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SUBDIRECT DECOMPOSITIONS OF DIGRAPHS

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ABSTRACT. Direct product decompositions of the covering graph $C(\overline{\mathcal{G}})$ of a digraph $\overline{\mathcal{G}}$ and direct product decompositions of $\overline{\mathcal{G}}$ were studied in [1]. The relations between a certain type of subdirect decompositions of $C(\overline{\mathcal{G}})$ and subdirect decompositions of $\overline{\mathcal{G}}$ will be studied in the present paper.

A *graph* $\mathcal{G} = (V, E)$ consists of a nonempty set V of vertices together with a prescribed set E of unordered pairs of distinct vertices of V. Each pair $\{x, y\} \in E$ is an *(undirected) edge* of the graph $\mathcal{G}$ and shall be denoted by xy.

A *digraph* $\overline{\mathcal{G}} = (V, \overline{E})$ consists of a nonempty set V of vertices together with a prescribed set $\overline{E}$ of ordered pairs of distinct vertices. Each ordered pair $(x, y) \in \overline{E}$ is a *(directed) edge* of the digraph $\overline{G}$ and shall be denoted by $\overline{xy}$.

Let I be a nonempty set and $\mathcal{G}_i = (V_i, E_i)$, $i \in I$ be graphs. Let V be the cartesian product of the sets V_i ($V = \prod_{i\in I} V_i$). The elements of V will be denoted $a = (a_i)$, $i \in I$, where $a_i = a(i) \in V_i$. Let $\mathcal{G}$ be a graph whose set of vertices is V and whose set of edges consists of those pairs $\{x, y\}$, $x, y \in V$ which satisfy the following condition: there is $i \in I$ such that $x_i y_i \in E_i$ and $x_j = y_j$ for each $j \in I \setminus \{i\}$. Then $\mathcal{G}$ is said to be the *direct product of the graphs* $\mathcal{G}_i$, $i \in I$ and we write $\mathcal{G} = \prod_{i\in I} \mathcal{G}_i$.

The direct product of digraphs is defined similarly.

For all further notions concerning digraphs and graphs we refer the reader to [2].

Let $\prod_{i\in I} \mathcal{G}_i = (V, E)$. If $W \subseteq V$, then we denote $O_i(W) = \{a_i a \in W\}$.

Let $\prod_{i\in I} \mathcal{G}_i = (V, E)$ be the direct product of graphs $\mathcal{G}_i = (V_i, E_i)$ ($i \in I$). If $W \subseteq V$ and $O_i(W) = V_i$ for each $i \in I$, then a graph $\mathcal{G} = (W, F)$, where

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$F = \{ab \in Ea, b \in W\}$, will be called a *subdirect product* of the graphs $\mathcal{G}_i$. If $\mathcal{G}$ is a subdirect product of graphs $\mathcal{G}_i$ we write $\mathcal{G} = (\text{sub}) \prod_{i \in I} \mathcal{G}_i$.

Subdirect products of digraphs are defined similarly.

Remark. If $W = V$, then $(\text{sub}) \prod_{i \in I} \mathcal{G}_i = \prod_{i \in I} \mathcal{G}_i$.

The subgraph of a graph $\mathcal{G} = (V, E)$ induced by a set $W \subseteq V$ will be denoted by $\mathcal{G}\langle W \rangle$.

Remark. Since a graph $(\text{sub}) \prod_{i \in I} \mathcal{G}_i$ is in fact a subgraph of the graph $\prod_{i \in I} \mathcal{G}_i$ induced by a suitable set W with $O_i(W) = V_i$ for each $i \in I$, then $(\text{sub}) \prod_{i \in I} \mathcal{G}_i = (\prod_{i \in I} \mathcal{G}_i)\langle W \rangle$.

If a mapping $f V_1 \to V_2$ is an isomorphism of a graph $\mathcal{G}_1 = (V_1, E_1)$ onto a graph $\mathcal{G}_2 = (V_2, E_2)$, then we shall write $\mathcal{G}_1 \overset{\mathrm{f}}{\simeq} \mathcal{G}_2$ or shortly $\mathcal{G}_1 \simeq \mathcal{G}_2$.

If $\mathcal{G} \overset{\mathrm{f}}{\simeq} (\text{sub}) \prod_{i \in I} \mathcal{G}_i$ then we shall say that $(\text{sub}) \prod_{i \in I} \mathcal{G}_i$ is a *subdirect decomposition* of the graph $\mathcal{G}$ (with respect to the mapping f).

In the present paper every subdirect decomposition $(\text{sub}) \prod_{i \in I} \mathcal{G}_i$, where $\mathcal{G}_i = (V_i, E_i)$, is supposed to be nontrivial (i. e. $|V_i| > 1$ for each $i \in I$).

Analogous terminology and notation are used for digraphs.

Let $\overline{\mathcal{G}} = (V, \overline{E})$ be a digraph. By the *covering graph* of $\overline{\mathcal{G}}$ we mean the graph $C(\overline{\mathcal{G}}) = (V, E)$ where $ab \in E$ iff $\overline{ab} \in \overline{E}$.

The following two lemmas are easy to verify.

Lemma 1. *Let* $\overline{\mathcal{G}}_1 = (V_1, \overline{E}_1)$, $\overline{\mathcal{G}}_2 = (V_2, \overline{E}_2)$ *be digraphs. If* $\overline{\mathcal{G}}_1 \overset{\mathrm{f}}{\simeq} \overline{\mathcal{G}}_2$ *then* $C(\overline{\mathcal{G}}_1) \overset{\mathrm{f}}{\simeq} C(\overline{\mathcal{G}}_2)$.

Lemma 2. *Let* $\prod_{i \in I} \overline{\mathcal{G}}_i = (V, \overline{E})$ *be the direct product of digraphs* $\overline{\mathcal{G}}_i$, $i \in I$ *and let* $W \subseteq V$. *Then* $C((\prod_{i \in I} \overline{\mathcal{G}}_i)\langle W \rangle) = (\prod_{i \in I} C(\overline{\mathcal{G}}_i))\langle W \rangle$.

Lemma 1 and Lemma 2 imply the following

Theorem 1. *Let* $\overline{\mathcal{G}}$, $\overline{\mathcal{G}}_i$, $i \in I$ *be digraphs and* $\overline{\mathcal{G}} \overset{\mathrm{f}}{\simeq} (\text{sub}) \prod_{i \in I} G_i$. *Then* $C(\overline{\mathcal{G}}) \overset{\mathrm{f}}{\simeq} (\text{sub}) \prod_{i \in I} C(\overline{\mathcal{G}}_i)$.

Definition. Let $\overline{\mathcal{G}} = (V, \overline{E})$ be a digraph and let $C(\overline{\mathcal{G}}) \overset{\mathrm{f}}{\simeq} (\text{sub}) \prod_{i \in I} \mathcal{G}_i$, where $\mathcal{G}_i = (V_i, E_i)$, $i \in I$. We shall say that the subdirect decomposition $(\text{sub}) \prod_{i \in I} \mathcal{G}_i$ of the graph $C(\overline{\mathcal{G}})$ induces a subdirect decomposition of the digraph $\overline{\mathcal{G}}$ if there exist such digraphs $\overline{\mathcal{G}}_i = (V_i, \overline{E}_i)$ that $C(\overline{\mathcal{G}}_i) = \mathcal{G}_i$ for each $i \in I$ and $\overline{\mathcal{G}} \overset{\mathrm{f}}{\simeq} (\text{sub}) \prod_{i \in I} \overline{\mathcal{G}}_i$.

A subdirect decomposition of $C(\overline{\mathcal{G}})$ does not induce a decomposition of $\overline{\mathcal{G}}$ in general. The digraph $\overline{\mathcal{G}} = (\{a,b,c,d\}, \{\overline{ab}, \overline{bc}, \overline{cd}, \overline{da}\})$ is not isomorphic to the subdirect product of any two digraphs but its covering graph is isomorphic to the subdirect (direct) product of two complete graphs K_2.

We are going to investigate when a subdirect decomposition of $C(\overline{\mathcal{G}})$ induces a subdirect decomposition of $\overline{\mathcal{G}}$.

Let $\mathcal{G} = (V,E)$ be a graph. If there exists a four-element set $W = \{a,b,c,d\} \subseteq V$ such that $\mathcal{G}\langle W\rangle = (W, \{ab, bc, cd, ad\})$, then we say that the graph $\mathcal{G}\langle W\rangle$ is a *square* (in $\mathcal{G}$) and we denote it by $\mathcal{S}(a,b,c,d)$. If $\overline{\mathcal{G}}$ is a digraph and $C(\overline{\mathcal{G}}\langle W\rangle) = \mathcal{S}(a,b,c,d)$, then the digraph $\overline{\mathcal{G}}\langle W\rangle$ is called a *square* (in $\overline{\mathcal{G}}$) and will be denoted by $\overline{\mathcal{S}}(a,b,c,d)$.

An edge ab of a graph $\prod_{i\in I}\mathcal{G}_i$ ($(\text{sub})\prod_{i\in I}\mathcal{G}_i$) will be called a *k-edge* whenever $a_j = b_j$ for each $j \in I\setminus\{k\}$.

We say that ordered pairs (a,b) and (c,d) of vertices of a direct product $\prod_{i\in I}\mathcal{G}_i$ (subdirect product $(\text{sub})\prod_{i\in I}\mathcal{G}_i$) are *r-equivalent* and write $(a,b) \overset{\text{r}}{\sim} (c,d)$ if ab and cd are r-edges and $a_r = c_r$, $b_r = d_r$.

It is easy to see that if $(a,b) \overset{\text{r}}{\sim} (c,d)$ then $(b,a) \overset{\text{r}}{\sim} (d,c)$.

A square $\mathcal{S}(a,b,c,d)$ in $\prod_{i\in I}\mathcal{G}_i$ ($(\text{sub})\prod_{i\in I}\mathcal{G}_i$) will be called an *r-square* whenever all its edges are r-edges for some $r\in I$. If such $r\in I$ does not exist, it will be called a *mixed square*.

Let $C(\overline{\mathcal{G}}) \overset{\text{f}}{\simeq} \prod_{i\in I}\mathcal{G}_i$. We shall say that the edge $\overline{ab}$ of the digraph $\overline{\mathcal{G}}$ and the edge ab of the covering graph $C(\overline{\mathcal{G}})$ are k-edges (with respect to the isomorphism f) if $f(a)f(b)$ is a k-edge of the graph $\prod_{i\in I}\mathcal{G}_i$. In an analogous way the other notions concerning the direct product $\prod_{i\in I}\mathcal{G}_i$ can be introduced for the digraph $\overline{\mathcal{G}}$ and the covering graph $C(\overline{\mathcal{G}})$.

In [1] it was proved that if $\mathcal{S}(a,b,c,d)$ is a mixed square, then there exist $r,s\in I$, $r\neq s$ such that ab, cd are r-edges and bc, ad are s-edges (cf. Lemmas 2, 3, 4 in [1]).

Lemma 3 [1]. *Let $\mathcal{S}(a,b,c,d)$ be a mixed square in $\prod_{i\in I}\mathcal{G}_i$, where ab is an r-edge and bc is an s-edge. Then $(a,b) \overset{\text{r}}{\sim} (d,c)$, $(b,c) \overset{\text{s}}{\sim} (a,d)$.*

Since $(\text{sub})\prod_{i\in I}\mathcal{G}_i = (\prod_{i\in I}\mathcal{G}_i)\langle W\rangle$, the above mentioned facts hold also for the subdirect products.

Let $(\text{sub})\prod_{i\in I}\mathcal{G}_i = (V,E)$ be a subdirect product of graphs $\mathcal{G}_i = (V_i, E_i)$ and let $(a_i), (b_i) \in V$, $i\in I$. We shall say that the subdirect product $(\text{sub})\prod_{i\in I}\mathcal{G}_i$ is *orientable* if the following condition is fulfilled:

If $a_k b_k \in E_k$ then there exists a k-edge $(a_i)(b_i)\in E$, $i\in I$.

Example. Let $\mathcal{G} = (\{a,b,c\}, \{ab, bc\})$, $\mathcal{G}' = (\{1,2,3,4\}, \{12, 23, 34\})$ be graphs. Let $W = \{(a,a), (a,b), (a,c), (b,a), (c,a), (c,c)\}$ and $W' =$

$= \{(1,1),(1,2),(1,3),(2,1),(3,1),(4,4)\}$. Then the subdirect product $(\mathrm{sub})\prod_{i\in\{1,2\}}\mathcal{G}_i = (\prod_{i\in\{1,2\}}\mathcal{G}_i)\langle W\rangle$, where $\mathcal{G}_i = \mathcal{G}$, $i \in \{1,2\}$, is orientable and the subdirect product $(\mathrm{sub})\prod_{i\in\{1,2\}}\mathcal{G}'_i = (\prod_{i\in\{1,2\}}\mathcal{G}'_i)\langle W'\rangle$, where $\mathcal{G}'_i = \mathcal{G}'$, $i \in \{1,2\}$, is not orientable. Let us notice that $(\mathrm{sub})\prod_{i\in\{1,2\}}\mathcal{G}_i \simeq (\mathrm{sub})\prod_{i\in\{1,2\}}\mathcal{G}'_i$.

All subdirect products considered in the next are assumed to be orientable.

Lemma 4. *Let* $C(\overline{\mathcal{G}}) \overset{\mathrm{f}}{\simeq} (\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$, *where* $\overline{\mathcal{G}} = (V,\overline{E})$ *and* $\mathcal{G}_i = (V_i, E_i)$. *The subdirect decomposition* $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$ *of* $C(\overline{\mathcal{G}})$ *induces a subdirect decomposition of* $\overline{\mathcal{G}}$ *if and only if for any two r-equivalent ordered pairs* (a,b), (c,d) *of vertices of* $\overline{\mathcal{G}}$ *the following condition is fulfilled:*

$$(1) \qquad \overline{ab} \in \overline{E} \quad \text{if and only if} \quad \overline{cd} \in \overline{E}.$$

Proof. It suffices to define $\overline{\mathcal{G}}_i$ for each $i \in I$ by $\overline{\mathcal{G}}_i = (V_i, \overline{E}_i)$, where $\overline{f(a)_i f(b)_i} \in \overline{E}_i$ if and only if there exists an i-edge $\overline{ab} \in \overline{E}$.

A subdirect product $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i = (W,E) = \mathcal{G}$ is said an *l-product* if the following condition is fulfilled:

If $a,b,c,d \in W$ and $(a,b) \overset{\mathrm{r}}{\sim} (c,d)$, then there exist a nonnegative integer n and vertices $x^0 = a, x^1, \ldots, x^n = c$, $y^0 = b, y^1, \ldots, y^n = d \in W$ such that $\mathcal{G}\langle x^j, x^{j+1}, y^{j+1}, y^j\rangle$ is a mixed square $\mathcal{S}(x^j, x^{j+1}, y^{j+1}, y^j)$ for each $j \in \{0,1,\ldots,n-1\}$.

Remark. If $\mathcal{G} = \prod_{i\in I}\mathcal{G}_i$ is a connected graph, then the direct product $\prod_{i\in I}\mathcal{G}_i$ is an l-product (cf. Lemma 6 in [1]).

The following theorem is a generalization of a result from [1].

Theorem 2. *Let* $C(\overline{\mathcal{G}}) \overset{\mathrm{f}}{\simeq} (\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$, *where* $\overline{\mathcal{G}} = (V,\overline{E})$ *is a digraph and* $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$ *is an l-product. The subdirect decomposition* $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$ *of* $C(\overline{\mathcal{G}})$ *induces a subdirect decomposition of* $\overline{\mathcal{G}}$ *if and only if the following condition is fulfilled:*

(2) If $\overline{\mathcal{S}}(a,b,c,d)$ *is a mixed square in* $\overline{\mathcal{G}}$, *then there exists* $i \in \{1,2,3\}$ *with* $\overline{\mathcal{S}}(a,b,c,d) \simeq \overline{\mathcal{S}}_i$, *where*
$\overline{\mathcal{S}}_1 = (\{a,b,c,d\}, \{\overline{ab}, \overline{bc}, \overline{dc}, \overline{ad}\})$,
$\overline{\mathcal{S}}_2 = (\{a,b,c,d\}, \{\overline{ab}, \overline{ba}, \overline{bc}, \overline{cd}, \overline{dc}, \overline{ad}\})$,
$\overline{\mathcal{S}}_3 = (\{a,b,c,d\}, \{\overline{ab}, \overline{ba}, \overline{bc}, \overline{cb}, \overline{cd}, \overline{dc}, \overline{da}, \overline{ad}\})$.

Proof. Let the subdirect decomposition $(\mathrm{sub})\prod_{i\in I}\mathcal{G}_i$ of $C(\overline{\mathcal{G}})$ induce a subdirect decomposition of $\overline{\mathcal{G}}$ and $\overline{\mathcal{S}}(a,b,c,d)$ be its mixed square. Then, by

Lemma 3, there exist $r, s \in I$, $r \neq s$, such that $(a,b) \overset{\mathrm{r}}{\sim} (d,c)$, $(b,c) \overset{\mathrm{s}}{\sim} (a,d)$ and $(b,a) \overset{\mathrm{r}}{\sim} (c,d)$, $(c,b) \overset{\mathrm{s}}{\sim} (d,a)$. From Lemma 4 it follows that $\overline{ab} \in \overline{E}$ iff $\overline{dc} \in \overline{E}$, $\overline{bc} \in \overline{E}$ iff $\overline{ad} \in \overline{E}$ and $\overline{ba} \in \overline{E}$ iff $\overline{cd} \in \overline{E}$, $\overline{cb} \in \overline{E}$ iff $\overline{da} \in \overline{E}$. Thus there exists $i \in \{1,2,3\}$ with $\overline{\mathcal{S}}(a,b,c,d) \simeq \overline{\mathcal{S}}_i$. To prove the converse implication, suppose that (2) is fulfilled. With respect to Lemma 4, it suffices to prove that if $(x,y) \overset{\mathrm{r}}{\sim} (u,v)$, then (1) holds. Since (sub) $\prod_{i \in I} \mathcal{G}_i$ is an l-product, then there exist a nonnegative integer n and vertices $x^0 = x, x^1, \ldots, x^n = u$, $y^0 = y, y^1, \ldots, y^n = v \in V$ such that $\overline{\mathcal{G}}\langle x^j, x^{j+1}, y^{j+1}, y^j \rangle$ is a mixed square $\overline{\mathcal{S}}(x^j, x^{j+1}, y^{j+1}, y^j)$ in $\overline{\mathcal{G}}$ for each $j \in \{0, 1, \ldots, n-1\}$. If $n = 0$, then (1) holds, since $(x,y) = (u,v)$. If $n = 1$, then $\overline{\mathcal{S}}(x,u,v,y)$ is a mixed square and from (2) it follows (1). Now it is easy to complete the proof by induction on n.

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