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ON DECOMPOSITIONS OF COMPLETE GRAPHS INTO FACTORS WITH GIVEN DIAMETERS

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Abstract. Let $F_m(d_1, d_2, \dots, d_m)$ be the least positive integer n such that the complete graph K_n can be decomposed into m factors with the diameters $d_1, d_2, \dots, d_m$. The estimations for $F_m(d_1, d_2, \dots, d_m)$ are found.

By a factor of a graph G we mean a subgraph of G containing all the vertices of G. A system of factors of G such that every edge of G belongs to exactly one of them is called a decomposition of G. The symbol K_n denotes the complete graph with n vertices.

Let $m, d_1, d_2, \dots, d_m$ be natural numbers. The symbol (see [1]) $F_m(d_1, d_2, \dots, d_m)$ denotes the smallest natural number n such that the complete graph K_n can be decomposed into m factors with the diameters $d_1, d_2, \dots, d_m$; if such a natural number does not exist then put $F_m(d_1, d_2, \dots, d_m) = \infty$. In the case $d_1 = d_2 = \dots = d_m = d$ we shall write $F_m(d, d, \dots, d) = f_m(d)$. The significance of the function $F_m(d_1, d_2, \dots, d_m)$ resides in the validity of the following assertion (proved in [1]): K_n is decomposable into m factors with diameters $d_1, d_2, \dots, d_m$ if and only if $n \geq F_m(d_1, d_2, \dots, d_m)$.

J. Bosák, A. Rosa and Š. Znám ([1]) initiated the studies of decompositions of complete graphs into factors with given diameters. Many papers deal with the problem of [1] or with its various modifications.

The following result is proved in [1]: Let $m, d_1, d_2, \dots, d_m$ be natural numbers ≥ 3, then

$$F_m(d_1, d_2, \dots, d_m) \leq d_1 + d_2 + \dots + d_m - m.$$

This result can be strengthened. For $m = 3$ this was done in [1] (if $min\{d_1, d_2, d_3\} \geq 5$, then $F_3(d_1, d_2, d_3) \leq d_1 + d_2 + d_3 - 8$) and in [2]

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(if $min\{d_1, d_2, d_3\} > 65$, then $F_3(d_1, d_2, d_3) = d_1 + d_2 + d_3 - 8$). For $m > 3$ the following theorem gives a better result than that mentioned above.

Theorem 1. *Let* $m \geq 3$, $d_1 \geq d_2 \geq \cdots \geq d_m \geq 6$, $d_1 \geq 2m - 1$ *and* $d_3 \geq m - 1$. *Then*

$$F_m(d_1, d_2, \ldots, d_m) \leq d_1 + d_{k-1} + d_k + 3,$$

where k *is the maximum natural number such that* $3 \leq k \leq m$ *and* $d_k \geq m - 1$.

Proof. It is sufficient to show that the complete graph with $d_1 + d_{k-1} + d_k + 3$ vertices is decomposable into m factors $F_1, F_2, \ldots, F_m$ with the diameters $d_2, d_3, \ldots, d_{k-2}, d_{k+1}, d_{k+2}, \ldots, d_m, d_1, d_{k-1}, d_k$, respectively. Denote the vertices of the above mentioned graph by $u_1, u_2, \ldots, u_{d_1+1}, v_1, v_2, \ldots, v_{d_{k-1}+1}, w_1, w_2, \ldots, w_{d_k+1}$. Let $t = \left[\frac{d_1+1}{2}\right]$.
We shall consider the path P_i' of the length d_1 for $i = 1, 2, \ldots, m-2$

$$u_{i+1}u_iu_{i+2}u_{i-1}u_{i+3}u_{i-2}\ldots u_{i-t+2}u_{i+t}u_{i-t+1}$$

in the case that d_1 is an odd number or

$$u_{i+1}u_iu_{i+2}u_{i-1}u_{i+3}u_{i-2}\ldots u_{i+t}u_{i-t+1}u_{i+t+1}$$

in the case that d_1 is an even number. The subscripts j of u_j are taken as the integers $1, 2, \ldots, d_1 + 1 \mod(d_1 + 1)$. Now we are going to construct the factors F_i for $i = 1, 2, \ldots, m$.

a) The factor F_i for $i = 1, 2, \ldots, m - 3$ consists of
  1) a path P_i with the following four properties
    (i) the length of P_i is equal to the diameter of the factor F_i,
    (ii) we get P_i from P_i' by deleting (if necessary) some vertices at the beginning and the end of the path P_i',
    (iii) P_i contains the vertices u_{2i+1} and u_{2i+2},
    (iv) neither u_{2i+1} nor u_{2i+2} is one of the first two vertices or one of the last two vertices of P_i,
  2) the edges
$$u_{2i+1}v_j, j = 1, 2, \ldots, d_{k-1} + 1,$$
$$u_{2i+2}w_j, j = 1, 2, \ldots, d_k + 1,$$
  3) for any vertex u_j which does not belong to P_i
    (i) the edge v_iu_j if j is an even number or
    (ii) the edge w_iu_j if j is an odd number.

b) The factor F_{m-2} will contain the path P'_{m-2} and the edges

$$u_{2m-3}v_j, j = 1, 2, \dots, d_{k-1} + 1,$$
$$u_{2m-2}w_j, j = 1, 2, \dots, d_k + 1.$$

c) The factor F_{m-1} will contain the path

$$v_1 v_2 v_{d_{k-1}+1} v_3 v_4 \dots v_{d_{k-1}-1} v_{d_{k-1}}$$

and the edges

$$v_{d_{k-1}+1} u_{2j}, j = 1, 2, \dots, t,$$
$$v_{d_{k-1}+1} w_j, j = 1, 2, \dots, d_k + 1,$$
$$w_{d_k} u_{2j-1}, j = 1, 2, \dots, s, \quad s = \left[\tfrac{d_1+2}{2}\right].$$

d) The factor F_m will contain the path

$$w_1 w_2 w_{d_k+1} w_3 w_4 \dots w_{d_k-1} w_{d_k}$$

and the edges

$$w_{d_k+1} u_{2j-1}, j = 1, 2, \dots, s \quad s = \left[\tfrac{d_1+2}{2}\right],$$
$$w_{d_k+1} v_j, j = 1, 2, \dots, d_{k-1},$$
$$v_{d_{k-1}} u_{2j}, j = 1, 2, \dots, t,$$
$$v_1 v_{d_{k-1}+1}.$$

Now consider all the edges which so far we have not included into any of the factors $F_1, F_2, \dots, F_m$. Let those of them which are of the types $v_i w_j, v_i v_j, w_i w_j$ or $u_i u_j, u_i w_j$ or $u_i v_j$ belong to the factors F_{m-2} or F_{m-1} or F_m, respectively. It is easy to verify that the factors $F_1, F_2, \dots, F_m$ have the desired diameters and they form a decomposition of the complete graph with $d_1 + d_{k-1} + d_k + 3$ vertices. Q.E.D.

In [1] a lower bound for $f_3(d)$ was found :

$$f_3(d) > \frac{3+\sqrt{3}}{2} d - \frac{5+4\sqrt{3}}{2}.$$

For $m > 3$ the following theorem holds.

Theorem 2. *If $m > 3$ and $d \geq 2m - 1$, then*

$$f_m(d) \geq \frac{m+\sqrt{m}}{m-1} d - \frac{m+\sqrt{m}(2m-1)}{m-1}.$$

Proof. The maximum number of edges in a graph with n vertices and with the diameter d is ([1], Lemma 1)

$$d + 3(n-d-1) + \frac{(n-d-1).(n-d-2)}{2}.$$

The necessary condition for the existence of a decomposition of the complete graph K_n into m factors with diameter d is the inequality ([1], Theorem 2)

$$m[d+3(n-d-1)+\frac{(n-d-1).(n-d-2)}{2}]\geq\frac{n(n-1)}{2}$$

or, equivalently

$$(1)\qquad (m-1)n^2+(3m-2md+1)n+m(d^2-d-4)\geq 0.$$

In the following we shall use the idea from the proof of Lemma 6 in [1]. The quadratic function of the variable n defined by the left hand side of (1) takes negative values for $n_1=d$ and for $n_2=\frac{m+\sqrt{m}}{m-1}d-\frac{m+\sqrt{m}(2m-1)}{m-1}$ (with the exception of the case $d=2m-1$, when it takes the value 0). Since this function is convex and a graph with the diameter d has at least $d+1$ vertices, the theorem is proved.

Remark. For $d=2m-1$ the estimation in Theorem 2 is the best possible. In fact, in this case the right hand side of the inequality of Theorem 2 gives the value 2m and and by [3] (Theorem 3) $f_m(d)=2m$ for $m\geq 3$ and $3\leq d\leq 2m-1$.

Theorem 3. *If $d\geq 2m$ and $m>3$ then*

$$f_m(d)\leq\frac{5}{2}d+3.$$

Proof. We shall confine ourselves to the case when d is an odd number (in the case when d is an even number we can proceed in a similar way). It is sufficient to show that the complete graph with $\frac{5d+5}{2}$ vertices is decomposable into m factors with the diameter d. Denote the vertices of this complete graph by symbols $u_1,u_2,\ldots,u_{2k},v_1,v_2,\ldots,v_k,w_1,w_2,\ldots,w_k,t_1,t_2,\ldots,t_k$, where $k=\frac{d+1}{2}$.

a) The factor F_i for $i=1,2,\ldots,m-3$ will contain the path

$$u_{i+1}u_iu_{i+2}u_{i-1}u_{i+3}u_{i-2}\ldots u_{i+k}u_{i-k+1},$$

where the subscripts j of u_j are taken as the integers $1,2,\ldots,2k(mod 2k)$ and the edges

$$u_{2i+1}v_j, u_{2i+1}w_j, j=1,2,\ldots,k,$$
$$u_{2i+2}t_j, j=1,2,\ldots,k.$$

b) The factor F_{m-2} will contain the path

$$v_1v_2\dots v_kw_1w_2\dots w_k$$

and the edges

$$v_3u_{2j}, v_3t_j, t_ku_{2j-1}, \quad j=1,2,\dots,k.$$

c) The factor F_{m-1} will contain

1) the edges
t_3u_{2j-1}, v_1u_{2j}, $j=1,2,\dots,k$,
t_3v_j, $j=1,2,4,5,\dots,k\,(j\neq 3)$,
v_1v_3

2) and the path
(i) $t_1t_2\dots t_kw_1w_3\dots w_{k-1}w_2w_4\dots w_k$ if k is even or
(ii) $t_1t_2\dots t_kw_1w_3\dots w_kw_2w_4\dots w_{k-1}$ if k is odd.

d) The factor F_m will contain

1) the edges
$t_1u_{2j-1}, t_1w_j, w_1u_{2j}$, $j=1,2,\dots,k$

2) and the path
(i) $v_3v_5\dots v_{k-1}v_2v_4\dots v_kv_1t_1t_3\dots t_{k-1}t_2t_4\dots t_k$ if k is even or
(ii) $v_3v_5\dots v_kv_2v_4\dots v_{k-1}v_1t_1t_3\dots t_kt_2t_4\dots t_{k-1}$ if k is odd.

Now consider the edges which so far we have not included into any of the factors $F_1, F_2, \dots, F_m$. Let those of them which are of the types v_iw_j, v_it_j, w_it_j, v_iv_j, w_iw_j, t_it_j or u_it_j, u_iu_j or u_iv_j or u_iw_j belong to the factor F_1 or F_{m-2} or F_{m-1} or F_m, respectively. It is easy to check that the factors F_1, $F_2, \dots, F_m$ have the diameter d and they form a decomposition of the complete graph with $\frac{5d+5}{2}$ vertices.

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