$=(f(a),n)o(f(c),n)=(f_n(f(a),f(c)),n+1)$$

which implies $f(f_n(a,c))=f_n(f(a),f(c))$.

Since g is the automorphism on G, we conclude that f is the automorphism on A, and obviously $\phi(f)=g$.

One can easily check that ϕ is one-to-one, and that for any automorphisms $f,g\in Aut\mathcal{A}$

$$\phi(fog)(a,n)=((fog)(a),n)=(f(g(a)),n)=(\phi(f)o\phi(g))(a,n).$$

Hence, ϕ is an isomorphism.

3. We define a relation Ω on G as follows:

(13) $$(a,m)\Omega(b,n)\quad iff\quad m=n.$$

Obviously, Ω is the congruence of groupoid (G,o). Let $F: Con\mathcal{A}\to Con\mathcal{G}$ be the mapping defined by (9) (in the proof of Theorem 1). In the same way as in the proof of Theorem 1, one can prove that F is an isomorphism between $Con\mathcal{A}$ and the ideal $(\Omega]$ of the lattice $Con\mathcal{G}$. The proof is complete.

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Department of Mathematics, Matej Bel University, Tajovského 40, 975 49 Banská Bystrica, Slovakia

(Received January 6, 1992)