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# THE LATTICE OF ORDER VARIETIES

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Abstract. The list of all varieties of posets that cover the variety C ∨ A is given (C is the variety of all complete lattices and A is the variety of all antichains). Some results on varieties of posets containing the variety of antichains are established.

In the paper we obtain some new results on the lattice of varieties of posets which was introduced in [1]. Notations and terminology correspond to those from [1]. Throughout the paper the basic set of the poset is always assumed to be nonempty.

A poset $Q$ is *a retract* of poset $P$ (written $Q \lhd P$) if there are order-preserving maps $f : Q \to P$ (which is called a coretraction map) and $g : P \to Q$ (called a retraction map) such that $g \circ f$ is the identity map of $Q$ to itself.

There is another characterisation of a retract: a subposet $Q$ of a poset $P$ is a retract of $P$ if there is an order-preserving map $g : P \to Q$ which is identical on $Q$.

Let K be a class of partially ordered sets. The class of all retracts of posets from K will be denoted by $\mathbf{R}$(K) and the class of all direct products of nonvoid families of posets from K will be denoted by $\mathbf{P}$(K).

*An order variety* is a class V of ordered sets which contains all retracts of members of V and all direct product of nonvoid families of members of V (i.e. V is order variety iff $\mathbf{R}(\mathrm{V}) \subseteq \mathrm{V}$ and $\mathbf{P}(\mathrm{V}) \subseteq \mathrm{V}$).

For a class K of posets let $\mathrm{K}^\tau$ denote the smallest order variety containing each member of K. A variety $\mathrm{K}^\tau$ is called the order variety generated by K. In [1] it is proved that

$$\mathrm{K}^\tau = \mathbf{RP}(\mathrm{K}).$$

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The collection of all order varieties is a complete lattice ordered by inclusion.

There are only two atoms in this lattice:

C - the variety of all complete lattices ( C = $\{\underset{\sim}{2}\}^{\tau}$ where $\underset{\sim}{2}$ denotes the two-element chain )

A - the variety of all antichains (A = $\{2\}^{\tau}$ where 2 denotes the two-element antichain).

Any order relation on $P$ induces the relation of comparability on $P$ defined by $a \sim b$ if $a \leq b$ or $b \leq a$. The relation of comparability is reflexive and symmetric. We note that the transitive closure of a reflexive and symmetric relation is an equivalence relation. The blocks of this equivalence relation are called *connected components* of the order relation. A poset $P$ is called *a connected poset* if it has just one connected component. K is a class of connected posets if any member of K is connected.

Let $\{P_i : i \in I\}$ be a family of mutually disjoint posets. The *cardinal sum* of the family $\{ P_i : i \in I\}$ (written $\sum\limits_{i \in I} P_i$) is the poset with the universe $P = \bigcup\limits_{i \in I} P_i$ and the order relation $\leq$ on $P$ defined as follows: $a \leq b$ iff there exists an index $i_o \in I$ such that $a, b \in P_{i_o}$ and $a \leq b$ holds in $P_{i_o}$ .

Let $\sum$K denote the class of all isomorphic images of the cardinal sums of members of K.

It is easy to see that any poset is the cardinal sum of its connected components.

**Lemma 1.** *Let $A = \sum\limits_{i \in I} A_i$ and $B = \sum\limits_{j \in J} B_j$ be cardinal sums of nonvoid families $\{ A_i : i \in I\}$ and $\{ B_j : j \in J\}$ respectively. If for every $i \in I$ there exists $j_i \in J$ such that $A_i \triangleleft B_{j_i}$ and $j_{i_1} = j_{i_2}$ implies $i_1 = i_2$, then $A \triangleleft B$.*

*Proof.* Let $\varphi_i : B_{j_i} \to A_i$ be a retraction and
$\phi_i : A_i \to B_{j_i}$ be a coretraction for every $i \in I$.
We define the maps $g : \sum\limits_{j \in J} B_j \to \sum\limits_{i \in I} A_i$
and $f : \sum\limits_{i \in I} A_i \to \sum\limits_{j \in J} B_j$ as follows:

$$g(a) = \begin{cases} \varphi_i(a) \text{ if } a \in B_{j_i} \\ d \text{ otherwise} \end{cases}$$

where $d$ is any fixed element of $\sum\limits_{i \in I} A_i$ and

$$f(b) = \phi_i(b) \text{ if } b \in A_i.$$

The reader can straightforwardly verify that for every $x \in \sum_{i \in I} A_i$ there are indices $i, j_i$ such that $x \in A_i$ and $f(A_i) \subseteq B_{j_i}$ and $((g \circ f)(x) = (\varphi_i \circ \phi_i)(x) = x$ and $f$ and $g$ are order-preserving maps. This completes the proof of the lemma.

**Theorem 1.** *The variety A of all antichains is contained in a variety V iff V= $\sum$V.*

*Proof.* a) If the poset $P$ contains at least two connected components $P_1, P_2$ then choose two elements $x \in P_1$ and $y \in P_2$. The map $f : P \to \{x, y\}$ which maps each element $u \in P_1$ to $x$ and each element $v \in P - P_1$ to $y$ is a retraction map with fixed points $x$ and $y$. Thus
$2 \lhd P$.This implies that if V contains at least one disconnected poset then V contains A.If V=$\sum$V then V contains a disconnected poset,so A$\subseteq$V.

b) Let A$\subseteq$V and $P \in \sum$V.Then $P = \sum_{i \in I} P_i$, where $P_i \in$ V for every $i \in I$. This implies that $\prod_{i \in I} P_i \in$ V. Let $\widetilde{I}$ denote the antichain with the universe I and let $Q = (\prod_{i \in I} P_i) \times \widetilde{I}$. Obviously $Q$ is a member of V.(Note that $\widetilde{I} \in A \subseteq$ V.) We denote

$$Q_j = \{x \in Q : x = (f, j) \text{ where} f \in \prod_{i \in I} P_i\}.$$

It is easy to see that $Q = \sum_{i \in I} Q_i$. It is known that $U_{l_o} \lhd \prod_{l \in L} U_l$ for any family of posets $\{ U_l : l \in L\}$ and for any $l_o \in L$. The posets $P = \sum_{i \in I} P_i$ and $Q = \sum_{i \in I} Q_i$ satisfy the conditions from Lemma 1 and so $P \lhd Q$. This implies that $P \in$ V. The converse inclusion is obvious.

**Theorem 2.** *If V is an order variety then $\sum$ V is also an order variety.*

*Proof.* If there is a disconnected poset in V then the assertion follows from Theorem 1. So assume that V contains no disconnected poset.

a) Let $P \lhd \sum_{i \in I} P_i$ and $P_i \in$ V for any $i \in I$. Denote by $f$ a retraction and by $g$ a coretraction corresponding to $f$. Let $S_i$ denote the set $\{ x \in P : g(x) \in P_i\}$ .Obviously, $P = \sum_{i \in I} S_i$. Let the subset $J \subseteq I$ be given by : $j \in J$ iff $S_j \neq \emptyset$. Then $P = \sum_{j \in J} S_j$. For every $j \in J$ the poset $P_j$ is connected and so is the poset $g(f(P_j))$. Hence $g(f(P_j)) \subseteq P_j$. Thus we get

$f(P_j) \subseteq S_j$ for any $j \in J$. On the other hand $S_j = f(g(S_j)) \subseteq f(P_j)$. It implies that $f(P_j) = S_j$ for any $j \in J$. We have $S_j \lhd P_j$ with $f \restriction P_j$ as a retraction and $g \restriction S_j$ as a coretraction. This implies that $S_j \in$ V and $P \in \sum$V. Therefore **R**($\sum$ V)$\subseteq \sum$V.

b) Now we prove that **P**($\sum$ V)$\subseteq \sum$ V. The proof is based on fact that

$$\prod_{i\in I}(\sum_{j\in J_i} {P_i}^j)$$

and

$$\sum_{f\in \prod\limits_{i\in I} J_i} (\prod_{i\in I} {P_i}^{f(i)})$$

are isomorphic posets. The poset $\prod\limits_{i\in I} {P_i}^{f(i)}$ is the direct product of a family of connected sets from V and so $\prod\limits_{i\in I} {P_i}^{f(i)} \in$ V. Hence $\sum\limits_{f\in \prod\limits_{i\in I} J_i} (\prod\limits_{i\in I} {P_i}^{f(i)}) \in \sum$ V. This completes the proof.

**Corollary 1.** *In the lattice of order varieties $\sum V = A \vee V$ for any variety V.*

**Theorem 3.** *If V is a variety which contains no disconnected poset, then there is no variety W such that V$\subset$W $\subset \sum$V(i.e.,the variety V is covered by $\sum$V).*

*Proof.* Suppose V$\subseteq$W $\subseteq \sum$V. If every poset in W is connected then W=V. Otherwise A$\subseteq$W,thus W=$\sum$W and $\sum$V $\subseteq \sum$W=W. Hence W=$\sum$V.

**Corollary 2.** *If the system $V_c$ of all connected posets of the variety V is a variety, then $V_c$ is covered by V or $V_c = V$.*

*Proof.* This follows from the fact that V=$\sum$V$_c$ or V=V$_c$.

**Theorem 4.** *If V and W are varieties of connected posets and V is covered by W then $\sum$V is covered by $\sum$W.*

*Proof.* Obviously, V$\subseteq$W implies $\sum$ V $\subseteq \sum$W. Let V$_1$ be a variety such that $\sum$ V $\subseteq$ V$_1$ $\subseteq \sum$ W. Denote by K the system of all connected posets of V$_1$.Then V$\subseteq$ K $\subseteq$W. It is easy to see that **R**(K)$\subseteq$W, **P**(K)$\subseteq$W, **R**(K)$\subseteq$ V$_1$, **P**(K)$\subseteq$ V$_1$. This implies **RP**(K)$\subseteq$ V$_1$ and **RP**(K)$\subseteq$W. So, for the variety **RP**(K) generated by K we have V$\subseteq$**RP**(K)$\subseteq$W. If **RP**(K)=V then V$_1$ = $\sum$V. If **RP**(K)=W then V$_1$ = $\sum$W.

**Corollary 3.** *Let V and W be varieties which contain A and let V be covered by W. Let $V_c$ and $W_c$ denote the systems of all connected posets of V and W, respectively.If $V_c$ and $W_c$ are varieties, then $V_c$ is covered by $W_c$.*

*Proof.* If K is variety such that $\mathrm{V}_c \subseteq \mathrm{K} \subseteq \mathrm{W}_c$ ,$\mathrm{V}_c \neq \mathrm{K} \neq \mathrm{W}_c$ then $\sum \mathrm{V}_c \subseteq \sum \mathrm{K} \subseteq \sum \mathrm{W}_c$ and $\sum \mathrm{V}_c \neq \sum \mathrm{K} \neq \sum \mathrm{W}_c$, a contradiction.

Let $P$,$Q$ be ordered sets.We will denote by $P \oplus Q$ the ordinal sum of $P$ and $Q$. Further, let $P^d$ denote the dual poset of $P$.Let $F_3$ be the poset with the universe { 1,2,3} and the order relation given by : 2≤1, 3≤1 and 2,3 are noncomparable elements (i.e.,$F_3$ is a fence ).

**Theorem 5 ([1]).** *In the lattice of order varieties each of the following order varieties cover the variety C of all complete lattices*

$$\text{(L)}\quad \begin{array}{l} \{\alpha\}^\tau,\ \ \{\beta^d\}^\tau,\ \ \{\alpha \oplus \beta^d\}^\tau,\ \ \{F_3\}^\tau,\ \ \{{F_3}^d\}^\tau,\ \ \{{F_3}^d \oplus F_3\}^\tau, \\ \{\alpha \oplus F_3\}^\tau,\ \ \{{F_3}^d \oplus \beta^d\}^\tau,\ \ \sum C, \end{array}$$

*where $\alpha$ and $\beta$ are any regular ordinals. Moreover, the system of all varieties which cover C consists just of the varieties in (L).*

**Theorem 6.** *In the lattice of order varieties the variety $\sum C (= A \vee C)$ is covered only by the varieties of the form $\sum$ V for all V from the list (L) except of V= $\sum$C.*

*Proof.* Theorem 4 implies that these varieties cover $\sum$C. Let V be variety with $\sum \mathrm{C} \subseteq \mathrm{V}$ and $\sum \mathrm{C} \neq \mathrm{V}$. Then there exists a poset $P$ such that $P$ is not a complete lattice, $P$ is connected, $P \in$V and $\{P\}^\tau \subseteq$V. In [1] it is shown (see proof of Theorem 7.1) that there is a poset $Q$ generating a variety from the list (L) and $Q$ is a retract of $P$. Thus, we get $\sum \{Q\}^\tau \subseteq \sum \{P\}^\tau \subseteq \sum \mathrm{V} = \mathrm{V}$. This completes the proof.

**Theorem 7.** *The system of all varieties of connected posets is the ideal of the lattice of order varieties.*

*Proof.* Let V and W be varieties of connected posets. Then V∨W = **RP**(V∪W) (cf. [1]). If $P \in$ **RP**(V∪W) then $P \lhd Q_1 \times Q_2$, where $Q_1 \in$ **P**(V) and $Q_2 \in$ **P**(W). This implies that $Q_1$ and $Q_2$ are connected and $Q_1 \times Q_2$ is also connected.Every retract of connected poset is also connected.Let V and K be any varieties of posets.If V contains no disconnected poset and K⊆V ,then also K contains no disconnected poset.

## References

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