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(1,1)-FORMS AND CONNECTIONS ON A VECTOR BUNDLE TM

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ABSTRACT. In this paper the subset of the space of the (1,1)-forms on TM on which there exist natural operators over id_{TM} transforming these forms into connections on TM and all these operators are determined.

INTRODUCTION

Let (x^i, x_1^i) be a local map on the tangent bundle $p : TM \to M$. A connection Γ on TM can be determined by its horizontal form h_Γ that is a (1,1)-form on TM such that $T_p \cdot h_\Gamma = T_p$, $h_\Gamma(Y) = 0$, $Y \in VTM$, where T_p denotes the tangent map of p and VTM is the vector bundle of all vertical vectors on TM. Γ_j^i are called the Christoffel's functions in coordinates $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma_j^i(x, x_1) dx^j \otimes \partial/\partial x_1^i$.

Let $J = dx^i \otimes \partial/\partial x_1^i$ be the canonical (1,1)-form on TM. Grifone [1], 1972, defined a connection on TM as a such (1,1)-form φ on TM that $\varphi J = -J$, $J\varphi = J$. Its local form is $\varphi = dx^i \otimes \partial/\partial x^i + (\varphi_j^i dx^j - dx_1^i) \otimes \partial/\partial x_1^i$. Evidently $\varphi \mapsto \frac{1}{2}(\varphi + id_{TTM}) = h_\Gamma$ is a bijection between the Grifone forms and the horizontal forms of connections.

Let $(1,1) \equiv C^\infty T^*TM \otimes TTM$ or CTM be the space of all smooth (1,1) forms or of all connections on TM. In this paper we determine the domain of natural operators of zero order from (1,1) into CTM and construct the list of these operators. We use the natural bundle theory, see for example [2], [3].

OPERATORS OF ZERO ORDER FROM (1,1) INTO CTM

In the natural bundle theory the effort to determine all natural operators of zero order from (1,1) into CTM over id_{TM} is equivalent to the one to determine all smooth natural transformations from the bundle $T^*TM \otimes TTM$ into CTM over Id_{TM}. As the natural bundles $T^*TM \otimes TTM$ and CTM are associated to the principal fibre bundle H^2M of the frames of second order on M with the type fibres $S_1 = (T^*TR^m \otimes TTR^m)_0$ and

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$S_2 = (CTR^m)_0$, in order to determine all above mentioned transformations we need to find all L^2_m- equivariant maps from S_1 into S_2, where L^2_m is the differential group of the second order.

Let $\overline{x}^i = f^i(x^j)$ be a local diffeomorphism $f : R^m \to R^m$, $f(0) = 0$. Then $TTf : TTR^m \to TTR^m$ is of the form.

$$\overline{x}^i = f^i(x^j),\ \overline{x}^i_1 = f^i_j x^j_1,\ d\overline{x}^i = f^i_j dx^j,\ d\overline{x}^i_1 = f^i_{jk} x^j_1 dx^k_1 + f^i_j dx^j_1$$

where $f^i_j := \frac{\partial f^i(0)}{\partial x^j}$, $f^i_{jk} = \frac{\partial f^i(0)}{\partial x^i \partial x^k}$. Then we find the action of L^2_m on S_1 and on S_2 as follows; on S_1:

(1)
$$\overline{x}^i_j = f^i_j x^j_1,$$
$$\overline{a}^i_j = f^i_s a^s_k \tilde{f}^k_j - f^i_t b^t_k \tilde{f}^k_s f^s_{uv} x^u_1 \tilde{f}^v_j\ ,\ \overline{b}^i_s = f^i_q b^q_j \tilde{f}^j_s\ ,\ f^i_s \tilde{f}^s_j = \delta^i_j\ ,$$
$$\overline{c}^i_j = (f^i_{sk} x^s_1 a^k_t + f^i_k c^k_t)\tilde{f}^t_j - (f^i_{sk} x^s_1 b^k_u + f^i_k h^k_u)\tilde{f}^u_r f^n_{pq} x^p_1 \tilde{f}^q_j\ ,$$
$$\overline{h}^i_j = (f^i_{ks} x^k_1 b^s_u + f^i_k h^k_u)\tilde{f}^u_j\ ,$$

on S_2 :

$$\overline{x}^i_j = f^i_j x^j_1,\ \overline{\Gamma}^i_j = f^i_{ks} x^k_1 \tilde{f}^s_j + f^i_k \Gamma^k_s \tilde{f}^s_j\ ,$$

where $(x^i_j, a^i_j, b^i_j, c^i_j, h^i_j)$ or (x^i_j, Γ^i_j) are coordinates on S_1 or on S_2.

Let $\alpha = (a^i_j dx^j + b^i_j dx^j_1) \otimes \partial/\partial x^i + (c^i_j dx^j + h^i_j dx^j_1) \otimes \partial/\partial x^i_1$ be a (1,1)-form on TM. Then $J\alpha J = b^i_j dx^j \otimes \partial/\partial x^i_1$ is a L^1_m-tensor. There are two cases

$$1. \qquad J\alpha J = 0, \quad \text{i.e.} \quad b^i_j = 0.$$

We have two forms $a := J\alpha = a^i_j dx^j \otimes \partial/\partial x^i_1,\ h := \alpha J = h^i_j dx^j \otimes \partial/\partial x^i_1$. Denote $\overline{S}_1$ the subspace of S_1 of such (1,1)-forms α for which $J\alpha J = 0$.

A map $\Phi : \overline{S}_1 \to S_2 : \overline{x}^i_1 = x^i_1,\ \Gamma^i_j = \Phi^i_j(x^q_1, a^q_p, c^q_p, h^q_p)$ is L^2_m-equivariant iff

(2)
$$f^i_{ks} x^k_1 \tilde{f}^s_j + f^i_k \Phi^k_s(x^q_1, a^q_p, c^q_p, h^q_p) = \Phi^i_j(\overline{x}^q_1, \overline{a}^q_p, \overline{c}^q_p, \overline{h}^q_p),$$

where $\overline{x}^q_1, \overline{a}^q_p, \overline{c}^q_p, \overline{h}^q_p$ are given by the equations (1). The equations (2) according to the subgroup of homotheties $f = (k\delta^i_j = f^i_j,\ f^i_{jk} = 0), k \neq 0$, are of the form

$$\Phi^i_j(x^q_1, a^q_p, c^q_p, h^q_p) = \Phi^i_j(kx^q_1, a^q_p, c^q_p, h^q_p).$$

Therefore Φ^i_j are independent of (x^q_1). Then the Ker π^2_1-equivariancy, where $\pi^2_1 : L^2_m \to L^1_m$ is the subjet projection, gives

(3)
$$f^i_{kj} x^k_1 + \Phi^i_j(a^q_p, c^q_p, h^q_p) = \Phi^i_j(a^q_p, c^q_p + f^q_{uk} x^u_1 a^k_p - h^q_s f^s_{up} x^u_1, h^q_p).$$

Remark 1. If α is semibasic with value in VTM, i.e. if $a = 0, h = 0$ then (2) is not true. Then there is not a natural operator of zero order from the space of semibasic (1,1)-forms with values in VTM into the space CTM. This coincides with the result of Janyška [2] by which a (1,1)-form on M does not determine any linear connection.

It follows from (2) that $\frac{\partial \Phi^i}{\partial c^q_p}$ are constant on the Ker π^2_1-orbits the local equations of which are as follows:

(4) $$\overline{x}^i_1 = x^i_1,\ \overline{a}^i_j = a^i_j,\ \overline{c}^i_j = c^i_j + F^i_k a^k_j - h^i_s F^s_j,\ \overline{h}^i_j = h^i_j,\ F^i_k := f^i_{uk} x^u_1.$$

There are two cases:

1a) The map $\kappa : (y^s) \mapsto (\delta^v_s a^k_u - h^v_s \delta^k_u) y^s_k$, i.e. $y \mapsto ya - hy$, on $(R^m)^* \otimes R^m$ is regular. Then it follows from (4) that $\frac{\partial \Phi^i_j}{\partial c^q_p}$ are independent of c^p_q. Then

$$\Phi^i_j = A^{iu}_{jv}(a^p_q, h^q_p) c^v_u + B^i_j(a^q_p, h^q_p)$$

and the equation of the Ker π^2_1-equivariancy of Φ^i_j is of the form

$$F^i_j = A^{iu}_{jv}(\delta^v_s a^k_u - h^v_s \delta^k_u) F^s_k\ ,$$

i.e. (A^{iu}_{jv}) are the coordinates of the map κ^{-1}. It is easy to see that Φ is L^2_m-equivariant iff $x^i_j = B^i_j(a^q_p, h^q_p)$ is a L^1_m-equivariant map from $\mathcal{R}^{m*} \otimes \mathcal{R}^m \times \mathcal{R}^{m*} \otimes \mathcal{R}^m$ into $\mathcal{R}^{m*} \otimes \mathcal{R}^m$.

Consider a map γ on $T^*M \otimes T(TM)$ over id_{TM} defined by the rule $\gamma : z \mapsto za - \alpha z$, $\gamma(z^i_j dx^j \otimes \partial/\partial x^i + \eta^i_j dx^j \otimes \partial/\partial x^i_1) = (z^i_s a^s_j - a^i_s z^s_j) dx^j \otimes \partial/\partial x^i + (-c^i_s z^s_j + \eta^i_s a^s_j - h^i_s \eta^s_j) dx^j \otimes \partial/\partial x^i_1$. When κ is regular there is a unique horizontal form $z_0 = dx^i \otimes \partial/\partial x^i + \eta^i_j dx^j \otimes \partial/\partial x^i_1$ such that $\gamma(z_0) = 0$ in coordinates $\eta^i_j = A^{iu}_{jv} c^v_u$. Denote by Γ_α the connection determined by z_0. We have proved the following proposition:

Proposition 1. *All natural operators Φ from the space of (1,1)-forms α on TM such that $J\alpha J = 0$ and the map κ_α is regular into the space of connections on TM over id_{TM} are of the form*

$$\Phi = \Gamma_\alpha + B(a, h)$$

where B is a natural operator of zero order from $C^\infty[(T^*M \otimes VTM) x_{TM} (T^*M \otimes VTM)]$ into $C^\infty(T^*M \otimes VTM)$ over id_{TM}.

1b) Let κ be singular. The equations $\overline{c}^i_j = c^i_j + (\delta^i_s a^k_j - h^i_s \delta^k_j) F^s_k$ of the Ker π^2_1-orbit can be written in the form

(5) $$\begin{aligned} \overline{c}^i_j &= c^i_j + (\delta^i_1 a^1_j - h^i_1 \delta^1_j) F^1_1 \cdots + (\delta^i_m a^1_j - h^i_m \delta^1_j) F^m_1 + (\delta^i_1 a^2_j - h^i_1 \delta^2_j) F^1_2 + \ldots \\ &\cdots + (\delta^i_m a^2_j - h^i_m \delta^2_j) F^m_2 + \cdots + (\delta^i_m a^m_j - h^i_m \delta^m_j) F^m_m\ , \quad m = \dim M \end{aligned}$$

or shortly $\overline{C} = C + AF$.

Let the rank of κ be $h = sm + k,\ k < m, s < m$. Suppose that the first h columns of the matrix A are independent. Then the equations (5) are of the form

(6) $$\overline{C} = C + A_1 F^1 + \cdots + A_m F^h,$$

where $A_1, \dots, A_m$ denote the first h columns of the matrix A,

$$\begin{aligned} F^I &= F^I + t^I_{m+1} F^{m+1} + \dots + t^I_{m^2-h} F^{m^2-h}, \quad I = 1, \dots, h \\ F^1 &:= F^1_1,\ F^2 := F^2_1, \dots, F^{m^2} := F^m_m,\ C^1 := C^1_1, \dots, C^{m^2} := C^m_m \end{aligned}$$

and t^I_u are the coefficients of the linear combination of the column A_u according to the columns $A_1, \dots, A_h$. Without loss of generality we suppose that the first h rows of the matrix A are independent. Using the first h equations of (6) we get

$$F^\xi = \tilde{A}^\xi_\eta (\overline{C}^\eta - C^\eta),\ \eta,\ \xi = 1, \dots, h.$$

Then the last $m^2 - h$ equations of (6) give

$$\overline{C}^\beta = C^\beta + A^\beta_\xi \tilde{A}^\xi_\eta (\overline{C}^\eta - C^\eta), \quad \beta = h+1, \dots m^2.$$

Then the equations of the Ker π^2_1-orbits are of the form

$$\overline{C}^\beta - A^\beta_\xi \tilde{A}^\xi_\eta C^\eta = C^\beta - A^\beta_\xi \tilde{A}^\xi_\eta C^\eta, \quad \overline{a}^i_j = a^i_j,\ \overline{h}^i_j = h^i_j.$$

This means that

$$\frac{\partial \Phi^i_j}{\partial c^q_p} = \psi^{ip}_{jq}(a^u_v, h^u_v, C^\beta - A^\beta_\xi \tilde{A}^\xi_\eta C^\eta).$$

For $c^q_p = C^\beta$ we have

$$\Phi^i_j = \int \psi^i_{j\beta}(a^u_v, h^u_v, C^\beta - A^\beta_\xi A^\xi_\eta C^\eta) dC^\beta.$$

Then $\Phi^i_j = \varphi^i_{j\beta}(a,h) C^\beta + \varphi^i_j(a,h,C^\eta)$ if $\psi^i_{j\beta}$ are independent of $u^\beta = C^\beta - A^\beta_\xi \tilde{A}^\xi_\eta C^\eta$ and
$\Phi^i_j = G^i_j(a,h,u)$ if $\psi^i_{j\beta}$ are dependent of u^β. As $\frac{\partial \Phi^i_j}{\partial C^\xi}$ are dependent of a, h, u only, there is $\varphi^i_j = \varphi^i_{j\xi}(a,h) C^\xi$. We conclude $\Phi^i_j = \varphi^{iq}_{jp}(a,h) c^p_q + G^i_j(a,h,u)$.

Now, the Ker π^2_1-equivariancy leads to the equations

$$F^i_j = \varphi^{iu}_{jv}(a,h)[\delta^v_s a^k_u - h^v_s \delta^u_k] F^s_k$$

from which it follows that κ must be regular. We have proved the following:

Proposition 2. *If $J\alpha J = 0$ and κ is singular then there is no connection on TM which is determined by the natural operator of zero order from the space of (1,1)-forms α that $J\alpha J = 0$ and κ is singular into the space of connections on TM.*

Example. In the case of a Grifone form $a^i_j = \delta^i_j,\ h^i_j = -\delta^1_j,\ J\alpha J = 0,\ \kappa = 2id$ and so $\Gamma^i_j = \frac{1}{2}c^i_j$ are the Christoffel's functions of the connection Γ_α.

2) Let $J\alpha J \neq 0$. There are two cases:

2a) Let $J\alpha J$ be regular, $\det(b^i_j) \neq 0$. Consider $J\alpha = (a^i_j dx^j + b^i_j dx^j_1) \otimes \partial/\partial x^i_1$. Denote by Γ_α the connection the horizontal distribution of which is spanned on vectors $X,\ J\alpha(X) = 0$, i.e. $\Gamma^i_j = -\tilde{b}^i_k a^k_j$ are the Christoffel's functions of Γ_α. Then any conection Γ on TM is of the form $\Gamma_\alpha + \varphi$, where $\varphi = \varphi^i_j dx^j \otimes \partial/\partial x^i_1$ is a semibasic (1,1)-form on TM with values in VTM. Now we find all natural operators φ of zero order form $C^\infty(T^*TM \otimes TTM)$ into $C^\infty(T^*M \otimes VTM)$, i.e. we find all L^2_m-equivariant maps from $(T^*TR^m \otimes T(TR^m))_0 = S_1$ into $(T^*R^m \otimes VTR^m)_0 = S_3$. In coordinates, we find all functions $\varphi^i_j(a,b,c,h)$ which satisfy:

$$f^i_k \varphi^k_s(a,b,c,d)\tilde{f}^s_j = \varphi^i_j(\overline{a},\overline{b},\overline{c},\overline{d}), \tag{7}$$

where $\overline{a},\overline{b},\overline{c},\overline{d}$ are determined by (1). In the case of the Ker π^2_1-equivariancy the equations (7) are of the form $\varphi^i_j(a,b,c,h) = \varphi^i_j(\overline{a},\overline{b},\overline{c},\overline{h})$. Therefore functions φ^i_j are constant on the Ker π^2_1-orbits, the equations of which are

$$\begin{aligned} &\overline{a}^i_j = a^i_j - b^i_k F^k_j,\ \overline{c}^i_j = F^i_k a^k_j + c^i_j - (F^i_k b^k_u + h^i_u)F^u_j \\ &\overline{h}^i_j = F^i_s b^s_j + h^i_j,\ \overline{b}^i_j = b^i_j. \end{aligned} \tag{8}$$

Calculating $F^u_v = \tilde{b}^u_k(a^k_v - \overline{a}^k_v)$ we get the equations

$$\begin{aligned} &b^s_k \overline{h}^k_p + \overline{a}^s_k b^k_p = b^s_k h^k_p + a^s_k b^k_p \\ &\overline{c}^q_p - \overline{h}^q_s \tilde{b}^s_v \overline{a}^v_p = c^q_p - h^q_s \tilde{b}^s_k a^k_p \\ &\overline{b}^i_j = b^i_j \end{aligned}$$

of the Ker π^2_1-orbits. This means that the functions φ^i_j are constant on the Ker π^2_1-orbits iff they are of the form

$$\varphi^i_j = \varphi^i_j(b^q_p, b^q_t h^t_p + a^q_t b^t_p,\ c^q_p - h^q_s \tilde{b}^s_k a^k_p).$$

It is easy to see that these functions are L^2_m-equivariant iff they are L^1_m-equivariant. Calculating

$$\begin{aligned} &J\alpha^2 J = (a^i_s b^s_j + b^i_s h^s_j)dx^j \otimes \partial/\partial x^i_1, \\ &\alpha - \alpha b^{-1} T_p \alpha = (c^i_j - h^i_s \tilde{b}^s_k a^k_j)dx^j \otimes \partial/\partial x^i_1 \end{aligned}$$

we can conclude:

Lemma 1. *All natural operators of zero order from the space of the (1,1)-forms on TM such that $J\alpha J$ is regular into the space of the semibasic (1,1)-forms with values in VTM are of the form $\varphi(J\alpha J, J\alpha^2 J, \alpha - \alpha\beta^{-1}T\pi\alpha)$ where φ is an operator of zero order from $C^\infty((T^*M \otimes VTM)x_{TM}T^*M \otimes VTMx_{TM}T^*M \otimes VTM))$ into $C^\infty(T^*M \otimes VTM)$.*

It immediately gives:

Proposition 3. *All natural operators of zero order from the space of the (1,1)-forms on TM such that $J\alpha J$ is regular into the space of connections on TM are of the form*

$$\Gamma_\alpha + \varphi(J\alpha J, J\alpha^2 J, \alpha - \alpha\beta^{-1}T\pi\varphi)$$

where φ is the operator described in Lemma 1.

2b) Let $J\alpha J$ be singular, $\det(b^i_j) = 0$. We find all L^2_m-equivariant maps $\Gamma^i_j = \Phi^i_j(x^k_1, a^q_p, b^q_p, c^q_p, h^q_p)$ from S_1 into S_2. It follows from the homothety subgroup equivariancy that Φ^i_j are independent of x^k_1. The condition of the subgroup Ker π^2_1-equivariancy is of the form

$$F^i_j + \Phi^i_j(a^q_p, b^q_p, c^q_p, h^q_p) = \Phi^i_j(\overline{a}^q_p, \overline{b}^q_p, \overline{c}^q_p, \overline{h}^q_p)$$

where $\overline{a}^q_p, \overline{b}^q_p, \overline{c}^q_p, \overline{h}^q_p$ are determined by the equations (8) which are the equations of Ker π^2_1-orbits on S_1. It is easy to see that the functions $\frac{\partial \Phi^i_j}{\partial c^q_p}$ are constant on the Ker π^2_1-orbits. The equations (8) can be arranged in the form

$$\begin{aligned}
&\overline{a}^q_p = a^q_p - b^q_r F^r_p \\
&\overline{b}^q_p = b^q_p,\ \overline{a}^q_k \overline{b}^k_p + \overline{b}^q_s \overline{h}^s_p = a^q_k b^k_p + b^q_s h^s_p :\equiv H^q_p \\
&\overline{b}^q_s(\overline{c}^s_t \overline{b}^t_p + \overline{h}^s_t \overline{h}^t_p) + \overline{a}^q_t \overline{H}^t_p = b^q_s(c^s_t b^t_p + h^s_t h^t_p) + a^q_t H^t_p
\end{aligned}$$

By the analogous procedure as in the case 1b) based on the equations $\overline{a}^q_p = a^q_p - b^q_r F^r_p$ we can deduce

Proposition 4. *If a (1,1)-form is such that $J\alpha J \neq 0$ but singular then there is no connection on TM which is determined by the natural operator of zero order from the space of such (1,1)-forms α on TM that $J\alpha J \neq 0$ but singular, into CTM.*

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ON A CONGRUENCE LATTICE REPRESENTATIONS

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Abstract. In [2] the following problem is formulated: Is it true that for every n-unary algebra $\mathcal{A}$ (n finite), there exists a 2-unary algebra $\mathcal{B}$ with $Con\mathcal{A} \cong Con\mathcal{B}$? This paper contains contributions to the solution of this problem. Certain results concerning the mentioned problem can be found in [3]. In this paper some results of [3] are generalized. Moreover, results for the lattice of subuniverses and the authomorphism group are presented.

Representations of congruence lattice have been considered by many authors. The survay of basic results on this topic may be found in [1], [2] and [4].

In this paper the set of all positive integers will be denoted by N. Further, $Con\mathcal{A}$, $Sub\mathcal{A}$ and $Aut\mathcal{A}$ denote the congruence lattice, the lattice of subuniverses and the automorphism group of an algebra $\mathcal{A}$, respectively.

Let $(A, f_1, ..., f_n)$ be a unary algebra and let a be any (but fixed) element of $\mathcal{A}$. We define unary operations f, g on the set $B = A \times \{1, 2, ..., n+3\}$ as follows:

(1) $f(x,k) = (x, k+1)$ for $k \in \{1,2\}$, $f(x,3) = (x,1)$,
(2) $f(x,k) = (f_{k-3}(x), k)$ for $k \in \{4, ..., n+3\}$,
(3) $g(x,1) = (a,1)$, $g(x, n+3) = (x,1)$,
(4) $g(a,2) = (a,2)$, $g(a,k) = (a, k+1)$ for $k \in \{3, ..., n+2\}$,
(5) $g(x,k) = (x, k+1)$ for $x \neq a$, $k \in \{2, ..., n+2\}$

(see Fig.1).

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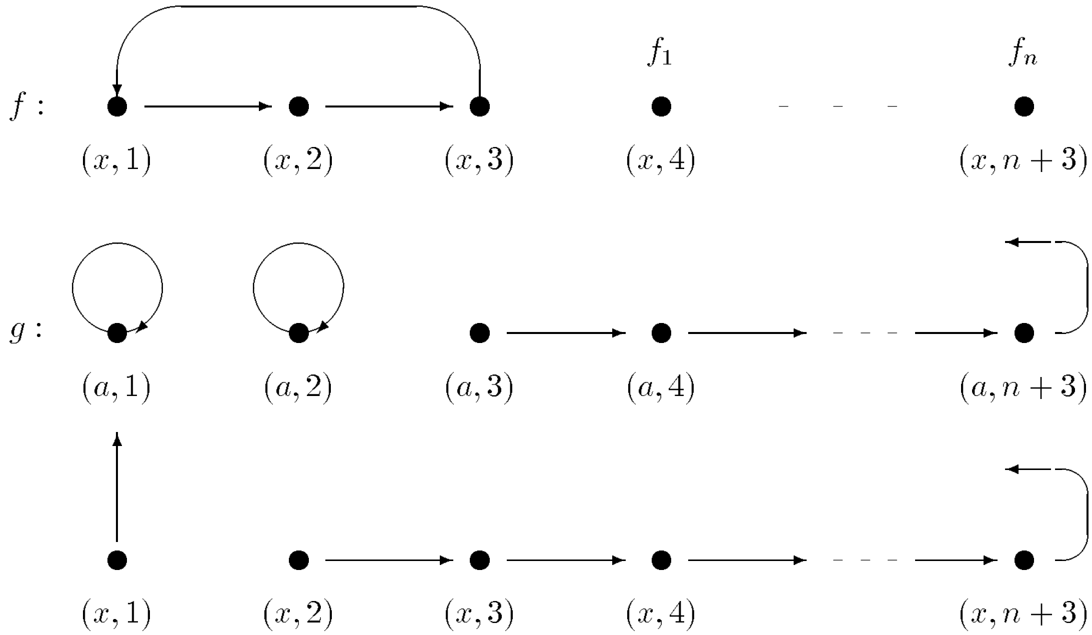

Fig. 1

Lemma 1. *Let* $(A, f_1, ..., f_n)$ *be a unary algebra,* $a \in A$ *and* (B, f, g) *be the unary algebra defined above, i.e., the operations* f, g *satisfy* $(1)-(5)$. *For any congruence relation* Φ *of the algebra* (B, f, g) *the following holds:*

(i) $(x,p)\,\Phi\,(y,p)$ *iff* $(x,q)\,\Phi\,(y,q)$ *for any* $p, q \in \{1, ..., n+3\}$,

(ii) $(a,p)\,\Phi\,(y,q)$ *and* $(a,p) \neq (y,q)$ *imply* $(a,i)\,\Phi\,(y,j)$ *for all* $i, j \in \{1, ..., n+3\}$,

(iii) $(x,p)\,\Phi\,(y,q)$ *and* $p \neq q$ *imply* $(a,1)\,\Phi\,(y,1)$.

Proof. (i). Let $(x,p)\Phi(y,p)$. Then

(a) $$f^k(x,p)\Phi f^k(y,p) \qquad \text{and}$$

(b) $$g^k(x,p)\Phi g^k(y,p) \qquad \text{for any } k \in N.$$

If $p \in \{1, 2, 3\}$ then from (a) we get

(c) $$(x,s)\Phi(y,s) \quad \text{for every} \quad s \in \{1, 2, 3\}.$$

Using (b), from $(x,3)\Phi(y,3)$ we get (c) for all $s \in \{4, ..., n+3\}$. If $p \in \{4, ..., n+3\}$ then from (b) we get $(x,1)\Phi(y,1)$ and then we proceed in the same way as in the previous case.

(ii). The assumption $(a,p)\Phi(y,q)$ implies

(d) $$f^k(a,p)\Phi f^k(y,q) \quad \text{and}$$

(e) $$g^k(a,p)\Phi g^k(y,q) \quad \text{for any} \quad k \in N.$$

Now we consider the following cases.

a) Let $y = a$ (i.e., $p \neq q$), $p, q \in \{1,2,3\}$. Then we get $(a,2)\Phi(a,3)$ from (d); thus, $g^k(a,2)\Phi g^k(a,3)$, $k \in N$, and, consequently, $(a,2)\Phi(a,s)$ for all $s = 3, 4, ..., n+3, 1, 2$.

b) Let $y = a$ and $p \leq 3 < q$. If $p = 2$ then from (e) we get $(a,2)\Phi(a,1)$, i.e., the case a). If $p = 1$ then from $f(a,1)\Phi f(a,q)$ we get $(a,2)\Phi(f_{q-3}(a),q)$, $g^k(a,2)\Phi g^k(f_{q-3}(a),q)$ and $(a,2)\Phi(a,1)$, respectively, i.e., the case a) again. For $p = 3$ we use $f^2(a,3)\Phi f^2(a,q)$, and then we proceed as in the previous case.

c) Let $y = a$ and $p, q \in \{4, ..., n+3\}$. Then we get the case b) using (e) since Φ is symmetric.

d) Let $y \neq a$, $p, q \in \{1,2,3\}$. Then (d) implies $(a,2)\Phi(y,s)$ for some $s \in \{1,2,3\}$. Further, from $g^k(a,2)\Phi g^k(y,s)$, $k \in N$, we obtain $(a,2)\Phi(a,1)$ and then by a)

(f) $$(a,i)\Phi(a,j) \quad \text{for all} \quad i, j \in \{1, ..., n+3\}.$$

The assumption $(a,p)\Phi(y,q)$ together with (f) and (i) implies

(g) $$(a,i)\Phi(y,j) \quad \text{for} \quad i, j \in \{1, ..., n+3\}.$$

e) Let $y \neq a$, $p \leq 3 < q$. From (d) we get $(a,2)\Phi(z,q)$ for some $z \in A$. Thus, we obtain $g^k(a,2)\Phi g^k(z,q)$ for each $k \in N$ and so $(a,2)\Phi(a,1)$, and then we proceed as in the case d).

f) Let $y \neq a$ and $q \leq 3 < p$. We obtain $(z,p)\Phi(y,2)$ from (d) for some element $z \in A$. Thus, $g^k(z,p)\Phi g^k(y,2)$ for each $k \in N$ and, consequently, $(a,1)\Phi(y,s)$ for some $s \in \{4, ..., n+3\}$, hence we have the case e).

g) Let $y \neq a$ and $p, q \in \{4, ..., n+3\}$. Using (e) we obtain $(a,s)\Phi(y,1)$ or $(a,1)\Phi(y,s)$ for some s, which are again the previous cases.

(iii). Let $(x,p)\Phi(y,q)$ and $p \neq q$. For $y = a$ the statement is evident; for $x = a$ we use (ii).

Now let $x \neq a$, $y \neq a$. For $p = 1$ the assumption and the operation g imply $(a,1)\Phi(y,s)$, where $s = q+1$ or $s = 1$. Then by (ii) we get

(h) $$(a,1)\Phi(y,1).$$

If $1 < p < q$ then using the operation g we get $(x,s)\Phi(a,1)$ for some s. We obtain (h) by using (ii) and the hypothesis. If $q < p$ then analogously we obtain (h) by using (ii).

Theorem 1. *Let* $(A, f_1, ..., f_n)$ *be a unary algebra. Let there exist* $a \in A$ *such that*

$$(6) \qquad a\Theta x \text{ and } a \neq x \quad \text{imply} \quad a\Theta f_i(x),\ i = 1, ..., n$$

for any congruence relation $\Theta \in Con\mathcal{A}$ *and any element* $x \in A$.
Then there exists a unary algebra (D, f, g) *such that* $Con\mathcal{A} \cong Con\mathcal{D}$.

Remark. An algebra $(A, f_1, ..., f_n)$ satisfies the assumptions of Theorem 1 if one of the following conditions holds:

(i) a is a fixed point of every operation f_i, $i = 1, 2, ..., n$.

(ii) If $x \neq a$ then $\Theta(x, a) = \nabla_A$
(∇_Ameans the largest element of$Con\mathcal{A}$).

(iii) If $i \in \{1, ..., n\}$ and $f_i(a) \neq a$ then $\Theta(a, f_i(a)) \leq \Theta(a, x)$
for every element $x \in A$, $x \neq a$.

Proof. Let (B, f, g) be an algebra whose operations are defined by $(1) - (5)$. We define a mapping $F : Con\mathcal{B} \to Con\mathcal{A}$ as follows

$$(7) \qquad xF(\Phi)y \qquad \text{iff} \qquad (x, 1)\Phi(y, 1)$$

for any congruence relation $\Phi \in Con\mathcal{B}$.

a) We prove that the mapping F is well-defined, i.e., that $F(\Phi) \in Con\mathcal{A}$. Obviously, $F(\Phi)$ is an equivalence on A. Let $xF(\Phi)y$. Then by (7) and (i) we get $(x, p)\Phi(y, p)$ for all $p \in \{4, ..., n + 3\}$. So, we get
$f(x, p)\Phi f(y, p)$, $(f_{p-3}(x), p)\Phi(f_{p-3}(y), p)$, $(f_{p-3}(x), 1)\Phi(f_{p-3}(y), 1)$ (by (i))
and $f_{p-3}(x)F(\Phi)f_{p-3}(y)$, respectively, for $p = 4, ..., n + 3$.
Hence, $F(\Phi) \in Con\mathcal{A}$.

b) Using Lemma 1 we have
$\Phi_1 \leq \Phi_2$ iff $F(\Phi_1) \leq F(\Phi_2)$
for any congruences $\Phi_1, \Phi_2 \in Con\mathcal{B}$.

Now we consider two cases.

c1) Let a be the fixed point of all operations $f_1, ..., f_n$, i.e., $f(a, p) = (a, p)$ holds for all $p = 4, ..., n + 3$. We define the relation Ω on B as follows:

$$(8) \qquad (x, p)\Omega(y, q) \quad \text{iff} \quad (x, p) = (y, q) \quad \text{or} \quad x = y = a.$$

Obviously, Ω is a congruence relation of the algebra (B, f, g).

Now we prove that F is a mapping of the interval $[\Omega, \nabla_B]$ of the lattice $Con\mathcal{B}$ onto $Con\mathcal{A}$. Let $\Theta \in Con\mathcal{A}$. We define a relation Φ on B by the rule:
$(x, p)\Phi(y, q)$ iff $x\Theta y$ and $(y\Theta a$ or $p = q)$.
It is easy to show that Φ is an equivalence relation on B.

Let $x\Theta y\Theta a$. Then by (6) we have
$f_i(x)\Theta f_j(y)\Theta a$ for all $i, j \in \{1, 2, ..., n\}$,
and, consequently, $f(x, p)\Phi f(y, q)$. Obviously, we also get $g(x, p)\Phi g(y, q)$.

Let $p=q$ and $x\Theta y$. Then, $f_i(x)\Theta f_i(y)$ for all $i\in\{1,...,n\}$ which implies

(k) $$f(x,p)\Phi f(y,p) \qquad \text{for all } p\in\{4,...,n+3\}.$$

Obviously, (k) also holds for $p\in\{1,2,3\}$. Further, it is clear that $g(x,p)\Phi g(y,p)$. Thus, $\Phi\in Con\mathcal{B}$ and evidently $\Phi\geq\Omega$, $F(\Phi)=\Theta$.

Assume that $\Phi_1,\Phi_2\in[\Omega,\nabla_B]$, $\Phi_1\neq\Phi_2$. Then there are $(x,p),(y,q)\in B$ such that $(x,p)\Phi_1(y,q)$ but $(x,p)\,(B^2-\Phi_2)\,(y,q)$. We can suppose $x\neq a\neq y$. If $p=q$ then $xF(\Phi_1)y$ but $x\,(A^2-F(\Phi_2))\,y$ by Lemma 1. If $p\neq q$ then
$(x,1)\Phi_1(a,1)\Phi_1(y,1)$
by Lemma1 and hence $xF(\Phi_1)aF(\Phi_1)y$. If $xF(\Phi_2)aF(\Phi_2)y$ i.e., $(x,1)\Phi_2(a,1)\Phi_2(y,1)$, then $(x,p)\Phi_2(y,q)$, in view of Lemma 1(ii), since $x\neq a\neq y$, a cotradiction. Thus, the restriction of the mapping F to the interval $[\Omega,\nabla_B]$ is an isomorphism, and, consequently, $Con\mathcal{B}/\Omega\cong Con\mathcal{A}$. Hence, the statement holds for the algebra $(D,f,g)=(B/\Omega,f,g)$.

c2) Let f_i, $1\leq i\leq n$, be such an operation that $f_i(a)=b\neq a$. Now we prove that we can take $(D,f,g)=(B,f,g)$. If $a\Theta x$ for some element $x\in A$, $x\neq a$, then by (6) $a\Theta f_i(x)$, thus, $a\Theta f_i(a)$, i.e., $a\Theta b$. Hence, $\Theta(a,b)\leq\Theta(a,x)$ for every element $x\in A$, $x\neq a$.

If $(a,1)\Phi(b,1)$, $\Phi\in Con\mathcal{B}$, then using Lemma 1 (ii) we get $(a,p)\Phi(b,q)$ for all $p,q\in\{1,...,n+3\}$. Conversely, from $(a,p)\Phi(a,q)$, $p\neq q$ we have $(a,1)\Phi(a,i+3)$ using Lemma 1 (ii). Hence we get $f(a,1)\Phi f(a,i+3)$ and then $(a,2)\Phi(f_i(a),i+3)$. Then by Lemma 1 (iii) we obtain $(a,1)\Phi(f_i(a),1)$, i.e., $(a,1)\Phi(b,1)$. Using Lemma 1 again we conclude that F is an injection.

To prove that F is onto, suppose that $\Theta\in Con\mathcal{A}$. If $a\Theta b$ we define a relation Φ on B as follows:
$(x,p)\Phi(y,q)$ iff $x\Theta y$ and ($y\Theta a$ or $p=q$).
In the other case, the relation Φ is defined on B by the rule:
$(x,p)\Phi(y,q)$ iff $p=q$ and $x\Theta y$.
One can prove in a similar way as in c1) that in both cases $\Phi\in Con\mathcal{B}$. Obviously, $F(\Phi)=\Theta$ by Lemma 1. Hence, F is the isomorphism between $Con\mathcal{B}$ and $Con\mathcal{A}$.

Lemma 2. *Let $(A,f_1,...,f_n)$ be a unary algebra, $a\in A$ and (B,f,g) be the unary algebra whose operations are defined by (1) - (5). If ϕ is an automorphism of the algebra (B,f,g), then*

(m) $$\phi(a,i)=(a,i) \qquad \textit{for all} \qquad i\in\{1,...,n+3\},$$

(n) $$\phi(x,i)=(y,j) \quad \textit{yields} \quad i=j \textit{ and } \phi(x,s)=(y,s)$$

for all $s\in\{1,...,n+3\}$.

Proof. (m). If $\phi\in Aut\mathcal{B}$, then there exist elements (x,i), (y,j) such that $\phi(x,i)=(y,j)$, $(x,i)\neq(a,2)$ and $(y,j)\neq(a,2)$. Then

(p) $$\phi g^k(x,i)=g^k(y,j) \qquad \text{for any} \quad k\in N,$$

which implies $\phi(a,1)=(a,1)$. Thus $\phi f(a,1)=f(a,1)$, i.e., $\phi(a,2)=(a,2)$ and similarly $\phi(a,3)=(a,3)$. From $\phi g^k(a,3)=g^k(a,3)$ we obtain $\phi(a,i)=(a,i)$ for all $i\in\{4,...,n+3\}$.

(n). Let $\phi(x,i)=(y,j)$ and $i\neq j$. Then $x\neq a$ and $y\neq a$ by (m). Then (p) is valid and from (p) we conclude the existence of positive integer p such that

$$\phi(x,p)=(a,1) \quad \text{or} \quad \phi(a,1)=(y,p)$$

which contradicts (m).

If $\phi(x,i)=(y,i)$, $x\neq a$ then $y\neq a$ and $\phi f^k(x,i)=f^k(y,i)$ and $\phi g^k(x,i)=g^k(y,i)$ imply $\phi(x,s)=(y,s)$ for every $s\in\{1,...,n+3\}$.

Theorem 2. *Let $(A,f_1,...,f_n)$ be a unary algebra, and $a\in A$ and (B,f,g) be the unary algebra whose operations are defined by (1) - (5). Then*

1) the group $Aut\mathcal{B}$ is isomorphic to the subgroup of those automorphisms $\phi\in Aut\mathcal{A}$ having a as a fixed point and

2) the lattice $Sub\mathcal{B}$ is isomorphic to the sublattice L of all subuniverses of the algebra $\mathcal{A}$ containing the element a.

Proof. 1) Define a mapping $F: Aut\mathcal{B}\to Aut\mathcal{A}$ as follows:

$$F(\phi)(x)=y \quad \text{iff} \quad \phi(x,1)=(y,1)$$

for any $\phi\in Aut\mathcal{B}$ and any $x,y\in A$. Since ϕ is a bijection on B such that (n) holds, we conclude that $F(\phi)$ is a bijection taking A onto A. By (m) we get $F(\phi)(a)=a$. If f_i is an operation on A and $F(\phi)(x)=y$, then $\phi(x,i+3)=(y,i+3)$ by (n), which implies

$$\phi f(x,i+3)=f(y,i+3), \qquad \phi(f_i(x),i+3)=(f_i(y),i+3),$$
$$\phi(f_i(x),1)=(f_i(y),1), \qquad F(\phi)f_i(x)=f_i(y)=f_iF(\phi)(x),$$

thus $F(\phi)\in Aut\mathcal{A}$.

If $\phi_1\neq\phi_2$, then $F(\phi_1)\neq F(\phi_2)$ by (n). It is easy to prove that $F(\phi_1\circ\phi_2)=F(\phi_1)\circ F(\phi_2)$ for any $\phi_1,\phi_2\in Aut\mathcal{B}$.

If $\psi\in Aut\mathcal{A}$ such that $\psi(a)=a$ then one can easily verify that the mapping $\phi: B\to B$ satisfying the condition

$$\phi(x,i)=(y,i) \quad \text{for all} \quad i\in\{1,...,n+3\} \quad \text{iff} \quad \psi(x)=y$$

is an automorphism of the algebra (B,f,g), and obviously $F(\phi)=\psi$.

2) Let $F: Sub\mathcal{B}\to Sub\mathcal{A}$ be a mapping defined as follows:

$$F(S)=\{s\in A; (s,1)\in S\} \quad \text{for any subuniverse } S\in Sub\mathcal{B}.$$

a) We prove that F is well-defined, i.e., $F(S)\in Sub\mathcal{A}$. Take $s\in F(S)$. From $(s,1)\in S$ we get $f^k(s,1)\in S$ and $g^k(s,1)\in S$, $k\in N$; therefore,

$$(s,i)\in S \quad \text{and} \quad (a,i)\in S$$

for all $i\in\{1,...,n+3\}$. From $(s,j+3)\in S$, $1\leq j\leq n$ we get $f(s,j+3)=(f_j(s),j+3)\in S$; thus, $(f_j(s),1)\in S$, i.e., $f_j(s)\in F(S)$.

b) If $\mathcal{A}_1$ is a subalgebra of the algebra $\mathcal{A}$ such that $a\in A_1$, then

$$B_1=\{(s,i); s\in A_1, \quad i\in\{1,...,n+3\}\}$$

is obviously a subuniversum of the algebra $\mathcal{B}$ and $F(\mathcal{B}_1)=\mathcal{A}_1$.

c) Clearly, F is one-to-one and

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \quad \text{iff} \quad F(\mathcal{S}_1) \subseteq F(\mathcal{S}_2)$$

for any subalgebras $\mathcal{S}_1, \mathcal{S}_2 \in Sub\mathcal{B}$, which completes the proof.

Remark 1. *If $a \in A$ is a fixed point of the operations $f_1, ..., f_n$ then in Theorem 2 one can replace the algebra (B, f, g) by the factor algebra $(B/\Omega, f, g)$ where Ω means the congruence relation given by (8).*

Corollary 1. *Let $(A, f_1, ..., f_n)$ be a unary algebra. If there exists an element $a \in A$ such that (6) holds and*

(i) $\phi(a) = a$ *for every automorphism* $\phi \in Aut\mathcal{A}$ *and*

(ii) *every subalgebra of the algebra $\mathcal{A}$ contains the element a*

then there exists a unary algebra (D, f, g) such that

$$Con\mathcal{A} \cong Con\mathcal{D}, \quad Aut\mathcal{A} \cong Aut\mathcal{D} \quad \text{and} \quad Sub\mathcal{A} \cong Sub\mathcal{D}.$$

Although the propositions are stated for algebras with finitely many operations, it is also true for algebras with countably many operations.

Theorem 3. *Let $(A, f_1, ..., f_n, ...)$ be a unary algebra and $a \in A$ an element such that*

(9) $$a\Theta x \quad \text{and} \quad a \neq x \quad \textit{imply} \quad a\Theta f_n(x)$$

for any congruence $\Theta \in Con\mathcal{A}$, any element $x \in A$ and any operation f_n, $n \in N$. Then there exists an algebra (D, f, g) such that

(j) $$Con\mathcal{D} \cong Con\mathcal{A},$$

(jj) $$Aut\mathcal{D} \cong Aut\mathcal{A}'$$

where $Aut\mathcal{A}'$ is a subgroup of all automorphisms $\phi \in Aut\mathcal{A}$ having the element a as a fixed point.

(jjj) $$Sub\mathcal{D} \cong Sub\mathcal{A}''$$

where $Sub\mathcal{A}''$ is a sublattice of all subuniverses of the lattice $Sub\mathcal{A}$ containing the element a.

Proof. We define operations f, g on the set $B = A \times N$ as follows (see Fig. 2) :

(1') $f(x, k) = (x, k+1)$ for $k \in \{1, 2\}$ or k odd, $k \geq 5$, and $f(x, 3) = (x, 1)$

(2') $f(x, 2k) = (f_{k-1}(x), 2k)$ for $k > 1$,

(3') $g(x, 1) = (a, 1)$,

(4') $g(x, 2) = (x, 1)$ for $x \neq a$ and $g(a, 2) = (a, 2)$

(5') $g(x, 2k+1) = (x, 2k+3)$, $g(x, 2k+2) = (x, 2k)$ for $k \geq 1$.

Analogously, as in previous parts, one can show that $f_n(a) = a$ for any $n \in N$ yields the crucial algebra (D, f, g) to be the algebra $(B/\Omega, f, g)$, where Ω means the congruence of the algebra (B, f, g) given by (8). If there exists $n \in N$ such that $f_n(a) \neq a$ then $(D, f, g) = (B, f, g)$.

Corollary 2. *(Kogalovskij, Soldatova). For any unary algebra $\mathcal{A}$ with a countable system of operations and a fixed point there exists a 2-unary algebra $\mathcal{B}$ for which*

1. *$Con\mathcal{B} \cong Con\mathcal{A}$ and*
2. *if $\mathcal{A}$ is finite then so is $\mathcal{B}$.*

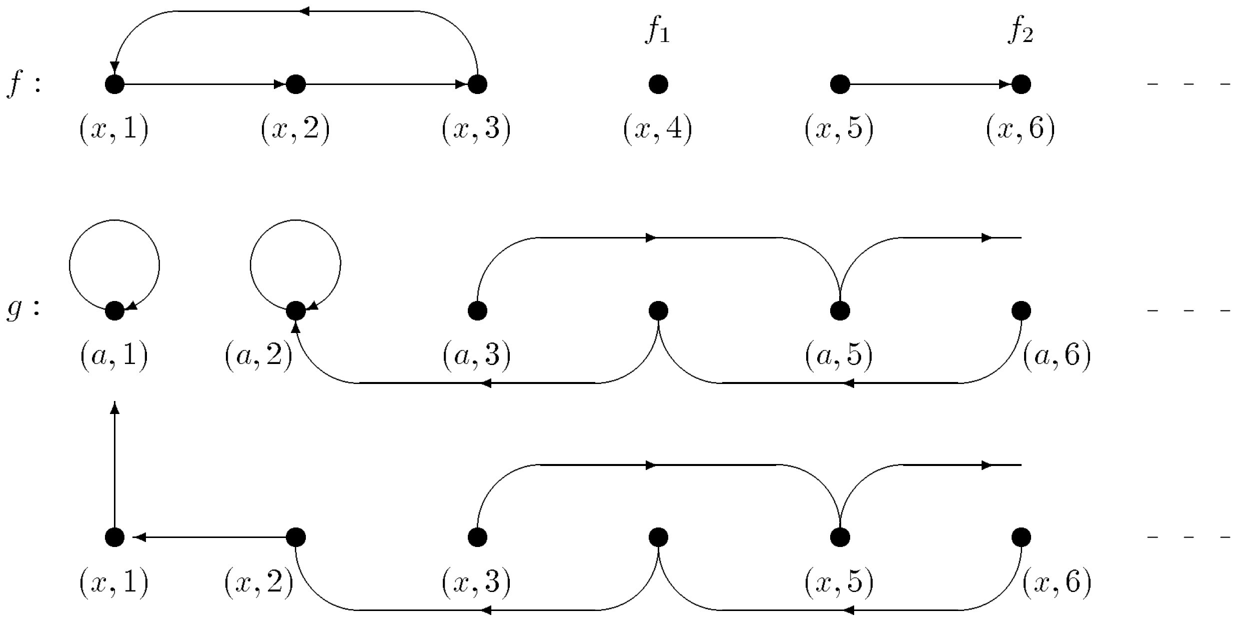

Fig. 2

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AFFINE COMPLETE STONE AND POST ALGEBRAS OF ORDER N

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Abstract. In this paper we characterize affine complete Stone algebras of order $\leq n$ ($n \geq 3$) and we show that the variety of Stone algebras of order ≤ 3 is affine complete. We also prove that each variety of Stone algebras of order $\leq n$ ($n \geq 2$) is locally affine complete. Finally, we show that the Post algebras (in the sense of [Ka-Mt 1972]) of order n ($n \geq 2$) are affine complete.

1. Introduction.

G. Grätzer in [G 1962] showed that every compatible function on a Boolean algebra $\boldsymbol{B}$ (i.e. function preserving the congruences of $\boldsymbol{B}$) can be represented by a polynomial of $\boldsymbol{B}$. Later on, in [G 1964] he characterized those bounded distributive lattices of which all compatible functions are polynomials. These were the first results leading to the study of affine complete algebras. By H. Werner [W 1971], an algebra $\boldsymbol{A}$ is called *affine complete* if all n-ary compatible functions on $\boldsymbol{A}$ are polynomials ($n \geq 1$). Further, an algebra $\boldsymbol{A}$ is said to be *locally affine complete* if any finite partial function in $A^n \to A$ (i.e. function whose domain is a finite subset of A^n) which is compatible (where defined) can be interpolated by a polynomial of A (see e.g. [P 1972] or [Kaa-P 1987]; in [Sz 1986] or [Kaa-Ma-S 1985] the notion 'locally affine complete' has another meaning).

G. Grätzer in [G 1968] (Problem 6) posed the problem of characterizing affine complete algebras. It seems to be very hard to answer such a question in general. A list of particular varieties in which all affine complete members were characterized was published in [C-W 1981], and probably the most recent list of such varieties can be found in [Ha-Pl 1994].

Much can be said about affine completeness if one is interested in varieties of algebras of which all members are affine complete, i.e. *affine complete varieties*. Affine complete varieties have been examined in [Kaa-P 1987] (see also [P 1972], [P 1979]). For a survey of the most recent results concerning affine complete varieties see [P 1993].

In this paper we deal with a special class of Stone algebras - Stone algebras of order n. We mainly deal with equationally definable Stone algebras of order $\leq n$ ($n \geq 2$) introduced by T. Katriňák and A. Mitschke [Ka-Mt 1972]. Stone algebras of order n represent one of the best-known generalizations of the Post algebras of order n, which are the algebras corresponding to n-valued propositional logic for $n \geq 2$.

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We first note that the Stone algebras of order ≤ 2 are affine complete. Then we prove that a Stone algebra $\boldsymbol{L}$ of order $\leq n$ ($n \geq 3$) is affine complete iff its dense filter $D(\boldsymbol{L})$ is an affine complete Stone algebra of order $\leq n-1$ and some extension property holds for partial compatible functions of $D(L)$. We show that the Stone algebras of order ≤ 3 are affine complete, hence this variety can be added, as a new member, to the list in [Ha-Pl 1994]. Afterwards we show that each variety of Stone algebras of order $\leq n$ ($n \geq 2$) is locally affine complete. In the last part of this paper we prove that the subdirectly irreducible Post algebra (in the sense of [Ka-Mt 1972]) of order n ($n \geq 2$) is primal. By [Hu 1971] this yields that each variety of Post algebras of order n ($n \geq 1$) is affine complete.

2. Preliminaries. A *(distributive) p-algebra* is an algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded (distributive) lattice and * is the unary operation of pseudocomplementation defined by

$$a^\star = \max\{x \in L \mid x \wedge a = 0\} \quad (a \in L).$$

A *Stone algebra* is a distributive p-algebra satisfying the identity $x^\star \vee x^{\star\star} = 1$.

In any Stone algebra $\boldsymbol{L}$, two subsets of L play an important role. The subset $D(\boldsymbol{L}) = \{x \in L \mid x^\star = 0\} = \{x \vee x^\star \mid x \in L\}$ of all *dense* elements of $\boldsymbol{L}$ which forms a filter in $\boldsymbol{L}$, and the subset $B(\boldsymbol{L}) = \{x \in L \mid x = x^{\star\star}\} = \{x^\star \mid x \in L\}$ of all *closed* elements of $\boldsymbol{L}$ which is a Boolean subalgebra of $\boldsymbol{L}$.

We first recall the definition of Stone algebras of order n ($n \geq 1$) as a subclass of the Stone algebras (see [Ba-D 1975; p. 206]):

Let $\boldsymbol{L}$ be a Stone algebra. $\boldsymbol{L}$ is a *Stone algebra of order 1*, if $\boldsymbol{L} = \mathbf{1}$. $\boldsymbol{L}$ is a *Stone algebra of order* n ($n \geq 2$), if $L \neq \mathbf{1}$ and $D(\boldsymbol{L})$ is a Stone algebra of order $n-1$.

Let $\boldsymbol{L}$ be a Stone algebra of order n ($n \geq 2$). We define
$D^0(\boldsymbol{L}) = \boldsymbol{L}$,
$D^i(\boldsymbol{L}) = D(D^{i-1}(\boldsymbol{L}))$ for $i = 1, \dots, n-1$.
Hence
$\boldsymbol{L} = D^0(\boldsymbol{L}) \supseteq D^1(\boldsymbol{L}) \supseteq \cdots \supseteq D^{n-1}(\boldsymbol{L}) = \mathbf{1}$
are Stone algebras. We shall denote the smallest element of $D^i(\boldsymbol{L})$ by d_i for $i = 1, \dots, n-1$, hence $D^i(\boldsymbol{L}) = [d_i)$. The chain $d_1 < \cdots < d_{n-1} = 1$ is said to be the *chain of smallest dense elements* of $\boldsymbol{L}$.

Before giving the definition of the Stone algebras of order n ($n \geq 2$) as an equational class, we recall some other necessary notions and results.

A *Brouwerian algebra* is an algebra $\boldsymbol{L} = (L; \vee, \wedge, *)$ where $(L; \vee, \wedge)$ is a lattice, and $*$ is the binary operation of relative pseudocomplementation defined by the rule
$x \leq y * z$ iff $x \wedge y \leq z$ for all $x, y, z \in L$.
One can show that $\boldsymbol{L}$ is distributive and has the greatest element $x * x$ denoted by 1. The class of all Brouwerian algebras is equational (see [Bi 1967] or [Ka-Mt 1972]).

A Brouwerian algebra satisfying the identity $x*y \vee y*x = 1$ is called a *relative Stone algebra.* If a Brouwerian algebra $\boldsymbol{L}$ has the smallest element 0, then the algebra $(L; \vee, \wedge, *, 0, 1)$ is called a *Heyting algebra.* Such algebra is also pseudocomplemented if one puts $x^\star = x*0$. The following rules of computation in Brouwerian (Heyting) algebras will be useful (see [Ba-D 1975; p. 174]):

(a) $x \leq y$ iff $x * y = 1$
(b) $y \leq x * y$
(c) $(x \vee y) * z = (x * z) \wedge (y * z)$
(d) $x * (y \wedge z) = (x * y) \wedge (x * z)$
(e) $(x * y)^\star = x^{\star\star} \wedge y^\star$.

Let $\boldsymbol{L}$ be a Brouwerian algebra and θ be a congruence relation on $\boldsymbol{L}$. The set $F_\theta = \{x \in L \mid x \equiv 1\ (\theta)\}$ is a filter of $\boldsymbol{L}$. Congruence relations of Brouwerian algebras can be characterized as follows (see [Ne 1965]):

2.1 Proposition. *Let $\boldsymbol{L}$ be a Brouwerian algebra. If θ is a congruence relation on $\boldsymbol{L}$, then*
$x \equiv y\ (\theta)$ *iff* $x \wedge d = y \wedge d$ *for a suitable $d \in F_\theta$. If F is a filter on $\boldsymbol{L}$, then the binary relation $\theta(F)$ defined by*
$x \equiv y\ (\theta(F))$ *iff* $x \wedge d = y \wedge d$ *for a suitable $d \in F$*
is a congruence relation on $\boldsymbol{L}$.

Thus the lattice of congruences of a Brouwerian algebra $\boldsymbol{L}$ is isomorphic to the lattice of all filters of $\boldsymbol{L}$, hence it is distributive. Further, it is well-known that Brouwerian (Heyting) algebras have a Congruence Extension Property (CEP).

In [Ka-Mt 1972], Stone algebras of order $\leq n$ $(n \geq 2)$ are characterized as follows:

2.2 Proposition ([Ka-Mt 1972; 5.2])**.** *An algebra $\boldsymbol{L} = (L; \vee, \wedge, *, e_0, \ldots, e_{n-1})$ is a Stone algebra of order $\leq n$ $(n \geq 2)$ with a chain $e_0 \leq \cdots \leq e_{n-1}$ of smallest dense elements if and only if it satisfies the lattice identities and the following list of identities:*
(1) $x \wedge [(x \wedge y) * z] = x \wedge (y * z)$
(2) $x \wedge [(y \wedge z) * z] = x$
(3) $x \wedge (x * y) = x \wedge y$
(4) $x * y \vee y * x = e_{n-1}$
(5) $e_{i+1} \wedge e_i = e_i$
(6) $e_{i+1} * e_i = e_i$
(7) $x \wedge e_{n-1} = x$
(8) $x \wedge e_0 = e_0$
(9) $e_{i+1} \wedge (x * e_i) * e_i = (x \wedge e_{i+1}) \vee e_i \quad (i \in \{0, \ldots, n-2\})$.

The lattice identities and (1)-(3) above characterize Brouwerian algebras. The identity (4) guarantees that $\boldsymbol{L}$ is a relative Stone algebra. The identities (5), (7) and (8) establish the chain $0 = e_0 \leq \cdots \leq e_{n-1} = 1$ of smallest dense elements, while (6) and (9) state that $[e_{i+1})$ is the filter of all dense elements of $[e_i)$.

It is known that the identity (4) is equivalent to the identity $x * y \vee (x * y) * y = e_{n-1}$. Thus putting $x^\star = x * e_0$ we immediately get $x^\star \vee x^{\star\star} = e_{n-1}$ which means that a Stone algebra of order $\leq n$ can be considered as a Stone algebra as well. Hence in any Stone algebra of order $\leq n$, the equation
(f) $x = x^{\star\star} \wedge (x \vee e_1)$
holds. Further, the subsets $B(\boldsymbol{L})$ of all 'closed' elements of $\boldsymbol{L}$ and $D(\boldsymbol{L}) = \{x \vee x^\star;\ x \in L\}$ of all 'dense' elements of $\boldsymbol{L}$ can be defined as above, and the formulas
(g) $(x \wedge y)^\star = x^\star \vee y^\star$
and
(h) $(x \vee y)^\star = x^\star \wedge y^\star$

are true in $\boldsymbol{L}$. If y is a closed element of a Stone algebra $\boldsymbol{L}$ of order $\leq n$ then the element $x * y$ is also closed and

(j) $x * y = x^{\star} \vee y$

holds (see [Ne 1965; Lemma 4.2]).

Finally, we mention the description of subdirectly irreducible Stone algebras of order $\leq n$ $(n \geq 2)$ given in [Ka-Mt 1972]. Let $\boldsymbol{L} = \{0 = a_0 < \cdots < a_{m-1} = 1\}$ $(1 \leq m \leq n)$ be an m-element chain. Obviously, $\boldsymbol{L}$ is a Stone algebra of order $\leq n$ if one puts $e_i = a_i$ for $i = 0, \ldots, m-1$ and $e_i = 1$ for $i = m, \ldots, n-1$. This algebra is usually denoted by $\boldsymbol{S_n(m)}$.

2.3 Proposition ([Ka-Mt 1972; 5.10]). *The only subdirectly irreducible Stone algebras of order* $\leq n$ $(n \geq 2)$ *are the algebras* $\boldsymbol{S_n(m)}$ *where* $1 \leq m \leq n$.

There are several ways to define Post algebras of order n (see [Ba-D 1975]). In this paper we use the equational definition of Post algebas of order n $(n \geq 2)$ presented in [Ka-Mt 1972; 5.3]:

2.4 Proposition. *An abstract algebra* $\boldsymbol{L} = (L, \vee, \wedge, *, +, e_0, \ldots, e_{n-1})$ *is a Post algebra of order* n $(n \geq 2)$ *if and only if it satisfies the lattice identities, the identities (1)-(9) of Proposition 2.2 and*

(10) $x \vee [(x \vee y) + z] = x \vee (y + z)$

(11) $x \vee [(y \vee z) + z] = x$

(12) $x \vee (x + y) = x \vee y$

(13) $x + y \wedge y + x = e_0$

(14) $e_{n-2} + e_{n-1} = e_{n-1}$.

Note that the identities (10)-(13) state that $\boldsymbol{L}$ is a dual relative Stone algebra, while (14) guarantees that the order of $\boldsymbol{L}$ is just n.

The subdirectly irreducible Post algebras of order n are characterized as follows:

2.5 Proposition ([Ka-Mt 1972; 5.13]). *Let* $\boldsymbol{L} = (L, \vee, \wedge, *, +, e_0, \ldots, e_{n-1})$ *be a non-trivial Post algebra of order* n $(n \geq 2)$. *The following conditions are equivalent:*

(1) $\boldsymbol{L}$ *is subdirectly irreducible;*

(2) L *consists of* n *elements;*

(3) $\boldsymbol{L}$ *is a chain.*

For these and other facts concerning Post algebras of order n see [Ka-Mt 1972] or [Ba-D 1975].

Finally, recall that an algebra $\boldsymbol{A}$ is said to be *primal*, if it is finite and every function on A is a term function of $\boldsymbol{A}$. Further, an algebra $\boldsymbol{A}$ is called *functionally complete* if every function on A is a polynomial of $\boldsymbol{A}$. H. Werner in ([W 1970] showed that a finite algebra $\boldsymbol{A}$ is functionally complete iff the discriminator is a polynomial of $\boldsymbol{A}$. Recall that the discriminator of an algebra $\boldsymbol{A}$ is the ternary function defined on A by the rule

$$d(x, y, z) = \begin{cases} z, & \text{if } x = y \\ x, & \text{if } x \neq y. \end{cases}$$

3. (Local) affine completeness.

We start with Stone algebras of order n as a subclass of Stone algebras having a smallest dense element. We recall a result from [Be 1982]:

3.1. Theorem ([Be 1982; Theorem 4]). *Let $\boldsymbol{L}$ be a Stone algebra having a smallest dense element. Then the following are equivalent:*

(1) *$\boldsymbol{L}$ is affine complete;*
(2) *$D(\boldsymbol{L})$ is an affine complete lattice;*
(3) *no proper interval of $D(\boldsymbol{L})$ is Boolean.*

Now let $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$ be a Stone algebra of order n ($n \geq 3$). Then by its definition, $D^{n-2}(\boldsymbol{L})$ is a Boolean algebra, thus $D(\boldsymbol{L})$ contains a Boolean interval. By Beazer's characterization above we get the following:

3.2 Proposition. *A Stone algebra $(L; \vee, \wedge, ^\star, 0, 1)$ of order n is not an affine complete Stone algebra for $n \geq 3$.*

3.3 Example. Let $L = \{0, d, 1\}$ be a 3-element chain. Considering L to be a p-algebra, $\boldsymbol{L}$ is a Stone algebra of order 3 with $D(\boldsymbol{L}) = \{d, 1\}$, $D^2(\boldsymbol{L}) = \{1\}$. By Proposition 3.2, $\boldsymbol{L}$ is not affine complete. We shall find a unary function on $\boldsymbol{L}$ which is compatible but not polynomial.

Take a function $f' : D(\boldsymbol{L}) \to D(\boldsymbol{L})$ defined by $f'(d) = 1$, $f'(1) = d$. Define $f : L \to L$ by the rule $f(x) = f'(x \vee d)$, i.e. $f(0) = f(d) = 1$, $f(1) = d$. Obviously, f is compatible. Suppose that f is a polynomial function of $\boldsymbol{L}$. Using the formulas (g) and (h), and the fact that $x^\star = 0$, $x^{\star\star} = 1$ for $x \in D(\boldsymbol{L})$, we get that $f' = f \restriction D(\boldsymbol{L})$ is a polynomial function of the lattice $D(\boldsymbol{L})$, thus an order-preserving function. This is, of course, a contradiction, hence f cannot be a polynomial function of the algebra $\boldsymbol{L}$. □

Next we shall deal with equationally definable Stone algebras of order $\leq n$ ($n \geq 2$) (see Proposition 2.2). First we present some preliminary lemmas.

3.4 Lemma. *Let $\boldsymbol{L} = (L; \vee, \wedge, *, e_0, \ldots, e_{n-1})$ be a Stone algebra of order $\leq n$ and let $\boldsymbol{L}' = (L; \vee, \wedge, *)$ be its Brouwerian reduct. Then $\boldsymbol{L}$ and $\boldsymbol{L}'$ have the same congruences.*

The proof is straightforward.

3.5 Lemma. *Let $(L; \vee, \wedge, *, e_0, \ldots, e_{n-1})$ be a Stone algebra of order $\leq n$ ($n \geq 2$) and let θ be a congruence of $\boldsymbol{L}$. Then $\theta \restriction D(\boldsymbol{L})$ is a congruence of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$.*

Proof. Evidently $\theta \restriction D(\boldsymbol{L})$ is a Brouwerian congruence as $D(\boldsymbol{L})$ is a Brouwerian subalgebra of $\boldsymbol{L}$. The statement now follows from Lemma 3.4. □

3.6 Lemma. *Let $(L; \vee, \wedge, *, e_0, \ldots, e_{n-1})$ be a Stone algebra of order $\leq n$ ($n \geq 2$) and let $s(\tilde{x})$ be a polynomial of $\boldsymbol{L}$. Then the function $s_1(\tilde{x}) : D(\boldsymbol{L})^n \to D(\boldsymbol{L})$ defined by $s_1(\tilde{x}) = s(\tilde{x}) \vee e_1$ is a polynomial function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$.*

Proof. We use induction on the length of the polynomial $s(\tilde{x})$. If $s(\tilde{x})$ is a variable or a constant symbol, then the statement is obvious. Now suppose that the statement holds for all polynomials $s(\tilde{x})$ of a length less than k ($k > 1$) and let $s(\tilde{x})$ be of the length k. If

$s(\tilde{x}) = p(\tilde{x}) \vee q(\tilde{x})$ ($s(\tilde{x}) = p(\tilde{x}) \wedge q(\tilde{x})$) for some polynomials $p(\tilde{x}), q(\tilde{x})$, then by induction hypothesis the statement is obviously true. Now let $s(\tilde{x}) = p(\tilde{x}) * q(\tilde{x})$. Gradually applying (f), (d), (j), (c), and (a) from Section 2, and the distributivity of L, we get

$$
\begin{aligned}
(k) \quad (p(\tilde{x}) * q(\tilde{x})) \vee e_1 &= (p(\tilde{x}) * [q(\tilde{x})^{\star\star} \wedge (q(\tilde{x}) \vee e_1)]) \vee e_1 \\
&= [p(\tilde{x}) * q(\tilde{x})^{\star\star} \wedge p(\tilde{x}) * (q(\tilde{x}) \vee e_1)] \vee e_1 \\
&= [(p(\tilde{x})^{\star} \vee q(\tilde{x})^{\star\star}) \wedge (p(\tilde{x}) \vee e_1) * (q(\tilde{x}) \vee e_1)] \vee e_1 \\
&= [(p(\tilde{x})^{\star} \vee e_1) \vee (q(\tilde{x})^{\star\star} \vee e_1)] \wedge [(p(\tilde{x}) \vee e_1) * (q(\tilde{x}) \vee e_1)]
\end{aligned}
$$

.

By induction hypothesis, $p(\tilde{x}) \vee e_1$ and $q(\tilde{x}) \vee e_1$ are polynomials of the algebra $D(\boldsymbol{L})$. Using (g), (h) and (e) from Section 2, we can transform $p(\tilde{x})^{\star}$ and $q(\tilde{x})^{\star\star}$ into polynomials of the Boolean algebra $(B(\boldsymbol{L}), \vee, \wedge, ^{\star}, e_0, e_{n-1})$, i.e. into forms

$$(l) \qquad \bigvee_{(i_1,\dots,i_n)\in\{1,2\}^n} (\alpha(i_1,\dots,i_n) \wedge x_1^{i_1} \wedge \cdots \wedge x_n^{i_n})$$

(see e.g. [Ba-D 1975; p. 92]) where x_i^1 and x_i^2 denote $x_i^{\star}$ and $x_i^{\star\star}$, respectively, $\alpha(i_1,\dots,i_n) \in B(\boldsymbol{L})$ and the join $\bigvee$ is taken over all n-tuples $(i_1,\dots,i_n) \in \{1,2\}^n$. Since the function $s_1(\tilde{x})$ is defined on D(**L**), we have in (l) $x_i^{\star} = e_0$, $x_i^{\star\star} = e_{n-1}$ for any $x_i \in D(\boldsymbol{L})$, $i = 1,\dots,n$. Thus $p(\tilde{x})^{\star}, q(\tilde{x})^{\star\star}$ can be represented by some constants $p, q \in B(\boldsymbol{L})$. This means that in (k) we get a polynomial of the algebra $D(\boldsymbol{L})$. The proof is complete. □

3.7 Lemma. *Let $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ ($n \geq 2$). Let θ_D be a congruence of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$. Then the equivalence θ on $\boldsymbol{L}$ defined by*
(m) $x \equiv y$ (θ) iff $x \vee e_1 \equiv y \vee e_1$ (θ_D) and $x^{\star} \vee e_1 \equiv y^{\star} \vee e_1$ (θ_D) and $x^{\star\star} \vee e_1 \equiv y^{\star\star} \vee e_1$ (θ_D) is a congruence of the algebra L.

Proof. Using the formulas (g) and (h) one can easily verify that θ is a congruence of the lattice L. Now let $x_i \equiv y_i$ (θ), $x_i, y_i \in L$, $i = 1, 2$. Similarly as in (k) of Lemma 3.6 we get

$$
\begin{aligned}
(x_1 * x_2) \vee e_1 &= [(x_1^{\star} \vee e_1) \vee (x_2^{\star\star} \vee e_1)] \wedge [(x_1 \vee e_1) * (x_2 \vee e_1)] \\
&\equiv [(y_1^{\star} \vee e_1) \vee (y_2^{\star\star} \vee e_1)] \wedge [(y_1 \vee e_1) * (y_2 \vee e_1)] \\
&= (y_1 * y_2) \vee e_1 \ (\theta_D).
\end{aligned}
$$

Using (e) we have

$$
\begin{aligned}
(x_1 * x_2)^{\star} \vee e_1 &= (x_1^{\star\star} \wedge x_2^{\star}) \vee e_1 = (x_1^{\star\star} \vee e_1) \wedge (x_2^{\star} \vee e_1) \\
&\equiv (y_1^{\star\star} \vee e_1) \wedge (y_2^{\star} \vee e_1) = (y_1 * y_2)^{\star} \vee e_1 \ (\theta_D),
\end{aligned}
$$

and applying (g), (h) to this, we finally get

$$
\begin{aligned}
(x_1 * x_2)^{\star\star} \vee e_1 &= [(x_1^{\star} \vee x_2^{\star\star}) \wedge e_{n-1}] \vee e_1 = (x_1^{\star} \vee e_1) \vee (x_2^{\star\star} \vee e_1) \\
&\equiv (y_1^{\star} \vee e_1) \vee (y_2^{\star\star} \vee e_1) = (y_1 * y_2)^{\star\star} \vee e_1 \ (\theta_D).
\end{aligned}
$$

Hence $x_1 * x_2 \equiv y_1 * y_2\ (\theta)$ which completes the proof of Lemma 3.7. □

In the following, $\tilde{x}$, $\tilde{x}^\star$ and $\tilde{x} \vee e_1$ will be abbreviations for $(x_1, \dots, x_n)$, $(x_1^\star, \dots, x_n^\star)$ and $(x_1 \vee e_1, \dots, x_n \vee e_n)$, respectively. Analogical meanings will have $\tilde{x}^\star \vee e_1$, $\tilde{x}^{\star\star}$ and $\tilde{x}^{\star\star} \vee e_1$. Futher, if we write e.g. $\tilde{x}^\star \vee e_1 \equiv \tilde{y}^\star \vee e_1\ (\theta)$, we mean that $x_i^\star \vee e_1 \equiv y_i^\star \vee e_1\ (\theta)$, $i = 1, \dots, n$.

The following condition defines an extension property for certain partial compatible functions of $D(\boldsymbol{L})$:

3.8 Definition. *Let $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ $(n \geq 2)$. We shall say that $\boldsymbol{L}$ satisfies a condition*

(D) *if for any compatible function $f : L^n \to L$ of the algebra $\boldsymbol{L}$, the partial function $f' : D(\boldsymbol{L})^{3n} \to D(\boldsymbol{L})$ such that*
$f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = f(\tilde{x}) \vee e_1 \quad (\tilde{x} \in L^n)$
and f' is undefined elsewhere can be extended to a total compatible function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$.

3.9 Remark. Let us verify that f' is a well-defined partial function which preserves the congruences of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ where defined. If θ_D is a congruence of the algebra $D(\boldsymbol{L})$ and $(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) \equiv (\tilde{y} \vee e_1, \tilde{y}^\star \vee e_1, \tilde{y}^{\star\star} \vee e_1)\ (\theta_D)$, then $\tilde{x} \equiv \tilde{y}\ (\theta)$ where θ is the congruence associated to θ_D in Lemma 3.7. Now $f(\tilde{x}) \equiv f(\tilde{y})\ (\theta)$ since f is compatible, thus $f(\tilde{x}) \vee e_1 \equiv f(\tilde{y}) \vee e_1\ (\theta_D)$. Hence $f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) \equiv f'(\tilde{y} \vee e_1, \tilde{y}^\star \vee e_1, \tilde{y}^{\star\star} \vee e_1)\ (\theta_D)$ what was to be proved. Using the same procedure with $\theta_D = \triangle_{D(\boldsymbol{L})}$, the smallest congruence on $D(\boldsymbol{L})$, one can show that f' is a well-defined partial function on $D(\boldsymbol{L})$. □

3.10 Proposition. *A Stone algebra $\boldsymbol{L} = (L; \vee, \wedge, *, e_0, e_1)$ of order ≤ 2 is affine complete.*

Proof. The Boolean algebra $\boldsymbol{L_1} = (L; \vee, \wedge, ^\star, e_0, e_1)$ is affine complete ([G 1962]). We shall show that $\boldsymbol{L}$ and $\boldsymbol{L_1}$ have the same congruences. Obviously, every congruence of the Heyting algebra $\boldsymbol{L}$ is also a congruence of the p-algebra $\boldsymbol{L_1}$. Conversely, let θ be a congruence of the p-algebra $\boldsymbol{L_1}$ and $x_i \equiv y_i(\theta)$, $x_i, y_i \in L$, $i = 1, 2$. Since all elements of $\boldsymbol{L_1}$ are closed, we get $x_1 * x_2 = x_1^\star \vee x_2 \equiv y_1^\star \vee y_2 = y_1 * y_2(\theta)$, thus θ is a congruence of the algebra $\boldsymbol{L}$ too. Hence a function $f : L^n \to L$ preserving the congruences of the algebra $\boldsymbol{L}$ also preserves the congruences of $\boldsymbol{L_1}$, thus f can be represented by a polynomial of the algebra $\boldsymbol{L_1}$. This polynomial, of course, can easily be rewritten as a polynomial of the algebra $\boldsymbol{L}$ replacing each $x^\star$ $(a^\star)$ by $x * e_0$ $(a * e_0)$. □

We get a characterization of affine complete Stone algebras of order $\leq n$ $(n \geq 3)$:

3.11 Theorem. *Let $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ $(n \geq 3)$. Then $\boldsymbol{L}$ is affine complete if and only if $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ is an affine complete Stone algebra of order $n - 1$ and (D) holds.*

Proof. Let $\boldsymbol{L}$ be affine complete and $f' : D(\boldsymbol{L})^n \to D(\boldsymbol{L})$ be a compatible function on the algebra D(**L**). We define a function $f : L^n \to L$ as follows:
$f(x_1, \dots, x_n) = f'(x_1 \vee e_1, \dots, x_n \vee e_1)$.
Clearly, $f \restriction D(\boldsymbol{L})^n = f'$. Using Lemma 3.5 one can show that f is compatible on $\boldsymbol{L}$. So by the assumption, the function f can be represented by a polynomial $s(\tilde{x}) =$

$s(x_1,\dots,x_n)$ of the algebra $\boldsymbol{L}$. Since in fact $f : L^n \to D(\boldsymbol{L})$, we have for any $\tilde{x} \in D(\boldsymbol{L})^n$, $f'(\tilde{x}) = f(\tilde{x}) = f(\tilde{x}) \vee e_1 = s(\tilde{x}) \vee e_1$, i.e. f' is a polynomial function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ by Lemma 3.6.

To show (D), let $f : L^n \to L$ be a compatible function of the algebra $\boldsymbol{L}$ and let $f' : D(\boldsymbol{L})^{3n} \to D(\boldsymbol{L})$ be the associated partial compatible function from Definition 3.8. Since $\boldsymbol{L}$ is affine complete, $f(x_1,\dots,x_n)$ can be represented by a polynomial $p(x_1,\dots,x_n)$ of $\boldsymbol{L}$. Define a function $f_1 : L^n \to L$ by

$f_1(\tilde{x}) = f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = p(\tilde{x}) \vee e_1$.

Let $a_1,\dots,a_m$ be all constant symbols appearing in $p(\tilde{x})$. Then $p(\tilde{x})$ can be meant as a term $t(\tilde{x}, \tilde{a}) = t(x_1,\dots,x_n,a_1,\dots,a_m)$ of the algebra $\boldsymbol{L_1} = (L; \vee, \wedge, *, a_1, \dots, a_m)$ where $t(x_1,\dots,x_{n+m})$ is a term of the Brouwerian algebra $(L; \vee, \wedge, *)$. Using the distributivity of L and the formula $(x * y) \vee e_1 = [(x^\star \vee e_1) \vee (y^{\star\star} \vee e_1)] \wedge [(x \vee e_1) * (y \vee e_1)]$ (see the proof of Lemma 3.6), one can transform $t(\tilde{x}, \tilde{a}) \vee e_1$ to a term $t_1(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1, \tilde{a} \vee e_1)$ of the algebra $(L; \vee, \wedge, *, ^\star, a_1, \dots, a_m, e_1)$ where $t_1(x_1,\dots,x_{3n+m})$ is a term of the algebra $(L; \vee, \wedge, *)$. Hence for any $\tilde{x} \in L^n$ we have

$f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = p(\tilde{x}) \vee e_1 = t(\tilde{x}, \tilde{a}) \vee e_1 = t_1(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1, \tilde{a} \vee e_1)$ $= p_1(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1)$ for some polynomial $p_1(x_1,\dots,x_{3n})$ of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$. This polynomial obviously represents the required total compatible function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ extending the partial function f'.

Conversely, let $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ be affine complete and (D) holds. Let $f : L^n \to L$ be a compatible function of the algebra $\boldsymbol{L}$. Using (f) we can write

$(m) \qquad f(\tilde{x}) = f(\tilde{x})^{\star\star} \wedge (f(\tilde{x}) \vee e_1)$.

We replace '$f(\tilde{x})^{\star\star}$' in (m) by a polynomial of $\boldsymbol{L}$. First we show that for any $\tilde{x} \in L^n$, $f(\tilde{x})^{\star\star} = f(\tilde{x}^{\star\star})^{\star\star}$. For any variable x_i we have

$x_i \wedge (x_i \vee x_i^\star) = x_i^{\star\star} \wedge (x_i \vee x_i^\star)$,

whence by Proposition 2.1, $x_i \equiv x_i^{\star\star}$ $(\theta(D(\boldsymbol{L}))$ where $\theta(D(\boldsymbol{L}))$ denotes the (Brouwerian) congruence associated to the filter D(**L**). Since f is compatible, we have $f(\tilde{x}) \equiv f(\tilde{x}^{\star\star})$ $(\theta(D(\boldsymbol{L})))$, thus by Proposition 2.1 again, there exists an element $d \in D(\boldsymbol{L})$ such that $f(\tilde{x}) \wedge d = f(\tilde{x}^{\star\star}) \wedge d$. So we get $f(\tilde{x})^{\star\star} = (f(\tilde{x}) \wedge d)^{\star\star} = (f(\tilde{x}^{\star\star}) \wedge d)^{\star\star} = f(\tilde{x}^{\star\star})^{\star\star}$ what was to be proved.

Now we define a function $f_1 : B(\boldsymbol{L})^n \to B(\boldsymbol{L})$ by the rule $f_1(\tilde{x}) = f(\tilde{x})^{\star\star}$. To show that f_1 is compatible on the algebra $(B(\boldsymbol{L}); \vee, \wedge, *, e_0, e_{n-1})$, let θ_B be a congruence of $B(\boldsymbol{L})$ and $x_i \equiv y_i$ (θ_B), $x_i, y_i \in B(\boldsymbol{L})$, $i = 1,\dots,n$. Obviously, $(B(\boldsymbol{L}); \vee, \wedge, *, e_0, e_{n-1})$ is a Heyting subalgebra of the Heyting algebra $(L; \vee, \wedge, *, e_0, e_{n-1})$. Since Heyting algebras have CEP, there exists an extension θ_L of the congruence θ_B to the Heyting algebra $\boldsymbol{L}$. Obviously, θ_L is a congruence of the algebra $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ too. Hence $x_i \equiv y_i$ (θ_L), $i = 1,\dots,n$, therefore $f(\tilde{x}) \equiv f(\tilde{y})(\theta_L)$ since f preserves the congruences of the algebra $\boldsymbol{L}$. It follows $f(\tilde{x})^{\star\star} \equiv f(\tilde{y})^{\star\star}(\theta_B)$, thus f_1 is compatible on $B(\boldsymbol{L})$. By Proposition 3.10, $(B(\boldsymbol{L}); \vee, \wedge, *, e_0, e_{n-1})$ is affine complete, thus f_1 can be represented by a polynomial $p(x_1,\dots,x_n)$ of the algebra $B(\boldsymbol{L})$. Hence in (m), $f(\tilde{x})^{\star\star} = f(\tilde{x}^{\star\star})^{\star\star} = f_1(\tilde{x}^{\star\star}) = p(\tilde{x}^{\star\star})$ for any $\tilde{x} \in L^n$. Finally, in the polynomial $p(\tilde{x}^{\star\star})$ we can put $x_i^{\star\star} = (x_i * e_0) * e_0$, $i = 1,\dots,n$, in order to get a polynomial of the algebra $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$.

To replace '$f(\tilde{x}) \vee e_1$' in (m) by a polynomial of $\boldsymbol{L}$, take the partial function f' from

Definition 3.8. By (D), f' can be extended to a total compatible function of the algebra $D(\boldsymbol{L})$, which can be represented by a polynomial $q(x_1, \dots, x_{3n})$ of $D(\boldsymbol{L})$ since $D(\boldsymbol{L})$ is affine complete. Hence in (m) $f(\tilde{x}) \vee e_1 = f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = q(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1)$ for any $\tilde{x} \in L^n$. Putting in $q(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1)$ again $x_i^\star = x_i * e_0$, $x_i^{\star\star} = (x_i * e_0) * e_0$, $i = 1, \dots, n$, we get the required polynomial of the algebra $\boldsymbol{L}$. The proof is complete. □

We have shown that the Stone algebras of order ≤ 2 are affine complete (Heyting) algebras. Now we prove that the Stone algebras of order ≤ 3 are affine complete too.

3.12 Proposition. *Any Stone algebra $(L; \vee, \wedge, *, e_0, e_1, e_2)$ of order ≤ 3 satisfies the condition (D).*

Proof. Let $D(\boldsymbol{L}) \neq \boldsymbol{1}$ and let S denote the domain of the partial compatible function f', i.e.

$$S = \{(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1);\ \tilde{x} \in L^n\} \subseteq D(\boldsymbol{L})^{3n}.$$

We define a polynomial function $p : D(\boldsymbol{L})^{3n} \to D(\boldsymbol{L})$ as follows:

$$(n) \quad p(x_1, \dots, x_{3n}) = \bigvee_{\tilde{a} \in S \cap \{e_1, e_2\}^{3n}} (f'(a_1, \dots, a_{3n}) \wedge y_1 \wedge \cdots \wedge y_{3n}),$$

where $y_i = \begin{cases} x_i * e_1, & \text{if } a_i = e_1 \\ x_i, & \text{if } a_i = e_2. \end{cases}$

We show that $f' \equiv p$ on $S \cap \{e_1, e_2\}^{3n}$. Let $\tilde{x} \in S \cap \{e_1, e_2\}^{3n}$. For $\tilde{a} = \tilde{x}$ we have $f'(\tilde{a}) \wedge y_1 \wedge \cdots \wedge y_{3n} = f'(\tilde{x}) \wedge e_2 = f'(\tilde{x})$. Now take $\tilde{a} \in S \cap \{e_1, e_2\}^{3n}$, $\tilde{a} \neq \tilde{x}$ and $a_j \neq x_j$ for some j, $1 \leq j \leq 3n$. Then

$$y_j = \begin{cases} x_j * e_1 = e_1, & \text{if } a_j = e_1 \\ x_j = e_1, & \text{if } a_j = e_2, \end{cases}$$

thus $f'(\tilde{a}) \wedge y_1 \wedge \cdots \wedge y_{3n} = f'(\tilde{a}) \wedge e_1 = e_1$. Hence in (n) we get $p(x_1, \dots, x_{3n}) = f'(x_1, \dots, x_{3n})$ what was to be proved. We assert that $f' \equiv p$ identically on the whole set S, thus that $p(x_1, \dots, x_{3n})$ is the required total compatible extension of the partial function f'. To show this, suppose on the contrary that there exists a $3n$-tuple $(d_1, \dots, d_{3n}) \in S$ such that $f'(d_1, \dots, d_{3n}) = a \neq b = p(d_1, \dots, d_{3n})$. Since $a, b \in D(\boldsymbol{L})$ and $D(\boldsymbol{L})$ is a subdirect product of copies of $\boldsymbol{2} = \{0, 1\}$, there exists a 'projection map' $h : D(\boldsymbol{L}) \to \{0, 1\}$, which is a $0,1$-homomorphism between the algebra $D(\boldsymbol{L})$ and some algebra $\boldsymbol{2} = \{0, 1\}$, such that $h(a) \neq h(b)$. Denote $h(S) = \{(h(x_1), \dots, h(x_{3n})) \in \{0, 1\}^{3n} \mid (x_1, \dots, x_{3n}) \in S\}$. Now define functions f_2', $p_2' : h(S) \cap \{0, 1\}^{3n} \to \{0, 1\}$ by the following rules:

$$f_2'(h(x_1), \dots, h(x_{3n})) = h(f'(x_1, \dots, x_{3n})),$$
$$p_2'(h(x_1), \dots, h(x_{3n})) = h(p'(x_1, \dots, x_{3n}))$$

where $(x_1, \dots, x_{3n}) \in S$. To show that f_2', p_2' are well-defined, suppose that $h(x_1) = h(y_1), \dots, h(x_{3n}) = h(y_{3n})$ for some $(x_1, \dots, x_{3n})$, $(y_1, \dots, y_{3n}) \in S$. Since f' (p') preserves the kernel Ker h $= \theta_D$ of the homomorphism h where defined, we get $f'(x_1, \dots, x_{3n}) \equiv f'(y_1, \dots, y_{3n})$ (θ_D), thus $f_2'(h(x_1), \dots, h(x_{3n})) = f_2'(h(y_1), \dots, h(y_{3n}))$ (analogously for p_2'). Obviously, $f_2' \equiv p_2'$ identically on $h(S) \cap \{0, 1\}^{3n}$, because $f' \equiv p'$ identically on $S \cap \{e_1, e_2\}^{3n}$ and $h(e_1) = 0$, $h(e_2) = 1$. Therefore

$h(a) = h(f'(d_1, \dots, d_{3n})) = f_2'(h(d_1), \dots, h(d_{3n})) = p_2'(h(d_1), \dots, h(d_{3n})) =$ $h(p'(d_1, \dots, d_{3n})) = h(b)$, a contradiction. Hence $f' \equiv p'$ identically on S and the proof is complete. □

3.13 Theorem. *A Stone algebra* $(L; \vee, \wedge, *, e_0, e_1, e_2)$ *of order* ≤ 3 *is affine complete.*

Proof. It follows from Theorem 3.11 and Propositions 3.10 and 3.12. □

3.14 Remark. In Example 3.3 we illustrated the fact that a 3-element chain considered as a p-algebra is not an affine complete Stone algebra of order 3. This means that it has too little polynomials 'to cover' all its compatible functions. Of course, if we replace in a Stone algebra of order n the operation * by $*$, the number of its polynomials will increase considerably. Theorem 3.13 says that for $n \leq 3$ they already 'cover' all compatible functions, i.e. the 3-element chain is an affine complete Heyting algebra. Note that the function f from Example 3.3 can be represented simply by the polynomial $p(\tilde{x}) = x * d$.

By Theorems 3.11 and 3.13, a Stone algebra of order ≤ 4 is affine complete iff $\boldsymbol{L}$ satisfies the condition (D). In general, we get the following result:

3.15 Corollary. *A Stone algebra* $\boldsymbol{L}$ *of order* $\leq n$ $(n \geq 4)$ *is affine complete if and only if* $D^i(\boldsymbol{L})$ *satisfies (D) for all* $i = 0, \ldots, n-4$.

The condition (D) above might actually be superfluous - we still do not know an example of a Stone algebra of order $\leq n$ in which (D) is not satisfied. Therefore we pose the following problem:

3.16 Problem. Find an example of a Stone algebra of order $\leq n$ in which the condition (D) is not satisfied or show that (D) in the characterizations above is superfluous.

The latter case would, of course, automatically mean that the variety of Stone algebras of order $\leq n$ is affine complete for $n > 3$ too.

However, we can show that each variety of Stone algebras of order $\leq n$ is locally affine complete $(n \geq 2)$:

3.17 Theorem. *Every Stone algebra* $(L; \vee, \wedge, *, e_0, \ldots, e_{n-1})$ *of order* $\leq n$ $(n \geq 2)$ *is locally affine complete.*

Proof. For $n = 2$ the result is well-known (see e.g. [P 1972]). Now let $n \geq 3$. Let $\boldsymbol{L}$ be locally affine complete and $f' : S \to D(\boldsymbol{L})$ be a partial compatible function of the algebra $D(\boldsymbol{L})$ where $S \subset D(\boldsymbol{L})^n$ is finite. One can easily verify that $f \equiv f'$ is a finite partial compatible function of the algebra $\boldsymbol{L}$ too. So by hypothesis, f can be interpolated in all elements of S by a polynomial $s(\tilde{x}) = s(x_1, \ldots, x_n)$ of the algebra $\boldsymbol{L}$. For any $\tilde{x} \in S$ we consequently have $f'(\tilde{x}) = f(\tilde{x}) = f(\tilde{x}) \vee e_1 = s(\tilde{x}) \vee e_1$, thus by Lemma 3.6, f' can be interpolated on S by polynomial function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$.

Conversely, let $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$ be locally affine complete and let $f : S \to L$ be a finite partial compatible function of the algebra $\boldsymbol{L}$. Using (f) we can write again

$$(m) \qquad f(\tilde{x}) = f(\tilde{x})^{\star\star} \wedge (f(\tilde{x}) \vee e_1) \quad (\tilde{x} \in S).$$

Analogously as in the proof of Theorem 3.11 one can show that $f(\tilde{x})^{\star\star}$ can be interpolated in all $\tilde{x} \in S$ by a polynomial of the algebra $(L; \vee, \wedge, *, e_0, \ldots, e_{n-1})$.

To replace '$f(\tilde{x}) \vee e_1$' in (m) by a polynomial of $\boldsymbol{L}$ for all $\tilde{x} \in S$, take the partial function f' defined by $f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = f(\tilde{x}) \vee e_1$ for all $\tilde{x} \in S$. Since f' is a finite partial compatible function of the locally affine complete algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$, it can be interpolated by a polynomial of $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$. The proof is complete. □

In the last part of this paper we show that for each $n \geq 2$, the variety of Post algebras of order n is affine complete. (As there are various definitions of Post algebras of order n in the literature, for some varieties of Post algebras of order n such result might already be known). Here we make use of the result of T.K. Hu [Hu 1971] which yields that a variety generated by a primal algebra is affine complete. First we show that the discriminator function on the subdirectly irreducible Post algebra $\boldsymbol{L}$ of order n is a polynomial of $\boldsymbol{L}$:

3.18 Proposition. *Let $(L, \vee, \wedge, *, +, e_0, \ldots, e_{n-1})$ be the subdirectly irreducible Post algebra of order n ($n \geq 2$). Then the discriminator is a polynomial of $\boldsymbol{L}$.*

Proof. Define a binary function $b(x, y)$ on L as follows:

$$b(x, y) = (x * y \wedge y * x) + e_{n-1}.$$

Obviously,

$$b(x, y) = \begin{cases} e_0, & \text{if } x = y \\ e_{n-1}, & \text{if } x \neq y. \end{cases}$$

One can easily verify that

$$d(x, y, z) = [b(x, y) \wedge x] \vee [b(b(x, y), e_{n-1}) \wedge z].$$

Hence the discriminator is a polynomial of $\boldsymbol{L}$. □

3.19 Theorem. *The variety of Post algebras of order n (in the sense of* [Ka-Mt 1972]*) is affine complete.*

Proof. From Proposition 3.18 it follows that the subdirectly irreducible Post algebra $\boldsymbol{L}$ of order n ($n \geq 2$) is functionally complete and from the fact that all constants $e_0, \ldots, e_{n-1}$ are the nullary operations of $\boldsymbol{L}$ we get that $\boldsymbol{L}$ is primal. Hence by Hu's result, the variety of Post algebras of order n is affine complete. □

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THE EDGE DISTANCE IN SOME FAMILIES OF GRAPHS

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Abstract. The edge distance between graphs is defined by the equality $d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + ||V_1| - |V_2||$ where $|A|$ is the cardinality of A and $E_{1,2}$ is an edge set of the maximal common subgraph of G_1 and G_2. Further, $\operatorname{diam} F_{p,q} = max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$ where $F_{p,q}$ denotes the set of all graphs with p vertices and q edges. In this paper we prove that for $p \geqq 10$ $\operatorname{diam} F_{p,p+1} = 2p - 8$ and $\operatorname{diam} F_{p,p+2} = 2p - 6$. At the end of the paper we give the answer to a problem recently posed by M. Šabo.

1. Preliminaries

A graph $G = (V, E)$ consists of a non-empty finite vertex set V and an edge set E. In this paper we consider undirected graphs without loops and multiple edges. A subgraph H of the graph G is a graph obtained from G by deleting some edges and vertices; notation: $H \subseteq G$. By $\triangle(G)$ we denote the maximal degree of vertices of the graph G. A graph G is a common subgraph of graphs G_1, G_2 if there exist graphs H_1, H_2 such that $H_1 \subseteq G_1$, $H_2 \subseteq G_2$ and $H_1 \cong G$, $H_2 \cong G$. A maximal common subgraph is a common subgraph which contains the maximal number of edges.

A distance of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined (see [2]) by

$$d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + ||V_1| - |V_2|| \tag{1}$$

where $|E_1|$, $|E_2|$, $|V_1|$, $|V_2|$ are the cardinalities of the edge sets and the vertex sets respectively, and $|E_{1,2}|$ is the number of edges of a maximal common subgraph of G_1 and G_2.

Throughout this paper, by $F_{p,q}$ we denote the set of all graphs with p vertices and q edges, $q \geq 1$. Further, $\operatorname{diam} F_{p,q} := \max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$. If $\operatorname{diam} F_{p,q} = d(G, H)$ and $c_{p,q}$ is the number of edges of the maximal common subgraph of the graphs G, H, then

$$\operatorname{diam} F_{p,q} = 2q - 2c_{p,q}. \tag{2}$$

We denote by v a firmly chosen vertex of the maximal degree in the considered graph G and by $v_1, v_2, \ldots, v_k$ the vertices adjacent to v (if $\triangle(G) = k$). We denote $U :=$

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$\{v_1, v_2, \ldots, v_k\}$ and $U' := V - \{v, v_1, \ldots, v_k\}$. The subgraph of the graph G induced by the vertex set X ($X \subset V$) we denote by $G(X)$ and the set of its edges by $E(G(X))$ or briefly by $E(X)$. The subgraph of the graph G which contains all edges with one vertex in the set U and the other in the set U' we denote by $G(U, U')$ and the set of its edges by $E(U, U')$.

2. Diameters of $F_{p,p+1}$ and $F_{p,p+2}$

Lemma 1. *If $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 3$ then G contains at least two of the graphs H_1, H_2, H_3 (Fig.1).*

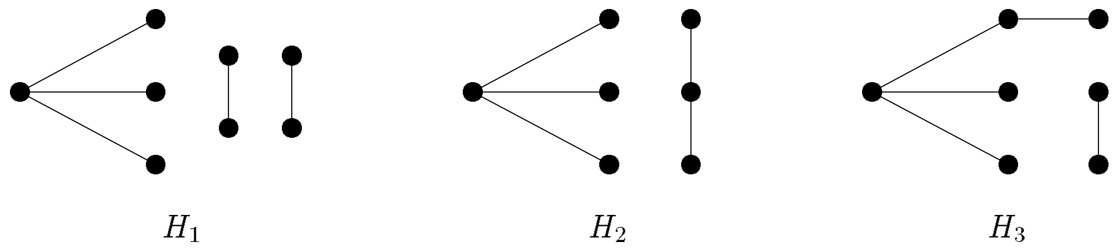


Fig.1

Proof. Since G has at least 11 edges and $\triangle(G) = 3$ then $|E(U')| \geqq 2$ from which it follows that G contains at least one of the graphs H_1 and H_2. Further we show that if G contains exactly one of the graphs H_1 and H_2 then G contains also the graph H_3. In fact, if G does not contain the graph H_1 then $G(U')$ has at most 3 edges. If G does not contain the graph H_2 then $G(U')$ has at most $[\frac{p-4}{2}]$ edges. In both cases $|E(U, U')| \geqq 2$ holds. Since $|E(U')| \geqq 2$ then from these facts it follows easily that G contains H_3.

Lemma 2. *If $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 3$ then G contains at least one of the graphs H_2 and H_4 (Figs.1,2).*

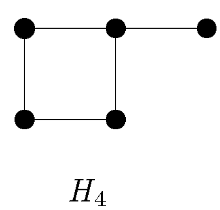
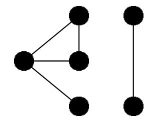


Fig.2

Proof. If G does not contain H_2 then $G(U')$ has the vertices of degree at most 1. It follows $|E(U)| + |E(U, U')| \geqq 5$ and this is possible only if $|E(U)| = 1$ or $|E(U)| = 0$ (since $\triangle(G) = 3$). The considered statement is easy to verify in both cases. In fact, if some vertex from U' has the degree at least 2 in $G(U, U')$ then G contains H_4. Otherwise G contains H_2.

Lemma 3. *If $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 3$ then G contains at least one of the graphs H_3 and H_5 (Figs.1,3).*



H_5

Fig.3

Proof. Obviously, G has a component H in which there are less vertices than edges. So, in H there is a vertex of degree 3 and we choose it as the vertex of degree 3 in the desired subgraph H_3 resp. H_5. Now it is sufficient to realize that H has at least 5 edges and G has at least 11 edges.

Lemma 4. *If $G \in F_{p,p+1}$ and $\triangle(G) = 3$ then G contains at least one of the graphs in Fig. 4.*

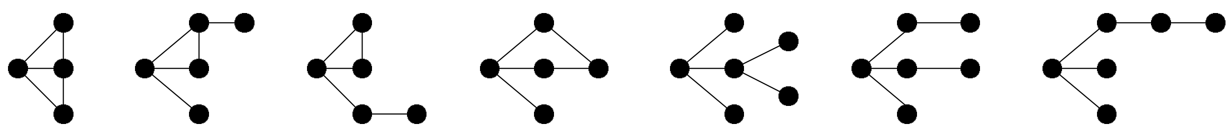


Fig.4

Proof. The graph G must have a component which has more edges than vertices. Obviously, this component contains a connected subgraph with five edges and with a vertex of degree 3. Since all such possibilities are listed in Fig. 4 the proof is finished.

Lemma 5. *Let $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 4$. If G contains neither the subgraph H_6 nor the subgraph H_7 (Fig. 5) then G contains at least one of the graphs in Fig. 4.*

H_6 H_7

Fig. 5

Proof. If G contains neither the subgraph H_6 nor the subgraph H_7 then $|E(U')| = |U'|+2$. So, the graph $G(U')$ must have a component having more edges than vertices. This component cannot have any vertex of degree 4, since otherwise the graph G would contain the subgraph H_6 or H_7, a contradiction. Then, by the same argument as in the proof of Lemma 4, the considered component contains at least one of the graphs in Fig. 4.

Lemma 6. *Each subgraph G of the graph K_6 with at least 11 edges contains the graphs H_6 and H_7 as well as each of the graphs in Fig. 4.*

Proof. It is sufficient to take into account that any vertex of a minimal degree in G has the degree at least 1 and the subgraph of the graph G induced by the set of the remaining vertices is the graph K_5 without at most two edges.

Lemma 7. *If $G_1, G_2 \in F_{p,p+1}, p \geqq 10$ and at least one of the graphs G_1, G_2 has a single non-trivial component which is a subgraph of the graph K_6 then $|E_{1,2}| \geqq 5$.*

Proof. Let, say, G_1 has the property that after removing its isolated vertices we get a subgraph of K_6. Let H be a component of the graph G_2 with more edges than vertices. If $\triangle(H) \geqq 5$ then H contains H_6 and it is sufficient to use Lemma 6. If $\triangle(H) \leqq 4$ then the statement is a consequence of Lemmas 4,5,6.

Theorem 8. *If $G_1, G_2 \in F_{p,p+1}, p \geqq 10$ and $\triangle(G_1) = \triangle(G_2) = 3$ then $|E_{1,2}| \geqq 5$.*

Proof. According to Lemma 1, the statement is obvious.

Theorem 9. *Let $G_1, G_2 \in F_{p,p+1}, p \geqq 10$, $\triangle(G_1) = 3$ and $\triangle(G_2) \geqq 4$. Then $|E_{1,2}| \geqq 5$.*

Proof. If $U' = \emptyset$ in the graph G_2 then $|E(U)| = 2$ and the considered statement is a consequence of Lemmas 1 and 2. So, in the sequel we suppose that $U' \neq \emptyset$. We distinguish three cases.

I. Let $E(U) \neq \emptyset$ and $E(U') \neq \emptyset$. We get the considered statement by Lemma 3 (obviously, the graph G_2 contains the graphs H_3 and H_5).

II. Let $E(U) = \emptyset$ and $E(U') \neq \emptyset$. Further we distinguish two subcases.

a) If $E(U, U') = \emptyset$ then $|E(U')| = |U'| + 2$ and $G_2(U')$ has a vertex of degree at least 3 and also two independent edges. Then the considered statement follows from Lemma 1.

b) Let $E(U, U') \neq \emptyset$. First of all we show that G_2 contains the subgraph H_2. If $\triangle(G_2) \geqq 5$ the statement follows from the fact that there is a vertex in U' which has the degree at least 2 (since $|E(U, U')| + |E(U')| = |U'| + 2$). If $\triangle(G_2) = 4$ the statement, obviously, holds if there is a vertex of U having the degree at least 3. In the opposite case it holds $|E(U')| \geqq |U'| - 2$ and since $|U'| - 2 > \frac{|U'|}{2}$, the graph $G_2(U')$ has a vertex of degree at least 2. So, G_2 really contains the graph H_2. Further it is possible to verify that G_2 contains at least one of the graphs H_3 and H_4. The considered statement follows from Lemmas 1 and 2.

III. Let $E(U') = \emptyset$.

a) Let $\triangle(G_2) \geqq 5$. If there is a vertex of U' having the degree at least 2 then G_2 contains H_2 and H_4, so the statement follows from Lemma 2. Further we can suppose that each vertex of U' has the degree at most 1. We get that $|E(U)| \geqq 2$.

a_1) First we suppose that $\triangle(G_2) \geq 6$. If there are two adjacent edges in $G_2(U)$ then the considered statement follows from Lemma 2. In the opposite case it is sufficient to use Lemma 3.

a_2) Now, we suppose that $\triangle(G_2) = 5$. If $E(U, U') = \emptyset$ then the considered statement follows from Lemma 7. If $E(U, U') \neq \emptyset$ then there is an edge from $E(U)$ and an edge from $E(U, U')$, which are not adjacent. We get the considered statement by Lemma 3.

b) Let $\triangle(G_2) = 4$. According to Lemma 7 we can assume that there are at least two vertices from U' having non-zero degrees. If $|E(U)| \geqq 2$ then G_2 contains the subgraphs H_3 and H_5 and the considered statement is a consequence of Lemma 3. If $|E(U)| \leqq 1$ then some vertex of U has the degree at least 2 in the graph $G_2(U, U')$ and some vertex of U' has the degree at least 2 in the graph $G_2(U, U')$. So, G_2 contains the graphs H_2 and H_4 and according to Lemma 2 the proof is finished.

Lemma 10. *If $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 4$ then G contains at least one of the graphs H_6 and H_8 (Figs. 5,6).*

H_8

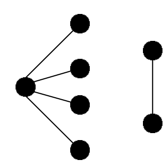


Fig. 6

Proof. It is sufficient to take into account that G has at least 11 edges and a graph with 5 vertices can have at most 10 edges.

Lemma 11. *If $G \in F_{p,p+1}, p \geqq 10$, $\triangle(G) = 4$ and G without its isolated vertices is not a subgraph of the graph K_6 then G contains at least one of the graphs H_2 and H_8 (Figs. 1,6).*

Proof. If G does not contain the subgraph H_8 then $E(U') = \emptyset$. Further, if there is a vertex from U having the degree at least 2 in $G(U, U')$ then G contains H_2. In the opposite case each vertex from U has the degree at most 1 in $G(U, U')$, i.e., $|E(U, U'| \leqq 4$. To finish the proof it suffices to consider all possible numbers of edges in $E(U, U')$.

Theorem 12. *If $G_1, G_2 \in F_{p,p+1}, p \geqq 10$ and $\triangle(G_1) = \triangle(G_2) = 4$ then $|E_{1,2}| \geqq 5$.*

Proof. The statement holds if both the graphs G_1, G_2 have the subgraph H_6 and also if they have the subgraph H_8. Then, without loss of generality we can assume that G_1 does not contain H_6 and G_2 does not contain H_8. Let us consider the graph H_7 in Fig. 5. If both the graphs G_1, G_2 contain H_7, the statement holds again. There are two possibilities in the opposite case:

a) G_1 contains H_7 and G_2 does not contain H_7. According to Lemma 10, the graph G_1 has the subgraph H_3 which is also contained in the graph G_2 because $|E(U,U')| = |U'| + 2$ and so $G_2(U,U')$ contains two independent edges.

b) G_1 does not contain H_7. According to Lemma 10, G_1 has the subgraph H_8. Further, the graph G_1 contains H_2, because $|E(U')| = |U'| + 2$ and so $G_1(U')$ contains a vertex of degree at least 3. Now it is sufficient to use Lemmas 7,11.

The proof is finished.

Theorem 13. *If* $G_1, G_2 \in F_{p,p+1}, p \geqq 10$, $\triangle(G_1) = 4$ *and* $\triangle(G_2) \geqq 5$ *then* $|E_{1,2}| \geqq 5$.

Proof. We can assume (according to Lemma 7) that neither the graph G_1 without its isolated vertices nor the graph G_2 without its isolated vertices is a subgraph of the graph K_6. Obviously, G_2 contains the graph H_8. If G_2 does not contain the graph H_6 then $|E(U)| = 0$ and also $|E(U,U')| = 0$. We get $|E(U')| = |U'| + 2$ and, obviously, $G_2(U')$ has a vertex of degree at least 3. This means that G_2 contains H_2. To finish the proof it is sufficient to use Lemmas 10 and 11.

Theorem 14. $\operatorname{diam} F_{p,p+1} = 2p - 8$ *for* $p \geqq 10$.

Proof. By Theorems 8, 9, 12 and 13 it suffices to find two graphs $G_1, G_2 \in F_{p,p+1}$ with $|E_{1,2}| = 5$. Such graphs are depicted in Fig. 7.

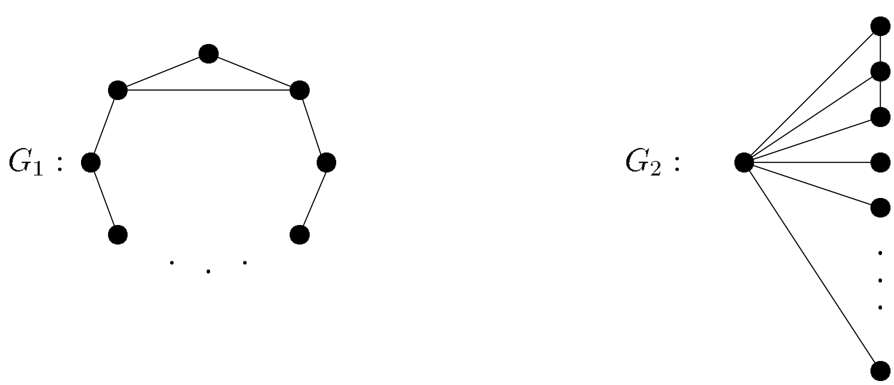


Fig. 7

Theorem 15. $\operatorname{diam} F_{p,p+2} = 2p - 6$ *for* $p \geqq 10$.

Proof. By Theorem 14 for any two graphs $G_1, G_2 \in F_{p,p+2}$ it holds $|E_{1,2}| \geqq 5$ (if $p \geqq 10$). Now it is sufficient to find two graphs $G_1, G_2 \in F_{p,p+2}$ for which $|E_{1,2}| = 5$ and the proof will be finished. Such graphs are depicted in Fig. 8.

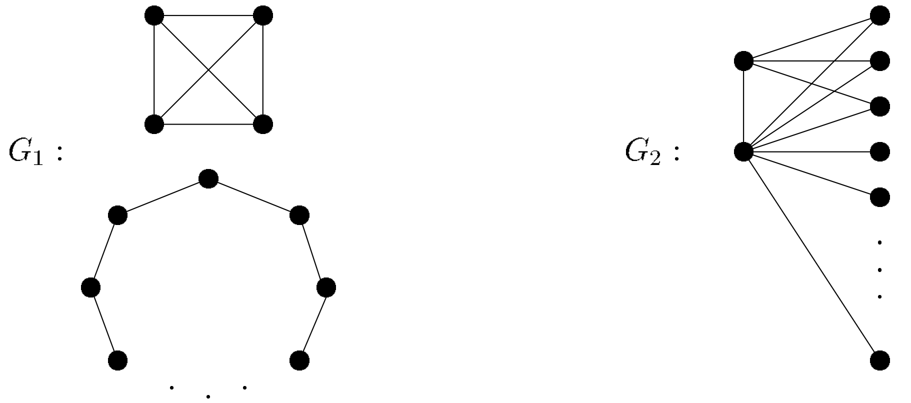


Fig. 8

Remark. By [1] $\operatorname{diam} F_{p,p+1} = 2p - 6$, if $5 \leqq p \leqq 9$. Further, obviously, $\operatorname{diam} F_{4,5} = 0$. These facts together with Theorem 14 mean that we know $\operatorname{diam} F_{p,p+1}$ for every p (there are graphs with p vertices and $p+1$ edges only for $p \geqq 4$). If we consider Theorem 15 and the following three values: $\operatorname{diam} F_{4,6} = 0$, $\operatorname{diam} F_{5,7} = \operatorname{diam} F_{5,3} = 2$, $\operatorname{diam} F_{6,8} = \operatorname{diam} F_{6,7} = 6$ (see [1] and [2]) then there are unknown $\operatorname{diam} F_{p,p+2}$ only for $p \in \{7, 8, 9\}$. In these cases $\operatorname{diam} F_{p,p+2} \in \{2p-6, 2p-4\}$.

3. $|E_{1,2}| = 2$

The following theorem gives the answer to the problem 6b which is listed in [2]. We denote the star with n edges by S_n (Fig. 9), the path with n edges by P_n and the circle with n edges by C_n.

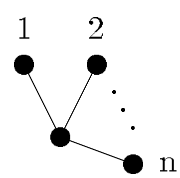


Fig. 9

Theorem 16. *If $G_1 \in F_{p_1,q_1}$ and $G_2 \in F_{p_2,q_2}$ then*

$$d(G_1, G_2) = q_1 + q_2 + |p_1 - p_2| - 4$$

if and only if the graphs G_1, G_2 satisfy one of the following three conditions (we do not pay attention to the isolated vertices; there can be any finite number of them):

1. $\triangle(G_1) = \triangle(G_2) = 1$
and one of the graphs G_1, G_2 has exactly two edges and the other one has at least two edges;

2. $\triangle(G_1) > 1$, $\triangle(G_2) > 1$
and at least one of the following conditions holds:

a) *one of the graphs* G_1, G_2 *is* S_2,

b) *one of the graphs* G_1, G_2 *is* S_n *for* $n \geqq 3$ *and the other one is a graph* G *for which* $\triangle(G) = 2$,

c) *one of the graphs* G_1, G_2 *is the graph* K_3 *and the other one is a graph which does not contain* C_3,

d) *one of the graphs* G_1, G_2 *is* P_3 *or* C_4 *and the other one is a graph which does not contain* P_3 *(i.e., each of its non-trivial components is* K_3 *or* S_n*),*

e) *one of the graphs* G_1, G_2 *is one of the graphs in Fig. 10 and the other one is a graph having only components of type* S_n *for* $n \leqq 2$;

3. $\triangle(G_i) = 1$, $\triangle(G_j) > 1$ $(\{i, j\} = \{1, 2\})$
and one of the following conditions holds:

a) $|E(G_i)| = 2$ *and* G_j *has at least two independent edges (i.e.,* G_j *has at least two non-trivial components or has a component which is different from* K_3 *and* S_n*),*

b) $|E(G_i)| \geqq 3$ *and* G_j *has exactly two non-trivial components and each of them is* K_3 *or* S_n,

c) $|E(G_i)| \geqq 3$ *and* G_j *has only one non-trivial component which is a subgraph of* K_5 *containing* P_3 *or* G_j *is one of the graphs in Fig. 11 (the graphs in Fig. 11 contain all line edges and an arbitrary subset of pointed edges).*

Proof. The case 1 is trivial. In Case 2 it is sufficient to take into account that at least one of the graphs G_1, G_2 does not contain the graph in Fig. 12 (and so, this graph is S_n or a subgraph of K_4). In case 3 the graph G_j, obviously, can have at most two non-trivial components if $|E(G_i)| \geqq 3$. If G_j has exactly one non-trivial component (denote it by H), we will distinguish two cases. The case $|V(H)| < 6$ is trivial. In the opposite case ($|V(H)| \geqq 6$) it is sufficient to distinguish whether H contains P_4 or not (obviously, H must contain P_3 and cannot contain P_5).

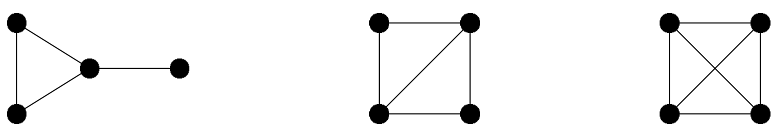


Fig. 10

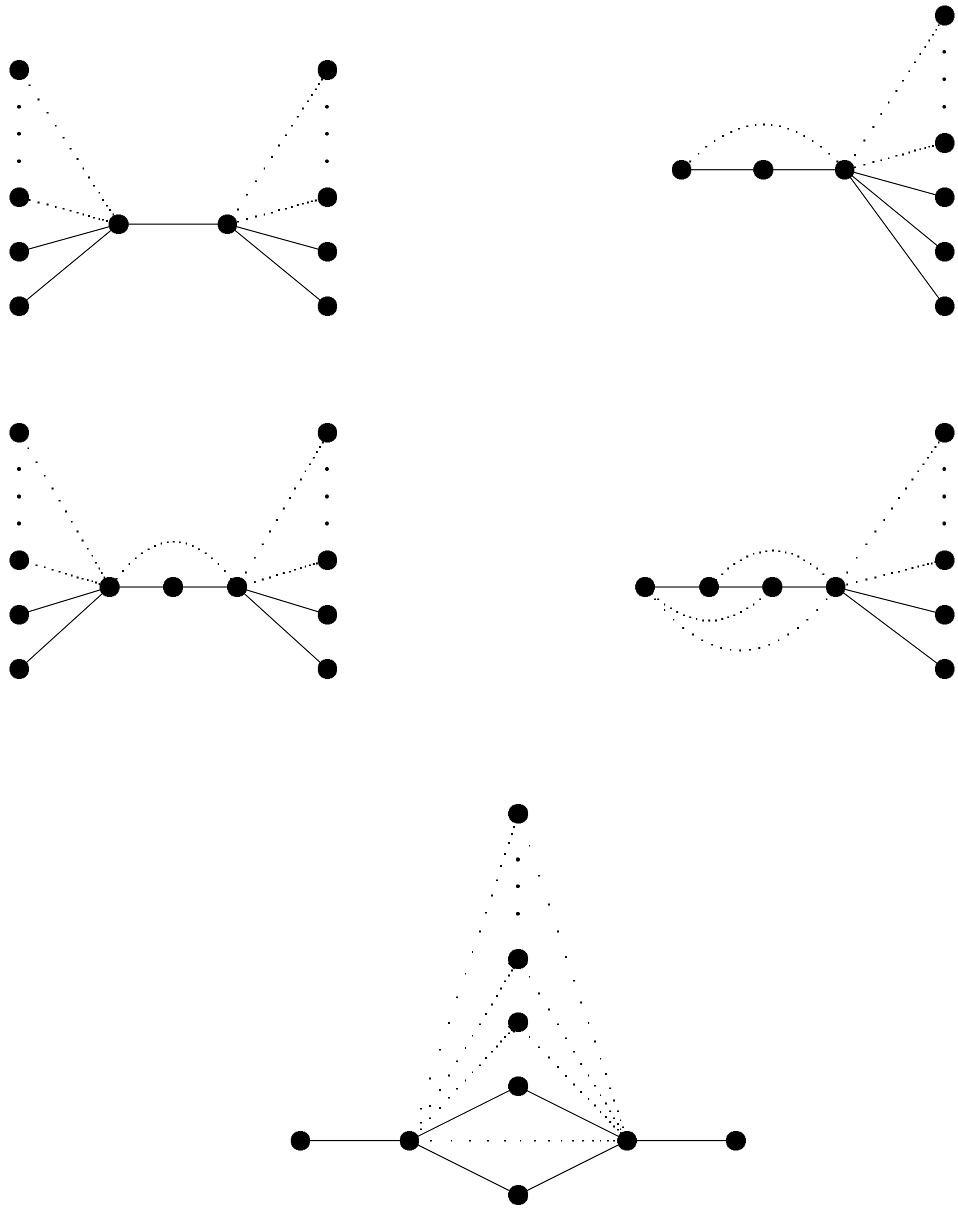


Fig. 11

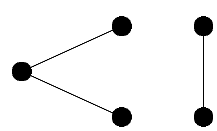


Fig. 12

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PRODUCTS OF STATES ON SOME KINDS OF TENSOR PRODUCTS

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Abstract. We study states on tensor products $B \otimes P$, where B is a horizontal sum of an arbitrary set of Boolean algebras and P is a bounded orthocomplemented poset. Such tensor products exist and they are (in a slightly more general case) constructed in [2]. It is shown that each pair of states on B and P generates a state (a product state) on $B \otimes P$ which in a certain way corresponds to these states.

It has been shown in [2] that for a horizontal sum ($\{0;1\}$-pasting) B of an arbitrary set of Boolean algebras and for a poset $(P, 0, 1, \perp)$ there exists a tensor product $B \otimes P$ and its construction has been given as well. We study states in such tensor products.

We will present the result of the above mentioned construction after a few neccessary definitions. By the abbreviation OCP we will understand a bounded orthocomplemented poset. In fact all the results of this work remain valid also in a little bit more general case, when P is only a bounded quasi-orthocomplemented poset. Since the differences are not substantial, we restrict ourselves to the more usual case of an OCP.

We recall, that if P is an OCP, $a, b \in P$, then a is orthogonal to b (we denote $a \perp b$) iff $a \leq b^{\perp}$.

Definition 1. Let P, Q, R be bounded OCPs. A mapping $\beta : P \times Q \to R$ is said to be a *bimorphism* if the following conditions are satisfied:

(1) for each orthogonal pair $a, b \in P$ and for any $c \in Q$ there is

$$\beta(a, c) \text{ and } \beta(b, c) \text{ are orthogonal,}$$

and

$$\beta(a \vee b, c) = \beta(a, b) \vee \beta(a, c),$$

(2) for each orthogonal pair $c, d \in Q$ and for any $a \in P$ there is

$$\beta(a, c) \text{ and } \beta(a, d) \text{ are orthogonal,}$$

and

$$\beta(a, c \vee d) = \beta(a, c) \vee \beta(b, d),$$

(3) $\beta(1, 1) = 1$.

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A tensor product for a horizontal sum (a $\{0,1\}$-pasting) of Boolean algebras and a OCP will be defined in a similar way to the tensor product of orthoalgebras used in [1].

Definition 2. Let B be a horizontal sum of an arbitrary set of Boolean algebras and let P be an OCP. Then a pair (T, τ) consisting of an OCP T and a bimorphism $\tau : B \times P \to T$ is said to be a *tensor product* of B and P iff the following conditions are satisfied:

(1) Each element of T is a finite join of mutually orthogonal elements of the form $\tau(a,b)$, where $a \in B, b \in P$,

(2) If L is an OCP fulfilling the previous property with the bimorphism $\beta : B \times P \to L$ then there is a morphism $\phi : T \to L$ such that $\beta = \phi \circ \tau$.

A morphism in the previous definition is understood in the usual way, i.e. it maps joins on joins and a unity element on a unity element. If no misunderstanding could occur, we will use T as a notation for the tensor product instead of (T, τ).

The construction of the tensor product $B \otimes P$ is based on the same idea as the construction of a sum for a Boolean algebra and a quantum logic which was introduced in [5].

Let S be a set consisting of all the elements of the type

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\},$$

where n is a natural number, $a_i \in B, b_i \in P$ and for each a_i the set of all a_j compatible with a_i (i.e. those from the same block of the horizontal sum) is an orthogonal partition of unity in B. We define a binary relation $\leq$ and an operation $\perp$ on S in the following way:

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\} \leq \{(c_1, d_1), (c_2, d_2), \ldots, (c_m, d_m)\}$$

iff $b_i \leq d_j$ whenever $a_i \wedge c_j \neq 0$, and

$$\{(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)\}^{\perp} = \{(a_1, b_1^{\perp}), (a_2, b_2^{\perp}), \ldots, (a_n, b_n^{\perp})\}.$$

As the next step we identify those $p, q \in S$ for which both $p \leq q$ and $q \leq p$ hold. The set of all the equivalence classes obtained by this identification will be denoted by T and its elements will be writen in square brackets. A routine verification shows that T is an OCP. The properties of this structure in case when B is a single Boolean algebra and P an orthomodular lattice is studied in [3] where there are results concerning mostly its completeness and in [4], where states and homomorphisms are studied.

Later we will make use of the following: If the elements

$$[(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)] \text{ and } [(c_1, d_1), (c_2, d_2), \ldots, (c_m, d_m)]$$

in T are mutually orthogonal, then without a loss of generality we may suppose that $a_1, a_2, \ldots, a_p, c_1, c_2, \ldots, c_q$ are from the same block of B, while all the remaining pairs a_i, c_j are from different blocks. If we assume this, then their least upper bound is the element

$$[(a_i \wedge c_j, b_i \vee d_j), (a_{p+1}, b_{p+1}), \ldots, (a_n, b_n), (c_{q+1}, d_{q+1}), \ldots, (c_m, d_m)],$$

where $i = 1, 2, \ldots, p$ and $j = 1, 2, \ldots, q$.

The following propositions are proved in [2]:

Proposition 3. *The mapping $\tau : B \times P \to T$ such that $\tau(a,b) = [(a,b),(a^{\perp},0)]$ is a bimorphism.*

Proposition 4. *Let B, P, T and τ have the same meaning as above. Then (T,τ) is a tensor product of B and P.*

The tensor product of B and P will be denoted by $B \otimes P$. Our aim is to study states on it. A state is understood in the usual way, i.e. a state on a bounded poset K is a mapping $s : K \to [0;1]$ such that $s(1) = 1$ and for each orthogonal pair $a, b \in K$ there is $s(a \vee b) = s(a) + s(b)$.

Proposition 5. *Let B, P, T and τ have the same meaning as above, let s_1 and s_2 be states on B and P respectively. Then there is a state s on T such that $s(\tau(a,1)) = s_1(a)$ and $s(\tau(1,b)) = s_2(b)$ for each $a \in B, b \in P$.*

Proof. Similarly to [5] we define the state S by the following rule:

$$s\left([(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)]\right) = \sum_{i=1}^{n} s_1(a_i)s_2(b_i).$$

As the unity element in T is $[(1,1)]$, we immediately have $s([(1,1)]) = 1$. Let now $[(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)]$ and $[(c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)]$ be orthogonal elements of T. Again we will suppose that $a_1, a_2, \ldots, a_p, c_1, c_2, \ldots, c_q$ are from the same block of B, while all the remaining pairs a_i, b_j are from different blocks. Then (see the remark before Proposition 3) we have

$$\begin{aligned} &s\left([(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)] \vee [(c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)]\right) = \\ &= s\left([(a_i \wedge c_j, b_i \vee d_j),(a_{p+1},b_{p+1}),\ldots,(a_n,b_n),(c_{q+1},d_{q+1}),\ldots,(c_m,d_m)]\right) = \\ &= \sum_{i=1}^{p}\sum_{j=1}^{q} s_1(a_i \wedge c_j)s_2(b_i \vee d_j) + \sum_{i=p+1}^{n} s_1(a_i)s_2(b_i) + \sum_{i=q+1}^{m} s_1(c_i)s_2(d_i). \end{aligned}$$

Making use of the additivity of s_2 and the fact that the sets $\{a_1, a_2, \ldots, a_p\}$ and $\{c_1, c_2, \ldots, c_q\}$ are partitions of unity we obtain that the term on the right-hand side is further equal to

$$\begin{aligned} &\sum_{i=1}^{p} s_1(a_1)s_2(b_i) + \sum_{i=1}^{q} s_1(c_i)s_2(d_i) + \sum_{i=p+1}^{n} s_1(a_i)s_2(b_i) + \sum_{i=q+1}^{m} s_1(c_i)s_2(d_i) = \\ &= s\left([(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)]\right) + s\left([(c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)]\right). \end{aligned}$$

Hence s is a state on T.

Moreover, if $a \in B$, then due to Proposition 3 we have $\tau(a,1) = [(a,1),(a^{\perp},0)]$ and evidently $s(\tau(a,1)) = s_1(a)$. If $b \in P$, then $\tau(1,b) = [(1,b)]$ and also in this case we have $s(\tau(1,b)) = s_2(b)$.

Therefore s is the state on $B \otimes P$ with the required properties and the proof is completed.

We have shown that each pair of states on B and P generates a corresponding state (a product state) on their tensor product $B \otimes P$. In fact there exist also states on $B \otimes P$ that are not generated by any such pair of states, even in the case when B is a Boolean algebra. For more details about states and homomorphisms on that structure see [4].

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THE DISTANCE POSET OF POSETS

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Abstract. In [3] a distance between isomorphism classes of ordered sets was introduced. Let $\mathcal{F}$ be a set of all (non-isomorphic) posets on a finite set P. For $(P,R),(P,S) \in \mathcal{F}$, we define $(P,R) \leq (P,S)$ if and only if there exists a bijective isotone map f of P onto itself. We will study the distance poset $(\mathcal{F}, \leq)$.

1. Introduction

In [4], [5] and [6] some properties of the distance graphs for some type of a metric for graphs and posets were investigated. In this paper some analogous results will be derived for a metric introduced in [3].

Throughout this paper all partially ordered sets are assumed to be finite. Let (P,R) be a partially ordered set (shortly poset). If $a, b \in P$, b covers a, then we will write $a \prec_R b$.

In [3] a metric on a system of isomorphism classes of posets, which have the same cardinality, is defined. Without loss of generality we can suppose that all posets are defined on the same (finite) set P. We will often write a poset R instead of a poset (P,R).

Let $B(P)$ be the set of all bijective maps of P onto itself. For any $f \in B(P)$ and posets $(P,R),(P,S)$ we denote by $d_f(R,S)$ the number defined by

$$d_f(R,S) = |f(R) \setminus S| + |S \setminus f(R)|, \tag{1}$$

where $f(R) = \{[f(a), f(b)];\ [a,b] \in R\}$ (cf. [3]). Since the posets (P,R) and $(P, f(R))$ are isomorphic, then

$$d_f(R,S) = |R| + |S| - 2|f(R) \cap S|. \tag{2}$$

The *distance* of the posets $(P,R),(P,S)$ is defined by

$$d(R,S) = \min\{d_f(R,S);\ f \in B(P)\}. \tag{3}$$

If we identify isomorphic posets, then (3) defines a *metric* on the set of all (finite, non-isomorphic) posets defined on the same set P.

If a map $f \in B(P)$ is an isotone map of a poset (P,R) onto a poset (P,S), then $f(R) \subseteq S$ and $d(R,S) = d_f(R,S) = |S| - |R|$ (cf. Remark 2 in [3]).

The following lemma is easy to verify (cf. Lemma 1 in [3]).

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Key words and phrases. Partially ordered set, distance, metric, distance poset.

Lemma 1.1. *For any posets $(P,R),(P,S)$ and any maps $f,g \in B(P)$ the following properties are satisfied:*

$$\begin{array}{lc} \text{(i)} & d_f(R,S) = d_g(R,S) \quad iff \quad |f(R) \cap S| = |g(R) \cap S|, \\ \text{(ii)} & d_f(R,S) < d_g(R,S) \quad iff \quad |f(R) \cap S| > |g(R) \cap S|, \\ \text{(iii)} & |f(R) \cap S| = |R \cap f^{-1}(S)|. \end{array}$$

The following three lemmas are obvious.

Lemma 1.2. *Let $(P,R),(P,S)$ be posets and $f \in B(P)$.*

a) *If $S \setminus f(R) \neq \emptyset$, then there exists $[a,b] \in S \setminus f(R)$ such that $a \prec_S b$.*
b) *If $f(R) \setminus S \neq \emptyset$, then there exists $[u,v] \in R$ such that $[f(u),f(v)] \in f(R) \setminus S$ and $u \prec_R v$.*

Lemma 1.3. *Let (P,R) be a poset. If for $a,b \in P$, $a \prec_R b$, then $(P, R \setminus \{[a,b]\})$ is also a poset.*

Lemma 1.4. *Let (P,R) be a poset. If $a,b \in P$, $a \prec_R b$, then $d(R, R \setminus \{[a,b]\}) = 1$.*

Let $(P,R),(P,S)$ be posets and let $f \in B(P)$. If $d_f(R,S) = d(R,S)$, f is said to be an *optimal map* of (P,R) onto (P,S) (cf. Definition in [3]). From Lemma 1.1 it follows that f is an optimal map if and only if $|f(R) \cap S|$ is maximal. Any isotone map $f \in B(P)$ is optimal (cf. Remark 2 in [3]).

Lemma 1.5. *Let $(P,R),(P,S)$ be posets and let $f \in B(P)$ be an optimal map of (P,R) onto (P,S). If $a \prec_S b$, $[a,b] \notin f(R)$, then f is an optimal map of (P,R) onto $(P, S \setminus \{[a,b]\})$ and $d(R, S \setminus \{[a,b]\}) = d(R,S) - 1$.*

Proof. Since $[a,b] \in S \setminus f(R)$, then

$$d_f(R, S \setminus \{[a,b]\}) = d_f(R,S) - 1 = d(R,S) - 1.$$

Now it is sufficient to prove that $d(R, S \setminus \{[a,b]\}) \geq d(R,S) - 1$. Suppose on the contrary that there exists a map $g \in B(P)$ with $d_g(R, S \setminus \{[a,b]\}) \leq d(R,S) - 2$. We distinguish two cases:

a) If $[a,b] \notin g(R)$, then $d_g(R, S \setminus \{[a,b]\}) = d_g(R,S) - 1$ and so $d_g(R,S) \leq d(R,S) - 1$, a contradiction.

b) If $[a,b] \in g(R)$, then $d_g(R, S \setminus \{[a,b]\}) = d_g(R,S) + 1$ and so $d_g(R,S) \leq d(R,S) - 3$, a contradiction. □

A map $f \in B(P)$ is an optimal map of (P,R) onto (P,S) if and only if f^{-1} is an optimal map of (P,S) onto (P,R) (cf. Lemma 1.1, (iii)). From this the next lemma follows (see Lemma 4 in [3]).

Lemma 1.6. *Let $(P,R),(P,S)$ be posets and let $f \in B(P)$ be an optimal map of (P,R) onto (P,S). If $a \prec_R b$ and $[f(a),f(b)] \notin S$, then f is an optimal map of $(P, R \setminus \{[a,b]\})$ onto (P,S) and $d(R \setminus \{[a,b]\}, S) = d(R,S) - 1$.*

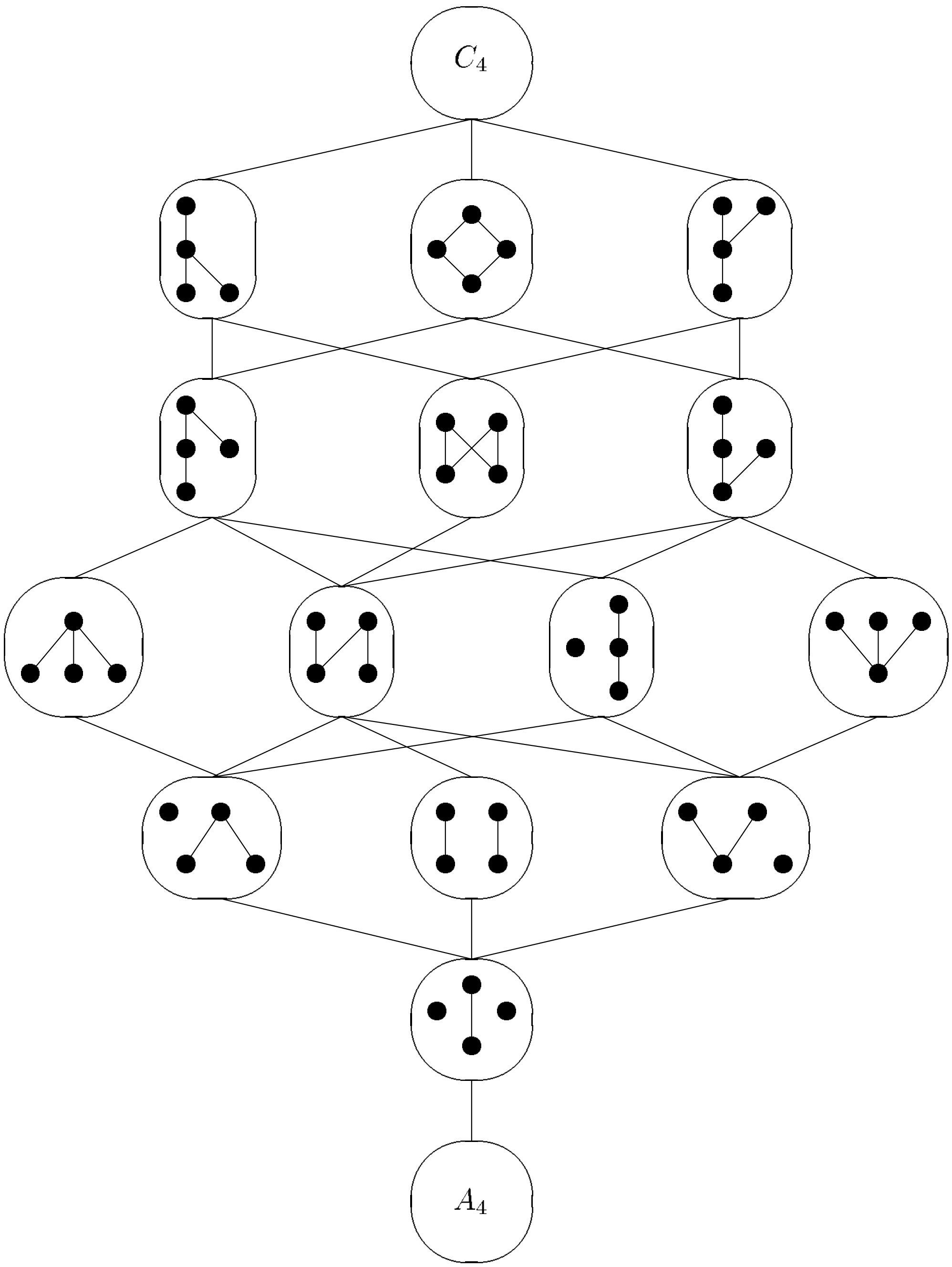
2. The distance poset

Let $\mathcal{F}_n$, $n \in N$, be a set of all (non-isomorphic) posets on a set P of cardinality n. For $(P, R), (P, S) \in \mathcal{F}_n$, we define

$$(P, R) \leq (P, S) \text{ if and only if there exists an isotone map } f \in B(P).$$

A binary relation $\leq$ is a partial order on $\mathcal{F}_n$. The poset $(\mathcal{F}_n, \leq)$ will be called the *distance poset* (of n-element posets). We shall study this poset.

Example. The following figure depicts the poset $(\mathcal{F}_4, \leq)$. C_4 is a four-element chain and A_4 is a four-element antichain.



Lemma 2.1. *Let $(\mathcal{F}_n, \leq)$ be the distance poset and let $(P,R),(P,S) \in \mathcal{F}_n$, $n \in N$. Then $d(R,S) = 1$ if and only if $(P,R) \prec (P,S)$ or $(P,S) \prec (P,R)$.*

The proof is simple; it will be omitted.

Lemma 2.2. *Let $(P,R),(P,S) \in \mathcal{F}_n$, $n \in N$, $(P,R) \leq (P,S)$ and $d(R,S) = m$, $m \in N$. Then all maximal chains from (P,R) to (P,S) have the length m.*

Proof. Let $f \in B(P)$ be an isotone map of (P,R) onto (P,S). Since the posets (P,R) and $(P,f(R))$ are isomorphic, then $d(R,S) = m = |S| - |R| = |S \setminus f(R)|$. Thus, by Lemma 2.1, for each maximal chain from $(P,R) = (P,f(R))$ to (P,S) there exists a sequence of ordered pairs $[a_1,b_1],\dots[a_m,b_m] \in S$ such that

$$C = ((P,S),(P,S \setminus \{[a_1,b_1]\}),(P,S \setminus \{[a_1,b_1],[a_2,b_2]\}),\dots,(P,S \setminus \{[a_1,b_1],\dots,[a_m,b_m]\})).$$

□

Now we recall some further notions and facts concerning posets and graphs.

If a poset (P,R) has the least element 0_P, then we define the *height* $h(a)$ of an element $a \in P$ as the length of the longest chain from 0_P to a. If a poset (P,R) has the least element and all maximal chains between the same endpoints have the same length, then we say that (P,R) is a *graded* poset. A poset (P,R) is said to have length n, denoted by $l(P) = n$, if the length of the longest chain in (P,R) is n. If (P,R) has the greatest element 1_P, then $l(P) = h(1_P)$.

A *graph* $G = (V,E)$ consists of a nonempty finite vertex set V together with a prescribed edge set E of unordered pairs of distinct vertices of V. Every edge can be written in the form ab, where $a,b \in V$.

Let $\delta(a,b)$ denote the distance from a to b (i.e. the length of the shortest path from a to b in a connected graph $G = (V,E)$), and let $\operatorname{diam} G = \max\{\delta(a,b);\ a,b \in V\}$ denote the diameter of G. The function δ is a metric. The *covering graph* $C(P)$ of a poset (P,R) is the graph whose vertices are the elements of P and whose edges are those pairs ab, $a,b \in P$, for which a covers b or b covers a. For elements a,b of a poset (P,R), $\delta(a,b)$ shall denote the distance from a to b in the covering graph $C(P)$ of (P,R).

From Lemma 2.2 we immediatelly obtain

Theorem 2.1. *The distance poset $(\mathcal{F}_n, \leq)$ is a graded poset with the least element $0_{\mathcal{F}_n} = A_n$ (an n-element antichain) and the greatest element $1_{\mathcal{F}_n} = C_n$ (an n-element chain).*

The following lemma is a part of Lemma 2.1 in [2].

Lemma 2.3. *Let (P,R) be a graded poset and let $a,b \in P$. Then*

$$\delta(a,b) = h(a) - h(b) \quad \text{if and only if} \quad [b,a] \in R.$$

Clearly, if a poset $(P,R) \in \mathcal{F}_n$, $n \in N$, then $h(P,R) = |R| - n$. From Lemma 2.3 we have

Lemma 2.4. *Let $(P,R),(P,S)$ be posets from $\mathcal{F}_n$, $n \in N$. If $(P,R) \leq (P,S)$, then $\delta(R,S) = d(R,S)$.*

The following theorem was motivated by the similar results of Zelinka in [5] and [6].

Theorem 2.2. *Let $(\mathcal{F}_n, \leq)$, $n \in N$ be the distance poset. If $(P,R),(P,S) \in \mathcal{F}_n$, then*

$$d(R,S) = \delta(R,S),$$

where $\delta(R,S)$ is the distance of vertices (P,R), (P,S) of the graph $C(\mathcal{F}_n)$.

Proof. Let $(P,R),(P,S) \in \mathcal{F}_n$. Then there exists a map $f \in B(P)$ such that

$$d(R,S) = d_f(R,S) = |f(R) \setminus S| + |S \setminus f(R)|.$$

Since $f(R) \cap S \subseteq f(R)$, $f(R) \cap S \subseteq S$, then

$$(P, f(R) \cap S) \leq (P, f(R)) = (P,R) \quad \text{and} \quad (P, f(R) \cap S) \leq (P,S).$$

From the triangle inequality for δ and from Lemma 2.4 it follows that

$$\begin{aligned}\delta(R,S) \leq& \delta(R, f(R) \cap S) + \delta(f(R) \cap S, S) = d(R, f(R) \cap S) + d(f(R) \cap S, S) = \\ =& d(f(R), f(R) \cap S) + d(f(R) \cap S, S) = |f(R) \setminus (f(R) \cap S)| + |S \setminus (f(R) \cap S)| = \\ =& |f(R) \setminus S| + |S \setminus f(R)| = d(R,S).\end{aligned}$$

Thus

(a) $$\delta(R,S) \leq d(R,S).$$

Let

$$R = R_{11}, R_{12}, \dots, R_{1i_1} = R_{21}, R_{22}, \dots, R_{2i_2}, \dots, R_{j1}, R_{j2}, \dots, R_{ji_j} = S$$

be a shortest path from (P,R) to (P,S) in $C(\mathcal{F}_n)$, where $\{R_{k1}, R_{k2}, \dots, R_{ki_k}\}$ is a chain in $(\mathcal{F}_n, \leq)$ for all $k \in \{1,2,\dots,j\}$. From Lemma 2.4 and from the triangle inequality for d it follows that

$$\begin{aligned}\delta(R,S) =& \delta(R, R_{1i_1}) + \delta(R_{21}, R_{2i_2}) + \cdots + \delta(R_{j1}, S) = \\ =& d(R, R_{1i_1}) + d(R_{21}, R_{2i_2}) + \cdots + d(R_{j1}, S) \geq d(R,S).\end{aligned}$$

Thus

(b) $$\delta(R,S) \geq d(R,S).$$

From (a) and (b) we have $\delta(R,S) = d(R,S)$. □

A simple induction yields the following corollary.

Corollary 2.1. *Let $(\mathcal{F}_n, \leq)$, $n \in N$ be the distance poset. If $(P,R),(P,S) \in \mathcal{F}_n$, then*

$$h(P,R) - h(P,S) \equiv d(R,S) \pmod 2$$

The next lemma is implicit in Alvarez [1].

Lemma 2.4. *Let (P,R) be a graded poset with the greatest element. Then*

$$\operatorname{diam} C(P) = \delta(0_P, 1_P) = l(P).$$

From Lemma 2.4, Lemma 2.3 and Theorem 2.1 we immediately get

Corollary 2.2. *Let $(\mathcal{F}_n, \leq)$, $n \in N$ be the distance poset. Then*

$$\operatorname{diam} C(\mathcal{F}_n) = \frac{n(n-1)}{2}.$$

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A VECTOR LATTICE VARIANT OF THE MARTINGAL THEOREM

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Abstract. The purpose of this paper is to give a variant of the martingal theorem for random variables with values in a vector lattice. The following theorem is known as the inverse martingal theorem, see [5,p.360]. The direct martingal theorem we do not use, because its generalization for vector lattice valued random variables is more difficult.

1. Introduction

Theorem 1.1. *(inverse martingal theorem) Let $(\Omega, \mathcal{S}, P)$ be a probability measure space, $\{\mathcal{S}_k\}$ be a decreasing sequence of σ-subalgebras of $\mathcal{S}$ and $\mathcal{S}_\infty$ be the intersection of $\{\mathcal{S}_k\}$. Then for any random variable $\xi : \Omega \to R$ with a finite expectation $E(\xi)$*

(i) $E(\xi|\mathcal{S}_k) \to E(\xi|\mathcal{S}_\infty)$ *almost certainly and*
(ii) $E|E(\xi|\mathcal{S}_k) - E(\xi|\mathcal{S}_\infty)| \to 0.$

The present paper generalizes the inverse martingal theorem for vector lattice valued random variables. The proof of the main result is very similar to the proof of the ergodic theorem published by author in [3] and it might be omitted. However, we give the proof for the sake of completness of results. Similarly as in [3] we use results of [2] about mean value and conditional mean value for vector lattice valued variable. Similar results for mean value were established in [6] under stricter conditions.

2. Vector lattices

More complete information about vector lattices may be found in [1] and [4].

A real vector space V is called a vector lattice if it has a partial ordering $\leq$ such that $(V, \leq)$ is a lattice and:

$\forall x, y, z \in V : x \leq y \implies x + z \leq y + z$

$\forall x, y \in V : \lambda \geq 0 : x \leq y \implies \lambda x \leq \lambda y$. Lattice operations are denoted by symbols $\vee$ and $\wedge$.

If $a \in V$ then the symbol $|a|$ denotes the element $a \vee (-a)$.

A vector lattice V is called σ-complete if every sequence $\{a_n\} \subset V$ bounded from above has a least upper bound which is denoted by the symbol $\bigvee_{n=1}^{\infty} a_n$ (or equivalently, every sequence $\{a_n\}$ bounded from below has a greatest lower bound which is denoted by $\bigwedge_{n=1}^{\infty} a_n$).

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Definition 2.1. Let V be a σ-complete lattice. A sequence $\{a_n\} \subset V$ is called *decreasing* to 0 if:

$\forall n : a_{n+1} \leq a_n$ and $\bigwedge_{n=1}^{\infty} a_n = 0$. We write $a_n \searrow 0$ in this case.

A sequence $\{x_n\} \subset V$ is called *converging* to $x \in V$ if there is a sequence $\{a_n\} \subset V$ decreasing to 0 such that $|x_n - x| \leq a_n$ for all n. We write $x_n \to x$ $(n \to \infty)$ in this case.

Proposition 2.2. *Let V be a σ-complete vector lattice.*

(i) *A sequence $\{x_n\} \subset V$ converges to $x \in V$ if and only if $\{x_n\}$ is bounded and* $x = \bigwedge_{n=1}^{\infty} \bigvee_{m=n}^{\infty} x_m = \bigvee_{n=1}^{\infty} \bigwedge_{m=n}^{\infty} x_m$

(ii) $a_n \searrow 0,\, b_n \searrow 0 \implies (a_n + b_n) \searrow 0$

(iii) $a_n \searrow 0, \lambda \geq 0 \implies \lambda a_n \searrow 0$

(iv) $x_n \to x, y_n \to y \implies (x_n + y_n) \to (x + y)$

(v) $x_n \to x \implies \lambda x_n \to \lambda x.$

The following lemma will be important in the proof of the main result in this paper.

Lemma 2.3. *Let V be a σ-complete vector lattice and $\{a_n\} \subset V, \{b_{n,k}\} \subset V$ be sequences such that:*

$\forall n, k : b_{n,k} \geq 0$

$\forall n : b_{n,k} \to 0$ $(k \to \infty)$

$a_n \searrow 0$ $(n \to \infty)$.

Put $c_k = \bigwedge_{n=1}^{\infty} (a_n + b_{n,k})$. *Then* $\forall k : c_k \geq 0$ *and* $c_k \to 0$ $(k \to \infty)$.

Proof. The inequality $c_k \geq 0$ for all k is obvious. The sequence $\{c_k\}$ is bounded because $0 \leq c_k \leq a_1 + b_{1,k}$ for all k and $b_{1,k} \to 0$ $(k \to \infty)$. It means that the element $\bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} c_j$ exists. We have:

$$\bigvee_{j=k}^{\infty} c_j = \bigvee_{j=k}^{\infty} \bigwedge_{n=1}^{\infty} (a_n + b_{n,j}) \leq \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j})$$

and

$$\bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} c_j \leq \bigwedge_{k=1}^{\infty} \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) = \bigwedge_{n=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) =$$

$$= \bigwedge_{n=1}^{\infty} (a_n + \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} b_{n,j}) = \bigwedge_{n=1}^{\infty} (a_n + 0) = \bigwedge_{n=1}^{\infty} a_n = 0.$$

Except for assumptions of lemma we used the obvious facts:

$$\bigvee_{j=k}^{\infty} \bigwedge_{n=1}^{\infty} (a_n + b_{n,j}) \leq \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j})$$

and

$$\bigwedge_{k=1}^{\infty} \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) = \bigwedge_{n=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}).$$

3. Integral and conditional mean value of vector lattice valued functions

In this section we give a summary of results of author's paper [2].

Let $(\Omega, \mathcal{S}, P)$ be a probability measure space and V be a σ-complete vector lattice. The symbol $F(\Omega, V)$ denotes the set of all functions $f : \Omega \to V$. Obviously, $F(\Omega, V)$ is a σ-complete vector lattice under natural operations and ordering.

Two functions $f, g \in F(\Omega, V)$ are called equivalent if there exists a set $A \in \mathcal{S}$ such that

$$P(A) = 0 \text{ and } \forall \omega \in \Omega \setminus A : f(\omega) = g(\omega).$$

The set of all equivalence classes is denoted by $\mathcal{F}(\Omega, \mathcal{S}, P, V)$ and it is a σ-complete vector lattice under natural operations and ordering. A function $f \in F(\Omega, V)$ is called simple if $f(\omega) = a_i$ for $\omega \in A_i$, where $\{A_i\}$ is a finite measurable partition of Ω and $a_i \in V$. We put $\displaystyle\int_\Omega f(\omega) dP(\omega) = \sum_{i=1}^{n} P(A_i) a_i$ in this case.

A class $\varphi \in F(\Omega, \mathcal{S}, P, V)$ is called simple if it contains some simple function f. We put $E(\varphi) = \int_\Omega \varphi dP = \int_\Omega f(\omega) dP(\omega)$ in this case.

The set of all simple functions is denoted by $L_0^\infty(\Omega, \mathcal{S}, P, V)$ and the set of all simple classes is denoted by $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$.

Let $\{f_n\} \subset F(\Omega, V)$ and $f \in F(\Omega, V)$. We say that a sequence $\{f_n\}$ converges to the function f uniformly almost everywhere if there exist $A \in \mathcal{S}, \{a_n\} \subset V$ such that:

$P(A) = 0$

$\forall \omega \in \Omega \setminus A : \forall n : |f_n(\omega) - f(\omega)| \leq a_n$

$a_n \to 0 \ (n \to \infty)$.

Obviously, the condition $a_n \to 0$ may be replaced by a stronger one $a_n \searrow 0$. We write $f_n \to f$ u.a.e. $(n \to \infty)$ in this case.

Let $\{\varphi_n\} \subset \mathcal{F}(\Omega, \mathcal{S}, P, V)$ and $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$. We say that the sequence $\{\varphi_n\}$ converges to a class φ uniformly almost everywhere if $f_n \to f$ u.a.e. for some $f_n \in \varphi_n$ and $f \in \varphi$. We write $\varphi_n \to \varphi$ u.a.e. $(n \to \infty)$ in this case.

Let $\mathcal{M}$ be a system of all vector subspaces of $\mathcal{F}(\Omega, \mathcal{S}, P, V)$ which contain the set $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$ and are closed with respect to the convergence which was described above. Obviously, $\mathcal{M}$ has the minimal element with respect to inclusion. This vector space is denoted by $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$.

Theorem 3.1.

(i) *$\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ is a vector sublattice of $\mathcal{F}(\Omega, \mathcal{S}, P, V)$, which is closed with respect to u.a.e. convergence.*

(ii) *There exists a unique nonnegative linear extension $\overline{E}$ of the set E onto $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$, which is continuous in the following sense: $\varphi_n \to \varphi$ u.a.e. $\implies \overline{E}(\varphi_n) \to \overline{E}(\varphi)$.*

Remark. We shall write $E(\varphi)$ or $\int_\Omega \varphi dP$ for $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ instead of $\overline{E}(\varphi)$.

In a similar way a conditional mean value operator can be constructed. Let $(\Omega, \mathcal{S}, P)$ be a probability measure space, $\mathcal{S}_0$ be a σ-algebra of $\mathcal{S}$ and $E(.|\mathcal{S}_0)$ be a conditional mean value operator for real functions. Take $\varphi \in \mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V); \varphi$ is an equivalence class of some simple function f of the form $\sum_{i=1}^{n} \chi_{Ai} a_i$. Denote by ψ the equivalence class of the function $\sum_{i=1}^{n} E(\chi_{Ai}|\mathcal{S}_0)a_i$. In this case $\psi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$.

Putting $E(\varphi|\mathcal{S}_0) = \psi$ we obtain a linear nonnegative operator

$$E(.|\mathcal{S}_0) : \mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V) \to \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V).$$

Theorem 3.2.

(i) *There exists a unique nonnegative linear extension*

$$\overline{E}(.|\mathcal{S}_0) : \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V) \to \mathcal{L}^\infty(\Omega, \mathcal{S}_0, P, V) \text{ of } E(.|\mathcal{S}_0).$$

(ii) *The operator $\overline{E}(.|\mathcal{S}_0)$ is continuous in the following sense: $\varphi_n \to \varphi$ u.a.e. $\implies$ $\overline{E}(\varphi_n|\mathcal{S}_0) \to \overline{E}(\varphi|\mathcal{S}_0)$ u.a.e.*

Remark. We shall write $E(\varphi|\mathcal{S}_0)$ instead of $\overline{E}(\varphi|\mathcal{S}_0)$.

We shall also use the pointwise convergence.

Let $\{f_n\} \subset F(\Omega, V)$ and $f \in F(\Omega, V)$. We say that the sequence $\{f_n\}$ converges to f almost everywhere if there exists a set $A \in \mathcal{S}$ such that $P(A) = 0$ and $\forall \omega \in \Omega \setminus A$: $f_n(\omega) \to f(\omega)$ $(n \to \infty)$. We write $f_n \to f$ a.e. in this case.

If $\{\varphi_n\} \subset \mathcal{F}(\Omega, \mathcal{S}, P, V)$ and $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$ then the notation $\varphi_n \to \varphi$ a.e. $(n \to \infty)$ means that $f_n \to f$ a.e. $(n \to \infty)$ for some $f_n \in \varphi_n$ and $f \in \varphi$.

4. The inverse martingal theorem for vector lattice valued random variables

Theorem 4.1. *Let $(\Omega, \mathcal{S}, P)$ be a probability measure space, V be a σ-complete vector lattice, $\{\mathcal{S}_k\}$ be a decreasing sequence of σ-subalgebras of $\mathcal{S}$ and $\mathcal{S}_\infty$ be the intersection of $\{\mathcal{S}_k\}$. Then for all $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$*

(i) *$E(\varphi|\mathcal{S}_k) \to E(\varphi|\mathcal{S}_\infty)$ a.e. and*

(ii) *$E|E(\varphi|\mathcal{S}_k) - E(\varphi|\mathcal{S}_\infty)| \to 0$.*

Proof. Denote by $\mathcal{M}$ the set of all $\varphi \in L^\infty(\Omega, \mathcal{S}, P, V)$, for which (i) is true. The martingal theorem for real functions implies that $\mathcal{M}$ contains $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$. It is sufficient to prove that $\mathcal{M}$ is closed with respect to u.a.e. convergence. Let $\{\varphi_n\} \subset \mathcal{M}$ be a sequence which converges to $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$ u.a.e.. Obviously, $\varphi \in \mathcal{L}(\Omega, \mathcal{S}, P, V)$ and $E(\varphi_n|\mathcal{S}_k) \to E(\varphi_n|\mathcal{S}_\infty)$ a.e. when $k \to \infty$ for all n.

Let f_n, f, g_n, g, h_{nk} and h be representants of the following equivalence classes φ_n, φ, $E(\varphi_n|\mathcal{S}_\infty)$, $E(\varphi|\mathcal{S}_\infty)$, $E(\varphi_n|\mathcal{S}_k)$ and $E(\varphi|\mathcal{S}_k)$. There is a set $A \in \mathcal{S}$ with $P(A) = 0$ and a sequence $\{a_n\} \subset V$ with $a_n \searrow 0$ such that $|f_n(\omega) - f(\omega)| \leq a_n$ for all $\omega \in \Omega \setminus A$. Since conditional mean value preserves ordering and constants, there are sets B_k and $B \in \mathcal{S}$ with $P(B_k) = 0$ and $P(B) = 0$ such that

$|g_n(\omega) - g(\omega)| \leq a_n$ for all $\omega \in \Omega \setminus B$ and

$|h_{nk}(\omega) - h(\omega)| \leq a_n$ for all $\omega \in \Omega \setminus B_k$.

By assumptions $h_{nk}(\omega) \to g_n(\omega)$ a. e. when $k \to \infty$ for any n; the exceptional ω form a set C_n with $P(C_n) = 0$. Denote $D = A \bigcup \left(\bigcup_{k=1}^{\infty} B_k \right) \bigcup \left(\bigcup_{n=1}^{\infty} C_n \right)$. For $\omega \in \Omega \setminus D$ denote $b_{nk} = |h_{nk}(\omega) - g_n(\omega)|$. We have

$$|h_k(\omega) - g(\omega)| \leq |h_k(\omega) - h_{nk}(\omega)| + |h_{nk}(\omega) - g_n(\omega)| + |g_n(\omega) - g(\omega)| \leq$$

$$\leq a_n + b_{nk} + a_n = 2a_n + b_{nk}.$$

Put $c_k = \bigwedge_{n=1}^{\infty} (2a_n + b_{n,k})$.

Then $|h_k(\omega) - g(\omega)| \leq c_k$ and $c_k \to 0$ by lemma 2.3. The proof of (i) is complete. Part (ii) can be proved by the same idea (it is not necessary to use the representants).

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FUNCTIONALLY COMPLETE ALGEBRAS

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Abstract. It is known that every funtionally complete algebra has the k-interpolation property for $k \geq 1$. We will prove that for each integer $k > 2$, every k-element algebra with the k-interpolation property is functionally complete.

1. Introduction

The polynomial function of an algebra (A, F) is represented by some "correctly arranged" string containing (possibly) the variables, the symbols "(" , ")", the constants (from A), and the symbols of the operations (from F). The algebra (A, F) is **functionally complete** iff every function $A^n \to A$ is polynomial.

The algebra (A, F) has the k-interpolation property iff for every integer $n > 0$, every k-tuple

$$(\overrightarrow{a_1}, \overrightarrow{a_2}, \ldots, \overrightarrow{a_k}) \in (A^n)^k$$

of pairwise distinct vectors and every k-tuple

$$(b_1, b_2, \ldots, b_k) \in A^k$$

there exists a n-ary polynomial function F such that

$$F(\overrightarrow{a_1}) = b_1,\ F(\overrightarrow{a_2}) = b_2,\ \ldots,\ F(\overrightarrow{a_k}) = b_k.$$

A lot of interesting information on the functional completness and the interpolation properties can be found in [1] or [2]. Further we will use the Composition theorem (Wille + Werner). By this theorem, if there exist two different elements $0, 1 \in A$, two binary polynomial operations "+" and "·" satisfying the identities

$$x + 0 = x,\ 0 + x = x,\ x \cdot 1 = x,\ x \cdot 0 = 0$$

and several unary polynomial functions (the unary functions g for which one value of g is equal to 1 and all other values of g are equal to 0), then (A, F) is functionally complete. (It is assumed that A is finite.) Further we will study only k-element algebras with the k-interpolation property, consequently, all unary functions will be polynomial. Moreover, each k-element algebra with the $2k$-interpolation property is functionally complete because the identities for "+" and "·" determine at most $2k$ values of each operation.

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Lemma 1.1. *Assume that (A, F) has the $(m+1)$-interpolation property. Then (A, F) has the m-interpolation property, too.*

Example 1.1. Let $(A, F) = (Z_2, +)$ be the additive group of the 2-element field. This algebra is not functionally complete (every unary function is polynomial but it is easy to prove that among 16 binary functions there are only 8 polynomial functions). It can be proved that this algebra has the 3-interpolation property (we leave it to the reader).

2. Three-element algebras

Assume that $A = \{0, 1, 2\}$ and the algebra (A, F) of any type has the 3-interpolation property. Every function $M : A^2 \to A$ can be represented by the following 3×3 matrix:

$$M = \begin{vmatrix} M(0,0) & M(0,1) & M(0,2) \\ M(1,0) & M(1,1) & M(1,2) \\ M(2,0) & M(2,1) & M(2,2) \end{vmatrix} .$$

Using such matrices, we can simply write the "interpolation polynomials". For example, the (possibly not existing) binary polynomial function G satisfying the equalities

$$G(0,0) = 0,\ G(0,1) = 1,\ G(1,2) = 2,\ G(2,0) = 1$$

will be represented by the following matrix ("$*$" denotes the non-determined value):

$$G = \begin{vmatrix} 0 & 1 & * \\ * & * & 2 \\ 1 & * & * \end{vmatrix}$$

and we will say that the matrix G is **polynomial** iff it represents some binary polynomial function G.

Lemma 2.1. *Every unary function on (A, F) is polynomial.*

Lemma 2.2. *Every 3×3 matrix over the set $\{0, 1, 2, *\}$ containing at most 3 numbers is polynomial.*

Lemma 2.3. *For 3×3 matrices K, L, M over the $\{0, 1, 2, *\}$, let us define the matrix KLM by the following way:*

$$KLM(x, y) = K(L(x, y), M(x, y)).$$

(The value $KLM(x, y)$ is not determined in the following three cases:
- *the value $L(x, y)$ is not determined,*
- *the value $M(x, y)$ is not determined,*
- *the values $L(x, y) = a$, $M(x, y) = b$ are determined, but $K(a, b)$ is not.) If the matrices K, L, M are polynomial, then the matrix KLM is polynomial, too.*

Lemma 2.4. *Let M be a polynomial matrix. Then the matrix K defined by*

$$K(x, y) = M(y, x)$$

is polynomial, too.

Lemma 2.5. *Every matrix of the form*

$$G = \begin{vmatrix} a & a & a \\ b & b & b \\ c & c & c \end{vmatrix}$$

(resp. the transposed matrix) is polynomial.

Proof. Let us define the unary function

$$f(0) = a,\ f(1) = b,\ f(2) = c.$$

By Lemma 2.1, the function f is polynomial. Moreover, $G(x, y) = f(x)$.

Lemma 2.6. *Assume that M is a polynomial matrix and that the matrix K can be obtained from M by one of the following operations:*

1) *some permutation of the rows (columns),*
2) *replacing of some row (column) by any other row (column).*

Then the matrix K is polynomial, too.

Proof. For example, we wish to exchange the first two rows of the matrix M. By Lemma 2.1, the unary function

$$f(0) = 1,\ f(1) = 0,\ f(2) = 2$$

is polynomial. Moreover, it holds $M(x, y) = M(f(x), y)$.
Another example - we wish to replace the column 0 by the column 2. It suffices to use the function

$$f(0) = 2,\ f(1) = 1,\ f(2) = 2$$

and the equality $M(x, y) = M(x, f(y))$.

Lemma 2.7. *Let M be a polynomial matrix and let f be an unary function. Then the matrix fM defined by*

$$fM(x, y) = f(M(x, y))$$

is polynomial, too.

Lemma 2.8. *The algebra (A, F) is functionally complete iff the following two matrices are polynomial:*

$$S = \begin{vmatrix} 0 & 1 & 2 \\ 1 & * & * \\ 2 & * & * \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 & * \\ 0 & 1 & * \\ 0 & 2 & * \end{vmatrix}.$$

Proof. Apply Lemma 2.1 and the Composition theorem. (The matrix S corresponds to "+" and the matrix P corresponds to "$\cdot$".) Here we assume that (A, F) has the 3-interpolation property.

Remark. In the following theorem, we will explicitly write all assumptions.

Theorem 2.1. *Assume that $A = \{0, 1, 2\}$ and that the algebra (A, F) has the 3-interpolation property. Then (A, F) is functionally complete.*

Proof. We are going to prove that the matrices S, P (Lemma 2.8) are polynomial. By Lemma 2.2, the matrices

$$S_1 = \begin{vmatrix} 0 & 1 & * \\ 1 & * & * \\ * & * & * \end{vmatrix}, \quad Q = \begin{vmatrix} 0 & 1 & * \\ 2 & * & * \\ * & * & * \end{vmatrix}$$

are polynomial. Applying Lemma 2.6, we succesively obtain the following polynomial matrices:

$$S_2 = \begin{vmatrix} 0 & 1 & * \\ 1 & * & * \\ 0 & 1 & * \end{vmatrix}, \quad S_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & * & 1 \\ 0 & 1 & 0 \end{vmatrix}, \quad S_4 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & * & 1 \end{vmatrix}.$$

Applying Lemma 2.4 and Lemma 2.6, we obtain the polynomial matrices

$$S_5 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & * \\ 0 & 0 & 1 \end{vmatrix}, \quad S_6 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & * \end{vmatrix}.$$

Direct calculations give (see Lemma 2.3)

$$QS_6S_3 = \begin{vmatrix} 0 & 1 & 2 \\ 1 & * & * \\ 2 & * & * \end{vmatrix} = S.$$

Proposition 2.1. *The matrix* $P' = \begin{vmatrix} 0 & 0 & * \\ 0 & 1 & * \\ * & * & * \end{vmatrix}$ *is polynomial.*

Let us continue in the proof of Theorem 2.1. By Proposition 2.1, Lemma 2.6 and Lemma 2.3, we obtain the polynomial matrices

$$P_1 = \begin{vmatrix} 0 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{vmatrix}, \quad P_2 = \begin{vmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 1 & * \end{vmatrix}, \quad P = QP_2P_1.$$

It remains only to prove Proposition 2.1. We know that the matrix Q is polynomial. The value $Q(1, 1)$ is not determined and it is easy to see that at least one of the following three matrices is polynomial:

$$P_3 = \begin{vmatrix} 0 & 1 & * \\ 2 & 0 & * \\ * & * & * \end{vmatrix}, \quad P_4 = \begin{vmatrix} 0 & 1 & * \\ 2 & 1 & * \\ * & * & * \end{vmatrix}, \quad P_5 = \begin{vmatrix} 0 & 1 & * \\ 2 & 2 & * \\ * & * & * \end{vmatrix}.$$

By Lemma 2.7, at least one of the following two matrices is polynomial:

$$P_6 = \begin{vmatrix} 0 & 1 & * \\ 0 & 0 & * \\ * & * & * \end{vmatrix}, \quad P_7 = \begin{vmatrix} 1 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{vmatrix}.$$

It suffices to apply Lemma 2.6.

Theorem 2.2. *Assume that* $A = \{0, 1, 2\}$ *and that the algebra* (A, F) *has the following properties:*

1) *every unary function is polynomial,*
2) *there exists a binary polynomial function* Q *such that*

$$Q(0,0) = 0, \;\; Q(0,1) = 1, \;\; Q(1,0) = 2.$$

Then (A, F) *is functionally complete.*

Proof. The binary function Q corresponds to the matrix Q in the proof of Theorem 2.1. Starting from Q, it is possible to derive S_1 and P_2 (apply Lemma 2.6 and Lemma 2.7).

Corollary. *Assume that* $A = \{0, 1, 2\}$ *and that the algebra* (A, F) *has the following properties:*

1) *among unary polynomial functions there exist at least one transposition, at least one 3-cycle and at least one function with exactly 2 values,*
2) *among binary polynomial functions there exists a function* G *and there exist* $a, b, c, d \in A$ *such that*

$$\{G(a,c), \;\; G(a,d), \;\; G(b,c), \;\; G(b,d)\} = A.$$

Then (A, F) *is functionally complete.*

3. Four-element and "big" algebras

In the case of 4-element algebras we can assume that $A = \{0, 1, 2, 3\}$ and the binary functions will be represented by 4×4 matrices. The results above (Lemma 2.1 – Lemma 2.8) must be modified (it is easy). The fundamental matrices in the modification of Lemma 2.8 will be

$$S = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & * & * & * \\ 2 & * & * & * \\ 3 & * & * & * \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 2 & * & * \\ 0 & 3 & * & * \end{vmatrix}.$$

Theorem 3.1. *Assume that* $A = \{0, 1, 2, 3\}$ *and that the algebra* (A, F) *has the 4-interpolation property. Then* (A, F) *is functionally complete.*

Proof. Starting from the "fundamental" polynomial matrix

$$Q = \begin{vmatrix} 0 & 1 & * & * \\ 2 & 3 & * & * \\ * & * & * & * \\ * & * & * & * \end{vmatrix},$$

we successively obtain the polynomial matrices

$$P_1 = \begin{vmatrix} 0 & 0 & * & * \\ 0 & 1 & * & * \\ * & * & * & * \\ * & * & * & * \end{vmatrix}, \quad P_2 = \begin{vmatrix} 0 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \\ 0 & 1 & * & * \end{vmatrix}, \quad P_3 = \begin{vmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 1 & * & * \end{vmatrix},$$

$P = QP_3P_2$,

$$S_1 = \begin{vmatrix} 0 & 1 & * & * \\ 1 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{vmatrix}, \quad S_2 = \begin{vmatrix} 0 & 1 & * & * \\ 0 & 1 & * & * \\ 1 & * & * & * \\ 1 & * & * & * \end{vmatrix}, \quad S_3 = \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & * & * \\ 1 & 1 & * & * \end{vmatrix},$$

$$S_4 = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & * & 1 & * \\ 0 & 1 & 0 & 1 \\ 1 & * & 1 & * \end{vmatrix}, \quad S_5 = QS_3S_4 = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & * & 3 & * \\ 2 & 3 & * & * \\ 3 & * & * & * \end{vmatrix}.$$

Trivially, S is a special case of S_5.

Theorem 3.2. *Assume that $A = \{0, 1, 2, 3\}$ and that the algebra (A, F) has the following properties:*

1) *every unary function is polynomial,*
2) *there exists a binary polynomial function Q such that $Q(0,0) = 0$, $Q(0,1) = 1$, $Q(1,0) = 2$, $Q(1,1) = 3$.*

Then (A, F) is functionally complete.

Corollary. *Assume that $A = \{0, 1, 2, 3\}$ and that the algebra (A, F) has the following properties:*

1) *among unary polynomial functions there exist at least one 4-cycle, at least one 3-cycle and at least one function with exactly 3 values,*
2) *among binary polynomial functions there exists a function G and there exist $a, b, c, d \in A$ such that $\{G(a,c),\ G(a,d),\ G(b,c),\ G(b,d)\} = A$.*

Then (A, F) is functionally complete.

Remark. The method of the proof of Theorem 3.1 can be applied more generally.

Theorem 3.3. *Assume that $A = \{0, 1, 2, \ldots, k-1\}, k > 5$ and that the algebra (A, F) has the k-interpolation property. Then (A, F) is functionally complete.*

The idea of the proof. First modify the results above (Lemma 2.1 – Lemma 2.8) and prove that there exists an integer m such that the following inequalities are satisfied:

$$m^2 \geq k, \ 2m \leq k.$$

The fundamental polynomial matrix Q can be defined by the following way:

$$Q(x, y) = mx + y,\ x < m,\ y < m,\ mx + y < k.$$

(All other values are not determined.) We leave the details to the reader. This method is not convenient in the case $k = 5$ because there exists no integer m satisfying the inequalities

$$m^2 \geq 5,\ 2m \leq 5.$$

Example 3.1. Put $k = 7$. The inequalities

$$m^2 \geq 7,\ 2m \leq 7$$

have the solution $m = 3$. Here the matrices S, P are:

$$S = \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & * & * & * & * & * & * \\ 2 & * & * & * & * & * & * \\ 3 & * & * & * & * & * & * \\ 4 & * & * & * & * & * & * \\ 5 & * & * & * & * & * & * \\ 6 & * & * & * & * & * & * \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * \\ 0 & 2 & * & * & * & * & * \\ 0 & 3 & * & * & * & * & * \\ 0 & 4 & * & * & * & * & * \\ 0 & 5 & * & * & * & * & * \\ 0 & 6 & * & * & * & * & * \end{vmatrix}$$

and the fundamental polynomial matrix is:

$$Q = \begin{vmatrix} 0 & 1 & 2 & * & * & * & * \\ 3 & 4 & 5 & * & * & * & * \\ 6 & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{vmatrix}.$$

The derivations of S, P from Q we leave to the reader. (Use the same method as in the case $k = 4$.)

4. Five-element algebras

Theorem 4.1. *Assume that $A = \{0, 1, 2, 3, 4\}$ and that the algebra (A, F) has 5-interpolation property. The (A, F) is functionally complete.*

Proof. The fundamental matrices in the modification of Lemma2.8 will be

$$S = \begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & * & * & * & * \\ 2 & * & * & * & * \\ 3 & * & * & * & * \\ 4 & * & * & * & * \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 4 & * & * & * \end{vmatrix}.$$

By the 5-interpolation property the following two matrices are polynomial:

$$Q = \begin{vmatrix} 0 & 1 & 2 & * & * \\ 3 & 4 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}, \quad R = \begin{vmatrix} 0 & 1 & 2 & * & * \\ 1 & * & * & * & * \\ 2 & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}.$$

From Q we derive polynomial matrices

$$S_1 = \begin{vmatrix} 0 & 1 & * & * & * \\ 1 & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}, \quad S_2 = \begin{vmatrix} 0 & 1 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 1 & * & * & * \\ 1 & * & * & * & * \\ 1 & * & * & * & * \end{vmatrix}, \quad S_3 = \begin{vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & * & * \\ 1 & 1 & 1 & * & * \end{vmatrix}$$

and from R we derive polynomial matrices

$$S_4 = \begin{vmatrix} 0 & 1 & 2 & * & * \\ 1 & * & * & * & * \\ 2 & * & * & * & * \\ 0 & 1 & 2 & * & * \\ 1 & * & * & * & * \end{vmatrix}, \quad S_5 = \begin{vmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & * & * & 1 & * \\ 2 & * & * & 2 & * \\ 0 & 1 & 2 & 0 & 1 \\ 1 & * & * & 1 & * \end{vmatrix}.$$

The matrix QS_3S_5 is a special case of S. It is more difficult to derive the matrix P. By the 5-interpolation property, the matrices

$$P_1 = \begin{vmatrix} * & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}, \quad Q' = \begin{vmatrix} 0 & * & * & * & * \\ * & 1 & 3 & * & * \\ * & 4 & 2 & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}$$

are polynomial. From P_1 we derive polynomial matrices

$$P_2 = \begin{vmatrix} * & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \end{vmatrix}, \quad P_3 = \begin{vmatrix} * & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 1 & * & * & * \end{vmatrix}.$$

Direct computations give

$$P_4 = Q'P_2P_3 = \begin{vmatrix} * & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 4 & * & * & * \end{vmatrix}.$$

In the case $P_4(0,0) = 0$, the proof is finished ($P_4 = P$). All other cases are equivalent and we can assume that $P_4(0,0) = 1$. Then we have the polynomial matrix

$$P_5 = \begin{vmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 4 & * & * & * \end{vmatrix}.$$

By the 4-interpolation property, the matrices

$$P_6 = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}, \quad U = \begin{vmatrix} 0 & 2 & * & * & * \\ 3 & 1 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}$$

are polynomial. From P_6 we derive the polynomial matrices

$$P_7 = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \end{vmatrix}, \quad P_8 = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{vmatrix}.$$

Direct computations give

$$P_9 = UP_7P_8 = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 0 & * & * & * \end{vmatrix}.$$

By the 5-interpolation property, the matrix

$$P_{10} = \begin{vmatrix} * & * & * & * & * \\ 0 & 1 & * & * & * \\ * & * & 2 & * & * \\ * & * & * & 3 & * \\ 4 & * & * & * & * \end{vmatrix}$$

is polynomial. From P_{10} we derive the polynomial matrix

$$P_{11} = \begin{vmatrix} 0 & 1 & * & * & * \\ 0 & 1 & * & * & * \\ * & * & 2 & * & * \\ * & * & * & 3 & * \\ 4 & * & * & * & * \end{vmatrix}$$

and direct computations give: $P = P_{11}P_5P_9$.

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