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CHARACTERIZATION OF UNIVERSAL QUASIGROUP IDENTITIES OF CANONICAL TYPE

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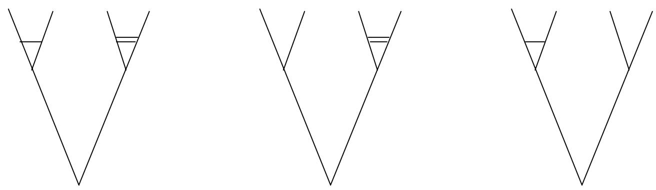
Abstract. Quasigroup identities of canonical type are defined. Conditions which are necessary and sufficient for such identities to be universal are found.

In 1968 Belousov [1] posed the conjecture that the variety of quasigroups is invariant under the isotopies if and only if it can be characterized by equalities whose coresponding diagrams satisfy the following two conditions:

(1) they can contain only forks of the following three types



(2) if the numbers 1, 2 and 3 are assigned to the tops of the forks as indicated below

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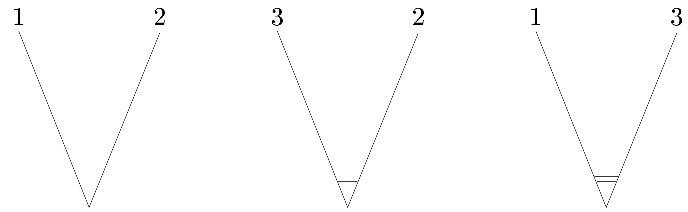


then whenever an element x has a position i, $i = 1, 2, 3$ in a fork then the element x has the same position i in any fork containing x.

In the present paper necessary and sufficient conditions for a class of quasigroup identities to be invariant under the isotopies are given.

Let $(Q; A)$ be a quasigroup. The right or the left inverse operation to the operation A will be denoted by ${}^rA$ or ${}^lA$, respectively. Using the unary functors r and l, we can assign the set $\Sigma_A = \{A, {}^rA, {}^lA, {}^{rl}A, {}^{lr}A, {}^{rlr}A\}$, to the quasigroup $(Q; A)$. Here ${}^{rl}A := {}^r({}^lA)$ and similarly for ${}^{rl}A$ and ${}^{rlr}A$. Denote by ι the identity map. Since $r^2 = l^2 = \iota$ and $rlr = lrl$, the set $\Sigma = \{\iota, r, l, rl, lr, rlr\}$ with the composition is a group isomorphic to the group of all permutations of a three-element set. Hence for every $\sigma \in \Sigma$, $A(x, y) = z$ if and only if ${}^\sigma A(\sigma x, \sigma y) = \sigma z$.

Throughout the paper X is a countable set of variables, $T(\Sigma_A)$ is the set of all terms of the language Σ_A over X and $R(\Sigma_A)$ is the set of all identities of the language Σ_A over X. For $w \in R(\Sigma_A)$ let W_w be the set of all variables contained in w and V_w the variety characterized by the identity w.

To every quasigroup identity $w \in R(\Sigma_A)$ a diagram can be assigned such that to every operation from Σ_A a vertex coresponds as shown in Fig.1.

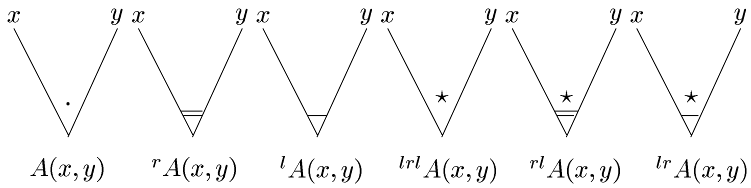


Fig.1

Moreover, to every variable occuring in the identity w we can assign one of the numbers 1, 2 and 3 according to Fig.2 (note that here the notation of vertices plays an important role).

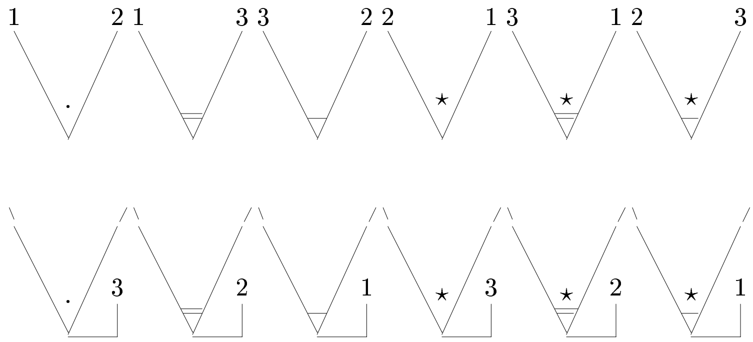


Fig.2

We give an example. Consider the identity

$$((zx) * y)\ {}^{rl}(.)z = (x\backslash y)(z/x)$$

where $(A) = (.)$, $({}^{r}A) = (\backslash)$, $({}^{l}A) = (/)$, $({}^{rlr}A) = (*)$ (i.e., the multiplicative notation is used). Then the diagram assigned to w is that in Fig.3.

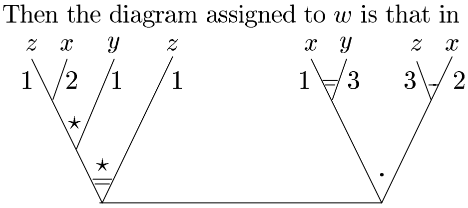


Fig.3

Definition 1. Every quasigroup identity $w \in R(\Sigma_A)$ of the form

$$^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \ldots, {}^{\sigma_n}A(x_n, x_{n+1}))\ldots) = x_o$$

where $n \in N$, $x_o, x_1, \ldots, x_{n+1} \in X$, $\sigma_1, \sigma_2, \ldots, \sigma_n \in \Sigma$ will be called an identity of canonical type.

Definition 2. An identity $w \in R(\Sigma_A)$ of canonical type will be called optimal, if it satisfies the following three conditions

(i) w contains no terms of the forms

$$B(x, {}^{r}B(x, t)),$$

$${}^{l}B(x, {}^{r}B(y,x)),$$

where $B \in \Sigma_A$, $t \in T(\Sigma_A)$, $x, y \in W_w$;

(ii) for every i, j with $2 \leq i \leq j \leq n$ there exists a quasigroup $Q \in V_w$ in which the identity

$$A_i(x_i, A_{i+1}(x_{i+1}, \ldots, A_n(x_n, x_{n+1}))\ldots) =$$

$$= A_j(x_j, A_{j+1}(x_{j+1}, \ldots, A_n(x_n, x_{n+1}))\ldots)$$

does not hold;

(iii) for every i with $2 \leq i \leq n$ there exists $Q \in V_w$ in which the identity

$$A_i(x_i, A_{i+1}, \ldots, A_n(x_n, x_{n+1}))\ldots) = x_{n+1}$$

does not hold.

An optimal quasigroup identity will be called nontrivial if it contains no isolated variable, i.e., if every variable $x \in W_w$ occurs at least twice in the identity w. In the sequel only nontrivial quasigroup identities will be considered.

Let (Q_1, B) and (Q_2, A) be quasigroups. An ordered triple $T = (\alpha, \beta, \gamma)$ of bijections of the set Q_1 onto the set Q_2 is called an isotopy of the quasigroup (Q_1, B) onto the quasigroup (Q_2, A) provided the following diagram commutes

$$\begin{array}{ccc} Q_1 \times Q_1 & \xrightarrow{B} & Q_1 \\ {\scriptstyle \alpha\times\beta}\downarrow & & {\scriptstyle \gamma}\downarrow \\ Q_2 \times Q_2 & \xrightarrow{A} & Q_2 \end{array}$$

The quasigroup $(Q_2; A)$ is also called an isotope of the quasigroup $(Q_1; B)$.

Let $T(\varphi_1, \varphi_2, \varphi_3)$ be an isotopy of a quasigroup $(Q; B)$ onto $(Q; A)$, (without loss of generality we can assume that $Q_1 = Q_2 =: Q$, since $Q_1 \cong Q_2$). An identity $w \in R(\Sigma_A)$ of the form

(w) $$ {}^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \ldots, {}^{\sigma_n}A(x_n, x_{n+1}))\ldots) = x_o$$

is called universal [2], if it is invariant under every quasigroup isotopy. In other words, w is an universal identity if it holds in some quasigroup $(Q; A)$

if and only if for any isotopy $T = (\varphi_1, \varphi_2, \varphi_3)$ onto quasigroup (Q, A), the identity

(Tw)

$$ {}^{\sigma_1}A(\varphi_{\sigma_1 1}x_1, \varphi_{\sigma_1 2}\varphi^{-1}_{\sigma_2 3}{}^{\sigma_2}A(\varphi_{\sigma_2 1}x_2, \dots, \varphi_{\sigma_{n-1}2}\varphi^{-1}_{\sigma_n 3}{}^{\sigma_n}A(\varphi_{\sigma_n 1}x_n, \varphi_{\sigma_n 2}x_{n+1}))\dots) = $$

$$ = \varphi_{\sigma_1 3}x_o $$

also holds in $(Q; A)$.

Definition 3. We will say that the diagram of a quasigroup identity satisfies the condition $\boxed{B1}$ if all forks contained in the diagram are of the following form (see Fig. 4):

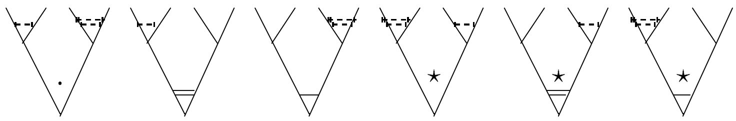
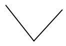
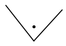
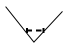
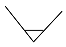
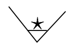
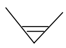
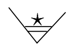


Fig. 4

Here  denotes any of the vertices  or 

Here  denotes any of the vertices  or 

Here  denotes any of the vertices  or 

Definition 4. We will say that the diagram of a quasigroup identity satisfies the condition $\boxed{B2}$ if every variable occuring at a top of the diagram has the same position in each of its occurences (see Fig.2).

Theorem 5. *Let an identity $w \in R(\Sigma_A)$ be nontrivial. If the diagram of the identity w satisfies the conditions $\boxed{B1}$ and $\boxed{B2}$ then w is invariant under the isotopies of quasigroups.*

Proof. Let $w \in R(\Sigma_A)$

(w) $$ {}^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \ldots, {}^{\sigma_n}A(x_n, x_{n+1}))\ldots) = x_o $$

holds in a quasigroup $(Q;\Sigma_A)$ and let $T = (\varphi_1, \varphi_2, \varphi_3)$ be an isotopy onto the qausigroup $(Q;\Sigma_A)$. We are going to prove that also the identity

(Tw)

$$ {}^{\sigma_1}A(\varphi_{\sigma_1 1}x_1, \varphi_{\sigma_1 2}\varphi_{\sigma_2 3}^{-1}\,{}^{\sigma_2}A(\varphi_{\sigma_2 1}x_2, \ldots, \varphi_{\sigma_{n-1}2}\varphi_{\sigma_n 3}^{-1}\,{}^{\sigma_n}A(\varphi_{\sigma_n 1}x_n, \varphi_{\sigma_n 2}x_{n+1}))\ldots) = $$
$$ = \varphi_{\sigma_1 3}x_o $$

holds in the quasigroup $(Q;\Sigma_A)$.
Write the identity Tw in the following more convenient form

(Tw)
$$ \begin{pmatrix} & \varphi_{\sigma_1 1} & & \varphi_{\sigma_2 1} & & & \varphi_{\sigma_n 1} & \varphi_{\sigma_n 2} & & \varphi_{\sigma_1 3} \\ & & \varphi_{\sigma_1 2}\varphi_{\sigma_2 3}^{-1} & & & \varphi_{\sigma_{n-1}2}\varphi_{\sigma_n 3}^{-1} & & & & \\ {}^{\sigma_1}A & (x_1, & {}^{\sigma_2}A & (x_2, & \ldots, & {}^{\sigma_n}A & (x_n, & x_{n+1})) & \ldots) = & x_o \end{pmatrix} $$

Here the coefficients (i.e., the bijections) from the isotopy $T(\varphi_1, \varphi_2, \varphi_3)$ corresponding to the variables occuring in the identity are written in the first row. The second row contains the coefficients (i.e. compositions of two bijections from the isotopy $T(\varphi_1, \varphi_2, \varphi_3)$) corresponding to the symbols of binary operations.

By the assumption the diagram of the identity w satisfies the condition $\boxed{B2}$. Thus every variable has the same position in each of its occurences. Therefore we can omit the first row and write the identity w using only two rows.

Since the diagram of the identity w satisfies also the condition $\boxed{B1}$, we have $\varphi_{\sigma_k 2} = \varphi_{\sigma_{k+1}3}$, i.e. $\varphi_{\sigma_k 2}\varphi_{\sigma_{k+1}3}^{-1} = \iota$ for any $k \in \{1, 2, \ldots, n-1\}$. Therefore also the second row can be omitted. Then the identities Tw and w have the same form and so it is obvious that Tw holds in a quasigroup if and only if w holds in it.

Theorem 6. *Let $w \in R(\Sigma_A)$ be a nontrivial identity. If the diagram of the identity w satisfies the condition $\boxed{B2}$ and does not satisfy the condition $\boxed{B1}$ then w is not a universal identity.*

Proof. Let $w \in R(\Sigma_A)$ be a nontrivial identity and let $T = (\varphi_1, \varphi_2, \varphi_3)$ be an isotopy onto a quasigroup $(Q; A)$. Write the identities w and Tw as follows:

(w) $$^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \dots, {}^{\sigma_n}A(x_n, x_{n+1}))\dots) = x_o$$

(Tw)
$$\begin{pmatrix} & \varphi_{\sigma_1 1} & & \varphi_{\sigma_2 1} & & & \varphi_{\sigma_n 1} & \varphi_{\sigma_n 2} & & \varphi_{\sigma_1 3} \\ & & \varphi_{\sigma_1 2}\varphi^{-1}_{\sigma_2 3} & & & \varphi_{\sigma_{n-1} 2}\varphi^{-1}_{\sigma_n 3} & & & & \\ {}^{\sigma_1}A & (x_1, & {}^{\sigma_2}A & (x_2, & \dots, & {}^{\sigma_n}A & (x_n, & x_{n+1})) & \dots) = & x_o \end{pmatrix}$$

Choose $T = (\varphi_1, \varphi_2, \varphi_3) = (\iota, \beta, \iota)$. The diagram of the identity w satisfies the condition $\boxed{B2}$. For the same reasons as in the proof of Theorem 1 we can omit the first row in Tw to get

(Tw) $$\begin{pmatrix} \beta^{\varepsilon_1} & & \beta^{\varepsilon_2} & & \dots & \beta^{\varepsilon_n} & \\ {}^{\sigma_1}A & (x_1, & {}^{\sigma_2}A & (x_2, & \dots, & {}^{\sigma_n}A & (x_n, x_{n+1}))\dots) = x_o \end{pmatrix}$$

Here $\varepsilon_1 = 0$ and $\varepsilon_i \in \{-1, 0, 1\}$ for $i \in \{2, \dots, n\}$.

The ordered n-tuple $< \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n >$ is said to be the signature of the identity Tw and every pair $< \varepsilon_j, \varepsilon_{j+m} >$ with $j, m \in \{1, 2, \dots, n-1\}$, $j + m \leq n$ a sign change in the signature provided $\varepsilon_j . \varepsilon_{j+m} = -1$ and simultaneously $\varepsilon_{j+1} = \varepsilon_{j+2} = \dots = \varepsilon_{j+m-1} = 0$.

We are going to show that there exist a permutation β and a quasigroup Q such that the identity w holds in Q but the identity Tw where $T = (\iota, \beta, \iota)$, does not hold in Q. Denote the number of sign changes in the signature $< \varepsilon_1, \dots, \varepsilon_n >$ of w by d and the number of non-zero elements of this signature by p. Without loss of generality we can assume that, for example,

$$\underbrace{\varepsilon_n = \varepsilon_{k_p} = 1, \quad \varepsilon_{n-1} = 0, \quad \varepsilon_{n-2} = \varepsilon_{k_{p-1}} = -1, \dots, \varepsilon_3 = 0, \quad \varepsilon_2 = \varepsilon_{k_1} = 1}_{\text{the signature has } \quad p \quad \text{non-zero elements}}$$

Then the following table can be assigned to the identity Tw

	$^{\sigma_n}A(x_n, x_{n+1})$		$\dots$	$^{\sigma_n}A(x_2, {}^{\sigma_3}A(x_3, {}^{p-1}y))$
		$^{\sigma_{n-2}}A(x_{n-2}, {}^{\sigma_{n-1}}A(x_{n-1}, {}^1y))$	$\dots$	
βx	^1y			^py
$\beta^{-1}x$		^2y	$\dots$	

Tab.

Here in the first row we have the arguments of the permutation β and in the second row the arguments of the permutation β^{-1}. The values of β or β^{-1} at these arguments are written in the third or fourth row, respectively. Since we want β to be a permutation, from the table we can see that all the values in the first and fourth row must be mutually different. Hence we get $\binom{p}{2}$ conditions. Similarly we get other $\binom{p}{2}$ conditions by taking into consideration that all the values in the second and third row must be mutually different. We are going to define β so that it satisfies all the mentioned conditions and, moreover, the condition

$$^{\sigma_1}A(x_1, \beta^{\varepsilon_2}\ {}^{\sigma_2}A(x_2, \dots, \beta^{\varepsilon_n}\ {}^{\sigma_n}A(x_n, x_{n+1}))\dots) \neq x_o,$$

i.e.

$$^{\sigma_1}A(x_1, {}^py) \neq x_o.$$

Exactly d conditions from all p(p-1)+1 ones have the following form

$$(q_k) \qquad {}^{\sigma_i}A(x_i, \dots, {}^{\sigma_j}A(x_j, {}^ky))\dots) \neq {}^ky$$

Since the identity w is optimal, for every condition (q_k), $k \in \{1, \dots, d\}$ there is a quasigroup $Q_k \in V_w$ and elements of this quasigroup for which (q_k) holds. By Birkhoff theorem, $Q := Q_1 \times Q_2 \times \dots \times Q_d \in V_w$. Then it is possible to define (we do not go into details) n+2 elements a_1, a_2, ... ,a_{n+1}, a_o from the quasigroup Q and a permutation β on Q such that

$$\beta^{\varepsilon_1 \sigma_1}A(a_1, \beta^{\varepsilon_2 \sigma_2}A(a_2, \dots, \beta^{\varepsilon_n \sigma_n}A(a_n, a_{n+1}))\dots) \neq a_o.$$

So the identity w holds in the quasigroup $Q = Q_1 \times Q_2 \times \dots \times Q_d$, but the identity Tw, $T = (\iota, \beta, \iota)$ does not hold in this quasigroup. Therefore w is not universal.

Theorem 7. *Let $w \in R(\{A, {}^rA, {}^lA\})$ be a nontrivial identity. If the diagram of the identity w satisfies the condition $\boxed{B1}$ and simultaneously does not satisfy the condition $\boxed{B2}$ then w is not universal.*

Proof. If the diagram of the identity

$$^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \dots, {}^{\sigma_n}A(x_n, x_{n+1}))\dots) = x_o$$

satisfies the condition $\boxed{B1}$, then for every $\sigma_i \in \{\iota, r, l, \}$ the following holds
- if $\sigma_1 \in \{\iota, r\}$, then $\sigma_{k+1} = r\sigma_k$, k=1,2, ... ,n-1;
- if $\sigma_1 = l$, then $\sigma_2 = r$ and $\sigma_{k+1} = r\sigma_k$, k=2,3, ... ,n-1.

Depending on σ_1 and parity of the length $l(w)$ of the identity w we obtain six possible forms of w (the multiplicative notation will be used):

1) if $\sigma_1 = \iota$ and $l(w)$ is an even number

$$(w_1) \qquad x_1(x_2\backslash(x_3(x_4\backslash\dots\backslash(x_{n-1}(x_n\backslash x_{n+1}))\dots) = x_o;$$

2) if $\sigma_1 = \iota$ and $l(w)$ is an odd number

$$(w_2) \qquad x_1(x_2\backslash(x_3(x_4\backslash\dots\backslash(x_{n-1}\backslash(x_n.x_{n+1}))\dots) = x_o;$$

3) if $\sigma_1 = r$ and $l(w)$ is an even number

$$(w_3) \qquad x_1\backslash(x_2(x_3\backslash\dots(x_{n-1}\backslash(x_n.x_{n+1}))\dots) = x_o;$$

4) if $\sigma_1 = r$ and $l(w)$ is an odd number

$$(w_4) \qquad x_1\backslash x_2(x_3\backslash\dots\backslash(x_{n-1}(x_n\backslash x_{n+1}))\dots) = x_o;$$

5) if $\sigma_1 = l$ and $l(w)$ is an even number

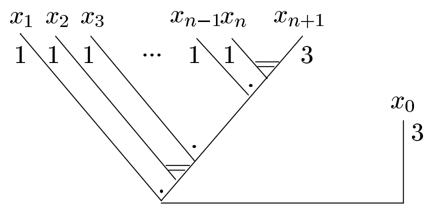
$$(w_5) \qquad x_1/(x_2\backslash(x_3(x_4\backslash\dots\backslash(x_{n-1}(x_n\backslash x_{n+1}))\dots) = x_o;$$

6) if $\sigma_1 = l$ and $l(w)$ is an odd number

$$(w_6) \qquad x_1/(x_2\backslash(x_3(x_4\backslash\dots(x_{n-1}\backslash(x_n.x_{n+1}))\dots) = x_o.$$

By using transformations which transform the identities fulfilled or not fulfilled in a quasigroup into the identities which are fulfilled or not fulfilled, respectively, in it, one can get the identity w_1 from w_5 and the identity w_2 from w_6. So it suffices to prove that the identities w_1, w_2, w_3 and w_4 are universal:

1a) If $x_{n+1} \neq x_o$, then the diagram of the identity w_1 has the following form:



Then there is at least one i and at least one j, $i, j \in \{1, 2, \ldots, n\}$ such that $x_i = x_{n+1}$ and $x_j = x_o$. Let, e.g., i=2 and j=3. Then

$$(Tw_1) \qquad \alpha x_1(\alpha x_2\backslash(\alpha x_3(\alpha x_4 \ldots (\alpha x_{n-1}\backslash\gamma x_2)) \ldots) = \gamma x_3.$$

Choose a quasigroup Q from the variety V_{w_1} and elements $a_1, a_2, \ldots, a_n, b \in Q$ such that $b \neq a_2 \neq a_3 \neq b$. After substituting these elements into the identity w_1 we get

$$a_1(a_2\backslash(a_3(a_4\backslash \ldots (a_{n-1}(a_n\backslash a_2)) \ldots) = a_3.$$

Choose the isotopy $T(\iota, \iota, \gamma)$ onto the quasigroup Q with
$\gamma a_3 = b$, $\gamma b = a_3$, $\quad \gamma x = x$, for all $x \in Q - \{a_3, b\}$.
Substitute these elements into

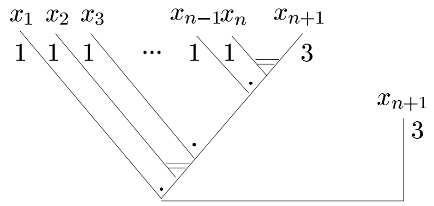
$$(Tw_1) \qquad x_1(x_2\backslash(x_3(x_4\backslash \ldots (x_{n-1}(x_n\backslash\gamma x_2)) \ldots) = \gamma x_3$$

to get

$$a_1(a_2\backslash(a_3(a_4\backslash \ldots (a_{n-1}(a_n\backslash a_2)) \ldots) = b.$$

Clearly, the identity Tw_1 does not hold in Q. Therefore w_1 is not universal.

1b) If $x_{n+1} = x_o$ then we have the following diagram of the identity w_1



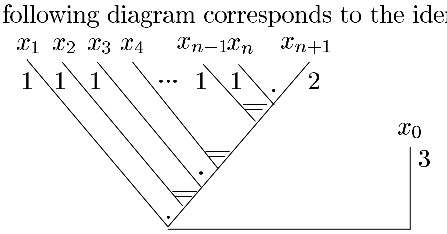
Since this diagram does not satisfy the condition $\boxed{B2}$, there is at least one $i \in \{1, 2, \ldots, n\}$ with $x_i = x_{n+1}$. Let, e.g. $x_2 = x_4 = x_{n+1}$. Then

$$(w_1) \qquad x_1(x_2\backslash(x_3\backslash(x_2(x_5\backslash \ldots (x_n\backslash x_2)) \ldots) = x_2$$

$$(Tw_1) \qquad x_1(x_2\backslash(x_3\backslash(x_2(x_5\backslash \ldots (x_n\backslash \gamma x_2)) \ldots) = \gamma x_2$$

where $\gamma x_2 =: y \notin W_{w_1}$.
Denote the left-hand side of the identity w_1 by f and consider the free quasigroup F whose free generators are the elements of the set $W_{w_1} \cup \{y\}$. In the quotient quasigroup $F\big|_{\Theta<f,x_2>}$ (where the relation $\Theta < f, x_2 >$ is the smallest congruence containing the element $[f, x_2]$) the identity w_1 holds whereas Tw_1 does not hold in it. Therefore w_1 is not universal.

2) The following diagram corresponds to the identity w_2



Consider quasigroup $(Q; ., \backslash, /)$ and take elements $a_o, a_1, \ldots, a_{n+1} \in Q$ in places of the variables $x_o, x_1, \ldots, x_{n+1}$. Choose an isotopy $T(\iota, \iota, \gamma)$ onto the quasigroup $(Q; ., \backslash, /)$ such that the permutation γ on Q has the property:

$\gamma a_o = b \neq a_o$. Then the identity Tw_2 does not hold in the quasigroup $(Q; ., \backslash, /)$ and so w_2 is not universal.

Analogously as in the case of the identity w_1 or w_2 one can prove that the identity w_3 or w_4, respectively, is not universal.

Theorem 8. *Let $w \in R(\Sigma_A)$ be a nontrivial identity. If the diagram of the identity w satisfies neither the condition $\boxed{B1}$ nor the condition $\boxed{B2}$ then w is not universal.*

Proof. Take $w \in R(\Sigma_A)$

$$\text{(w)} \qquad {}^{\sigma_1}A(x_1, {}^{\sigma_2}A(x_2, \dots, {}^{\sigma_n}A(x_n, x_{n+1}))\dots) = x_o$$

By the assumption the diagram of the identity w does not satisfy the condition $\boxed{B2}$. Then there is a variable $x_i \in W_w$ which occurs in the diagram in two different positions. Without loss of generality we may assume that one of them is the position 2. Consider the variety V_w and choose a suitable quasigroup $Q \in V_w$ (the choice will be specified later). Take an isotopy $T = (\iota, \beta, \iota)$ onto Q (β will be specified later)

(Tw)

$$\left(\begin{array}{cccccccccc} & \beta^{\varepsilon'_1} & & \beta^{\varepsilon'_2} & \dots & & \beta^{\varepsilon'_n} & \beta^{\varepsilon'_{n+1}} & & \beta^{\varepsilon'_o} \\ \beta^{\varepsilon_1} & & \beta^{\varepsilon_2} & & \dots & \beta^{\varepsilon_n} & & & & \\ {}^{\sigma_1}A & (x_1, & {}^{\sigma_2}A & (x_2, & \dots, & {}^{\sigma_n}A & (x_n, & x_{n+1})) & \dots) = & x_o \end{array}\right)$$

where $\varepsilon'_i \in \{0,1\}$ and $\varepsilon_j \in \{-1,0,1\}$, $i = 0,1,\dots,n+1$, $j = 1,2,\dots,n$. Then n-tuple $< \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n >$ can be regarded as the signature of Tw. Denote by p the number of non-zero elements of this signature and by d the number of its sign changes. Now analogously as in the proof of Theorem 6 consider p(p-1)+1 conditions and substitute suitable chosen elements a_1, $a_2, \dots, a_{n+1}, a_o$ from the quasigroup Q into the variables $x_1, x_2, \dots, x_{n+1}$, x_o.

Further, we get p(n+2) new conditions by requiring

$$t_m \neq a_k, \qquad k = 0,1,\dots,n+1, \quad m = 1,2,\dots,p,$$

where t_1, t_2, … ,t_p are terms from the first and from the fourth row in Tab. So together we have $\overline{p} := p(p-1) + 1 + p(n+2)$ conditions. Let $\overline{d}$ of them have the (q_k) form (see proof of Theorem 6). Then the wanted quasigroup $Q \in V_w$ will be of the form $Q := Q_1 \times \dots \times Q_{\overline{d}}$, where Q_1, … ,$Q_{\overline{d}}$ are suitable quasigroups chosen from V_w by using conditions (q_k),

$k = 1, 2, \ldots, \overline{d}$. Now we are going to define a permutation β on Q. Take a map β satisfying the above mentioned $\overline{p}$ conditions as well as the conditions $\beta a_k = a_k$, k=0,1, ... ,n+1. Then extend β to be a permutation on Q. Then Tw is not fulfilled in the quasigroup Q though w is fulfilled in it. Therefore w is not universal, which finishes the proof.

From Theorem 5 and Theorem 8 it follows that $\boxed{B1}$ is a necessary condition for a quasigroup identity $w \in R(\Sigma_A)$ to be universal. Similarly, from Theorem 6 and Theorem 7 we get that $\boxed{B2}$ is a necessary condition for a quasigroup identity $w \in R(\{A, {}^rA, {}^lA\})$ to be universal.

Hence we have

Theorem 9. *Let $w \in R(\{A, {}^rA, {}^lA)\}$ be a nontrivial identity. Then the diagram of the identity w satisfies the conditions $\boxed{B1}$ and $\boxed{B2}$ if and only if the identity w is universal.*

Recall that every quasigroup identity such that its length is at most 7 and neither it nor the identities obtained from it by any transformations contain a square, can be transformed into an identity of canonical type by using only transformations which do not change the universality of the identities [3]. This fact enlarges the class of quasigroup identities for which Theorem 8 gives a necessary and sufficient condition for the universality of the quasigroup identities.

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