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EDGE-LOCALLY HOMOGENEOUS GRAPHS

Roman Nedela

Abstract. In the paper we shall investigate the relationship between locally homogenerous graphs and edge-locally homogeneous graphs. A local version of the well-known theorem establishing that an edge-transitive graph is either vertex-transitive, or bipartite is proved. Further we aply the theory of covering spaces to derive some general results on the family of edge-locally G_0 graphs for a fixed graph G_0.

Introduction

In 1986 Zelinka [11] introduced the concept of edge-locally homogeneous graphs. It can be understood as an edge version of the concept of locally homogeneous graphs (or graphs with a constant link, see [1,4]). Let G be a graph and x be either a vertex, or an edge of G. Denote the subgraph of G induced on the set of vertices at distance 1 from x by $link(x, G)$. The graph G is called locally homogeneous, or locally G , if there exists a finite graph G_0 such that for each vertex v of G $link(v, G) \cong G_0$. Similarly, the graph G is called edge-locally homogeneous, or edge-locally G_0 , if there exists a finite graph G_0 such that for each edge e of G $link(e, G) \cong G_0$. Two main problems for edge-locally homogeneous graphs can be considered:

(a) For which finite graphs G_0 does there exist an edge-locally G_0 graph?,

(b) For a fixed finite graph G_0 what can be said about the set of all connected edge-locally G_0 graphs?

Zelinka in [11] showed some examples of edge-locally G_0 graphs. A lot of examples of such graphs can be obtained using the concept of edge-transitive graphs. Further, it was proved in [11] that there is no edge-locally C_5 graph. This result was generalized by Fronček [2] by proving that for n odd, $n \neq 3$, there is no edge-locally C_n graph. In contrast, it is proved in [5] that a finite edge-locally C_n graph exists for all the remaining

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values of n. Another result contained in [2] reads as follows: if G is a complete multipartite graph then an edge-locally G graph exists if and only if all parts of G contain the same number of vertices. In this paper we shall investigate a connection between the locally homogeneous graphs and edge-locally homogeneous graphs. Further we apply the concept of covering spaces to derive some results analogous to those given in [7].

Strongly edge-locally homogeneous graphs

For a given graph G and its edge e denote by $Link(e,G)$ the subgraph of G induced on the set of vertices at distance ≤ 1 from e. That means $e \in Link(e,G)$. Then the graph G will be called *strongly edge-locally homogeneous* if for any two edges e, f in G there is an isomorphism $\varphi Link(e,G) \to Link(f,G)$ mapping e onto f. The following observation is clear.

Proposition 1. *If a graph G is strongly edge-locally homogeneous then G is edge-locally homogeneous .*

In fact we know no edge-locally homogeneous graph which is not strongly edge-locally homogeneous as well. Thus the question, whether the opposite implication in Proposition 1 holds true, is open. The following theorem can be considered as a local version of the well-known theorem (see [3]) establishing that an edge-transitive graph is either vertex-transitive or bipartite.

Theorem 2. *Let G be a strongly edge-locally homogeneous graph. Then either G is locally homogeneous or bipartite.*

Proof. Let $e = uv$ be a fixed edge of G. Let $f = xy$ be an arbitrary edge of G. Since G is strongly edge-locally homogeneous, there is an isomorphism $\varphi Link(f,G) \to Link(e,G)$ mapping f onto e. Then either there is an automorphism ψ of $Link(e,G)$ mapping u to v, or there is no such an automorphism. In the first case either φ, or $\psi\varphi$ maps $link(x,G)$ onto $link(u,G)$. Since f, and consequently x, is chosen arbitrarily, G is locally homogeneous in this case. In the second case the set of vertices of G splits into two subsets U, V. A vertex x is in U (V) if and only if there is an isomorphism mapping $link(x,G)$ onto $link(u,G)$ (or onto $link(v,G)$, respectively). Since there is no automorphism of $Link(e,G)$ mapping u onto v, $U \cap V = \varnothing$. Clearly, each edge of G joins a vertex from U to a vertex in V, otherwise it would be an automorphism of $Link(e,G)$ mapping u to v. Thus G is bipartite. $\square$

It was noted by Zelinka that if G is bipartite then the edge-local homogeneity of G implies the strong edge-local homogeneity of G. The following proposition shows further properties of strongly edge-locally homogeneous graphs. A bipartite graph G is called *biregular* if the vertices in one part of G have degree p while the vertices in the second part of G are of degree q, for some integers p, q. An r-regular graph in which each edge lies in t triangles and q induced quadrangles will be called an (r, t, q)-graph.

Proposition 3. *Let G be a strongly edge-locally homogeneous graph. Then either G is an (r, t, q)-graph for some integers r, t, q, or it is a bipartite biregular graph.*

Proof. It follows directly from the definition of strong local homogeneity that each edge of G lies in the same number of triangles and in the same number of induced quadrangles. According to Theorem 2 G is either locally homogeneous, and consequently regular, or it is bipartite, and therefore biregular. □

It follows from Proposition 3 that strongly edge-locally homogeneous graph G containing at least one triangle is an (r, t, q) graph. Thus for each vertex u of G $link(u, G)$ is a t-regular graph on r vertices. Regular graphs with regular links of vertices were investigated by Šoltés in [9]. He proved there that an r-regular graph G with t-regular links of vertices is the complete $(1 + r/(r - t))$-partite graph, whose each part contains $r - t$ vertices, if $t < r < t + r\sqrt{\frac{8}{9}(t-1)} + \frac{4}{3}$.

Covering spaces of edge-locally homogeneous graphs

Let G be a graph. Denote by ΔG the simplicial complex, whose 0-simplexes are vertices of G, 1-simplexes are edges of G, 2-simplexes are bounded by triangles and induced quadrilaterals of G, and the incidence relation is given by the subgraph inclusion. That means that ΔG arises from G by gluing a 2-cell to each triangle and to each induced quadrangle of G. The following three propositions are analogous to Propositions 3, 4 and 5 in [7]. They follow from Theorems 1.2 and 1.5 in [6].

Proposition 4. *Let G be an edge-locally G graph, for some finite graph G . Let (X, p) be a connected covering space of ΔG. Then there is an edge-locally G graph H such that $p^{-1}(G) = H$ and $X = \Delta H$.*

Proposition 5. *Let $(G, \alpha)_n$ be a permutation voltage graph and G be a connected edge-locally G_0 graph. Then G_n^α is edge-locally G_0 if and only if the product of voltages in each triangle and quadrangle of G is 1.*

Proposition 6. *Let G be a connected edge-locally G_0 graph. Let (G, Γ, α) be an ordinary voltage graph. Then the derived graph G^α is edge-locally G_0 if and only if the product of voltages in each triangle and quadrangle of G is 1.*

The following proposition was motivated by the similar results of Vince [10] for locally homogeneous graphs. Call a subgroup B of the automorphism group $Aut\, G$ of a graph G strongly discontinous if for each $\varphi \in B$ and each vertex v of G the distance $\rho(v, \varphi(v)) \geq 5$.

Proposition 7. *Let G be edge-locally G_0 , for some finite graph G_0 . Let $\Gamma \subseteq Aut\, G$ be a strongly discontinous subgroup of $Aut\, G$. Then the regular quotient G/Γ is edge-locally G_0 .*

Proof. Consider $link([e], G/\Gamma)$ for some edge e in G. We show that the restriction $p' = p/link(e, G)$ of the covering projection mapping a vertex v onto [v] is an isomorphism mapping $link(e, G)$ onto $link([e], G/\Gamma)$. By its definition p' is onto. Since $Link(e, G)$ is a graph of diameter at most 3, by the assumption we have that p' is a bijection on the set of vertices of $link(e, G)$. Clearly, if $e = uv$ is an edge in $link(e, G)$ then $[u][v]$ is an edge in $link([e], G/\Gamma)$. On the other hand, let $[f] = [u][v]$ be an edge in $link([e], G/\Gamma)$, where $f = vw$ is the edge of G incident with v and mapped by p onto $[u][v]$. Suppose, on the contrary, that an edge uv is not in $link(e, G)$. Then $w \neq u$, $w \in [u]$ and the distance $\rho_G(u, w) \leq 4$, a contradiction with the assumption. Thus p' is an isomorphism of the graphs $link(e, G)$ and $link([e], G/\Gamma)$, and G/Γ is edge-locally G_0 . □

Note that if the graph G in Proposition 7 is strongly edge-locally G_0 then the graph G/Γ is strongly edge-locally G_0 as well. The following corollary allows us to build edge-locally homogeneous graphs from groups.

Corollary 8. *Let G be an edge-transitive graph. Let $\Gamma \subseteq Aut\, G$ be a strongly discontinous subgroup of the automorphism group $Aut\, G$. Then the regular quotient G/Γ is strongly edge-locally homogeneous.*

The following two theorems can be considered as edge variants of results in [7].

Theorem 9. *Let G be a finite graph and let the Euler characteristic $\chi(\Delta\, G) \leq 0$. Then for each $n \geq 1$ there exists an n-fold cover of $\Delta\, G$.*

Proof. Denote by v, e, f_3 and f_4 the number of vertices, edges, triangles and induced quadrangles in G, respectively. The fundamental group $\pi(\Delta\, G)$ is generated by $gen = e - v + 1$ generators satisfying $rel = f_3 + f_4$ relations.

By the assumption we have $rel = f_3 + f_4 < e - v + 1 = gen$. Thus $\pi(\Delta G)$ contains a subgroup of index n for each $n > 1$. It follows from the well-known correspondence betweeen the covers of a topological space and subgroups of its fundamental group that for each $n > 1$ there is an n-fold cover of ΔG. □

Theorem 10. *Let G_0 be a finite graph. Let G be a finite connected edge-locally G_0 graph and let $\chi(\Delta G) \leq 0$. Then*

(a) for each $n > 1$ there exists a connected edge-locally G_0 graph with $n.v(G)$ vertices,

(b) there exists an infinite connected edge-locally G graph.

Proof. Theorem 9 implies that there is an n-fold cover of ΔG for each $n > 1$. The statement (a) now follows from Proposition 4. Consider the universal cover $\tilde{X}$ of ΔG. By Proposition 4 $\tilde{X} \cong \Delta \tilde{G}$, where $\tilde{G}$ is edge-locally G_0 , and moreover, $\tilde{G}$ covers each the edge-locally G_0 graph constructed in the proof of part (a) of the theorem. Thus the number of vertices of $\tilde{G}$ is infinite. □

If G is strongly edge-locally homogeneous then the following proposition enumerates the cases for which $\chi(\Delta G) \leq 0$.

Proposition 11. *Let G be strongly edge-locally G_0 . Then $\chi(\Delta G) \leq 0$ if and only if either G is an (r, t, s)-graph, where $0 \leq 4t + 3s \leq 11$ and $r \geq 24/(12 - 4t - 3s)$, or G is bipartite biregular, $e(G_0) = s \leq 3$ and $e(G) \geq v(G)/(1 - s/4)$.*

Proof. Let v and e be the numbers of vertices and edges in G, respectively. Denote by f_3 and f_4 the numbers of triangles and induced quadrangles of G, respectively. Then $\chi(\Delta G) = v - e + f_3 + f_4$. According to Proposition 3 G is either an (r, s, t)-graph, or it is bipartite biregular. In the first case we have $e = vr/2$, $f_3 = et/3 = vrt/3$, and $f_4 = es/4 = vrs/8$. Thus the inequality $\chi(\Delta G) \leq 0$ is equivalent to the inequality $24 + r(4t + 3s - 12) \leq 0$ implying the first part of the statement. In the second case $f_3 = 0$ and the inequality $\chi(\Delta G) \leq 0$ is equivalent to the inequality $v \leq e(1 - s/4)$, where $s = f_4 = e(G)$. □

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Department of Mathematics, Matej Bel University, Tajovského 40, 975 49 Banská Bystrica, Slovakia

E-mail address: nedela@fhpv.umb.sk

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