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(1,1)-FORMS AND CONNECTIONS ON A VECTOR BUNDLE TM

ANTON DEKRÉT

ABSTRACT. In this paper the subset of the space of the (1,1)-forms on TM on which there exist natural operators over id_{TM} transforming these forms into connections on TM and all these operators are determined.

INTRODUCTION

Let (x^i, x_1^i) be a local map on the tangent bundle $p : TM \to M$. A connection Γ on TM can be determined by its horizontal form h_Γ that is a (1,1)-form on TM such that $T_p \cdot h_\Gamma = T_p$, $h_\Gamma(Y) = 0$, $Y \in VTM$, where T_p denotes the tangent map of p and VTM is the vector bundle of all vertical vectors on TM. Γ_j^i are called the Christoffel's functions in coordinates $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma_j^i(x, x_1) dx^j \otimes \partial/\partial x_1^i$.

Let $J = dx^i \otimes \partial/\partial x_1^i$ be the canonical (1,1)-form on TM. Grifone [1], 1972, defined a connection on TM as a such (1,1)-form φ on TM that $\varphi J = -J$, $J\varphi = J$. Its local form is $\varphi = dx^i \otimes \partial/\partial x^i + (\varphi_j^i dx^j - dx_1^i) \otimes \partial/\partial x_1^i$. Evidently $\varphi \mapsto \frac{1}{2}(\varphi + id_{TTM}) = h_\Gamma$ is a bijection between the Grifone forms and the horizontal forms of connections.

Let $(1,1) \equiv C^\infty T^*TM \otimes TTM$ or CTM be the space of all smooth (1,1) forms or of all connections on TM. In this paper we determine the domain of natural operators of zero order from (1,1) into CTM and construct the list of these operators. We use the natural bundle theory, see for example [2], [3].

OPERATORS OF ZERO ORDER FROM (1,1) INTO CTM

In the natural bundle theory the effort to determine all natural operators of zero order from (1,1) into CTM over id_{TM} is equivalent to the one to determine all smooth natural transformations from the bundle $T^*TM \otimes TTM$ into CTM over Id_{TM}. As the natural bundles $T^*TM \otimes TTM$ and CTM are associated to the principal fibre bundle H^2M of the frames of second order on M with the type fibres $S_1 = (T^*TR^m \otimes TTR^m)_0$ and

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$S_2 = (CTR^m)_0$, in order to determine all above mentioned transformations we need to find all L^2_m- equivariant maps from S_1 into S_2, where L^2_m is the differential group of the second order.

Let $\overline{x}^i = f^i(x^j)$ be a local diffeomorphism $f : R^m \to R^m$, $f(0) = 0$. Then $TTf : TTR^m \to TTR^m$ is of the form.

$$\overline{x}^i = f^i(x^j),\ \overline{x}^i_1 = f^i_j x^j_1,\ d\overline{x}^i = f^i_j dx^j,\ d\overline{x}^i_1 = f^i_{jk} x^j_1 dx^k_1 + f^i_j dx^j_1$$

where $f^i_j := \frac{\partial f^i(0)}{\partial x^j}$, $f^i_{jk} = \frac{\partial f^i(0)}{\partial x^i \partial x^k}$. Then we find the action of L^2_m on S_1 and on S_2 as follows; on S_1:

(1)
$$\begin{aligned}
&\overline{x}^i_j = f^i_j x^j_1,\\
&\overline{a}^i_j = f^i_s a^s_k \tilde{f}^k_j - f^i_t b^t_k \tilde{f}^k_s f^s_{uv} x^u_1 \tilde{f}^v_j\ ,\ \overline{b}^i_s = f^i_q b^q_j \tilde{f}^j_s\ ,\ f^i_s \tilde{f}^s_j = \delta^i_j\ ,\\
&\overline{c}^i_j = (f^i_{sk} x^s_1 a^k_t + f^i_k c^k_t)\tilde{f}^t_j - (f^i_{sk} x^s_1 b^k_u + f^i_k h^k_u)\tilde{f}^u_r f^n_{pq} x^p_1 \tilde{f}^q_j\ ,\\
&\overline{h}^i_j = (f^i_{ks} x^k_1 b^s_u + f^i_k h^k_u)\tilde{f}^u_j\ ,
\end{aligned}$$

on S_2 :

$$\overline{x}^i_j = f^i_j x^j_1,\ \overline{\Gamma}^i_j = f^i_{ks} x^k_1 \tilde{f}^s_j + f^i_k \Gamma^k_s \tilde{f}^s_j\ ,$$

where $(x^i_j, a^i_j, b^i_j, c^i_j, h^i_j)$ or (x^i_j, Γ^i_j) are coordinates on S_1 or on S_2.

Let $\alpha = (a^i_j dx^j + b^i_j dx^j_1) \otimes \partial/\partial x^i + (c^i_j dx^j + h^i_j dx^j_1) \otimes \partial/\partial x^i_1$ be a (1,1)-form on TM. Then $J\alpha J = b^i_j dx^j \otimes \partial/\partial x^i_1$ is a L^1_m-tensor. There are two cases

$$1. \qquad J\alpha J = 0, \quad \text{i.e.} \quad b^i_j = 0.$$

We have two forms $a := J\alpha = a^i_j dx^j \otimes \partial/\partial x^i_1$, $h := \alpha J = h^i_j dx^j \otimes \partial/\partial x^i_1$. Denote $\overline{S}_1$ the subspace of S_1 of such (1,1)-forms α for which $J\alpha J = 0$.

A map $\Phi : \overline{S}_1 \to S_2 : \overline{x}^i_1 = x^i_1,\ \Gamma^i_j = \Phi^i_j(x^q_1, a^q_p, c^q_p, h^q_p)$ is L^2_m-equivariant iff

(2)
$$f^i_{ks} x^k_1 \tilde{f}^s_j + f^i_k \Phi^k_s(x^q_1, a^q_p, c^q_p, h^q_p) = \Phi^i_j(\overline{x}^q_1, \overline{a}^q_p, \overline{c}^q_p, \overline{h}^q_p),$$

where $\overline{x}^q_1, \overline{a}^q_p, \overline{c}^q_p, \overline{h}^q_p$ are given by the equations (1). The equations (2) according to the subgroup of homotheties $f = (k\delta^i_j = f^i_j,\ f^i_{jk} = 0), k \neq 0$, are of the form

$$\Phi^i_j(x^q_1, a^q_p, c^q_p, h^q_p) = \Phi^i_j(kx^q_1, a^q_p, c^q_p, h^q_p).$$

Therefore Φ^i_j are independent of (x^q_1). Then the Ker π^2_1-equivariancy, where $\pi^2_1 : L^2_m \to L^1_m$ is the subjet projection, gives

(3)
$$f^i_{kj} x^k_1 + \Phi^i_j(a^q_p, c^q_p, h^q_p) = \Phi^i_j(a^q_p, c^q_p + f^q_{uk} x^u_1 a^k_p - h^q_s f^s_{up} x^u_1, h^q_p).$$

Remark 1. If α is semibasic with value in VTM, i.e. if $a = 0, h = 0$ then (2) is not true. Then there is not a natural operator of zero order from the space of semibasic (1,1)-forms with values in VTM into the space CTM. This coincides with the result of Janyška [2] by which a (1,1)-form on M does not determine any linear connection.

It follows from (2) that $\frac{\partial \Phi^i}{\partial c^q_p}$ are constant on the Ker π^2_1-orbits the local equations of which are as follows:

$$\overline{x}^i_1 = x^i_1,\ \overline{a}^i_j = a^i_j,\ \overline{c}^i_j = c^i_j + F^i_k a^k_j - h^i_s F^s_j,\ \overline{h}^i_j = h^i_j,\ F^i_k := f^i_{uk} x^u_1. \tag{4}$$

There are two cases:

1a) The map $\kappa : (y^s) \mapsto (\delta^v_s a^k_u - h^v_s \delta^k_u) y^s_k$, i.e. $y \mapsto ya - hy$, on $(R^m)^* \otimes R^m$ is regular. Then it follows from (4) that $\frac{\partial \Phi^i_j}{\partial c^q_p}$ are independent of c^p_q. Then

$$\Phi^i_j = A^{iu}_{jv}(a^p_q, h^q_p) c^v_u + B^i_j(a^q_p, h^q_p)$$

and the equation of the Ker π^2_1-equivariancy of Φ^i_j is of the form

$$F^i_j = A^{iu}_{jv}(\delta^v_s a^k_u - h^v_s \delta^k_u) F^s_k\ ,$$

i.e. (A^{iu}_{jv}) are the coordinates of the map κ^{-1}. It is easy to see that Φ is L^2_m-equivariant iff $x^i_j = B^i_j(a^q_p, h^q_p)$ is a L^1_m-equivariant map from $\mathcal{R}^{m*} \otimes \mathcal{R}^m \times \mathcal{R}^{m*} \otimes \mathcal{R}^m$ into $\mathcal{R}^{m*} \otimes \mathcal{R}^m$.

Consider a map γ on $T^*M \otimes T(TM)$ over id_{TM} defined by the rule $\gamma : z \mapsto za - \alpha z$, $\gamma(z^i_j dx^j \otimes \partial/\partial x^i + \eta^i_j dx^j \otimes \partial/\partial x^i_1) = (z^i_s a^s_j - a^i_s z^s_j) dx^j \otimes \partial/\partial x^i + (-c^i_s z^s_j + \eta^i_s a^s_j - h^i_s \eta^s_j) dx^j \otimes \partial/\partial x^i_1$. When κ is regular there is a unique horizontal form $z_0 = dx^i \otimes \partial/\partial x^i + \eta^i_j dx^j \otimes \partial/\partial x^i_1$ such that $\gamma(z_0) = 0$ in coordinates $\eta^i_j = A^{iu}_{jv} c^v_u$. Denote by Γ_α the connection determined by z_0. We have proved the following proposition:

Proposition 1. *All natural operators Φ from the space of (1,1)-forms α on TM such that $J\alpha J = 0$ and the map κ_α is regular into the space of connections on TM over id_{TM} are of the form*

$$\Phi = \Gamma_\alpha + B(a, h)$$

where B is a natural operator of zero order from $C^\infty[(T^*M \otimes VTM) x_{TM} (T^*M \otimes VTM)]$ into $C^\infty(T^*M \otimes VTM)$ over id_{TM}.

1b) Let κ be singular. The equations $\overline{c}^i_j = c^i_j + (\delta^i_s a^k_j - h^i_s \delta^k_j) F^s_k$ of the Ker π^2_1-orbit can be written in the form

$$\begin{aligned} \overline{c}^i_j = c^i_j &+ (\delta^i_1 a^1_j - h^i_1 \delta^1_j) F^1_1 \cdots + (\delta^i_m a^1_j - h^i_m \delta^1_j) F^m_1 + (\delta^i_1 a^2_j - h^i_1 \delta^2_j) F^1_2 + \dots \\ &\cdots + (\delta^i_m a^2_j - h^i_m \delta^2_j) F^m_2 + \cdots + (\delta^i_m a^m_j - h^i_m \delta^m_j) F^m_m\ , \quad m = \dim M \end{aligned} \tag{5}$$

or shortly $\overline{C} = C + AF$.

Let the rank of κ be $h = sm + k,\ k < m, s < m$. Suppose that the first h columns of the matrix A are independent. Then the equations (5) are of the form

$$\overline{C} = C + A_1 F^1 + \cdots + A_m F^h, \tag{6}$$

where $A_1, \ldots, A_m$ denote the first h columns of the matrix A,

$$F^I = F^I + t^I_{m+1} F^{m+1} + \cdots + t^I_{m^2-h} F^{m^2-h}, \quad I = 1, \ldots, h$$
$$F^1 := F^1_1,\ F^2 := F^2_1, \ldots, F^{m^2} := F^m_m,\ C^1 := C^1_1, \ldots, C^{m^2} := C^m_m$$

and t^I_u are the coefficients of the linear combination of the column A_u according to the columns $A_1, \ldots, A_h$. Without loss of generality we suppose that the first h rows of the matrix A are independent. Using the first h equations of (6) we get

$$F^\xi = \tilde{A}^\xi_\eta(\overline{C}^\eta - C^\eta),\ \eta,\ \xi = 1, \ldots, h.$$

Then the last $m^2 - h$ equations of (6) give

$$\overline{C}^\beta = C^\beta + A^\beta_\xi \tilde{A}^\xi_\eta(\overline{C}^\eta - C^\eta), \quad \beta = h+1, \ldots m^2.$$

Then the equations of the Ker π^2_1-orbits are of the form

$$\overline{C}^\beta - A^\beta_\xi \tilde{A}^\xi_\eta C^\eta = C^\beta - A^\beta_\xi \tilde{A}^\xi_\eta C^\eta, \quad \overline{a}^i_j = a^i_j,\ \overline{h}^i_j = h^i_j.$$

This means that

$$\frac{\partial \Phi^i_j}{\partial c^q_p} = \psi^{ip}_{jq}(a^u_v, h^u_v, C^\beta - A^\beta_\xi \tilde{A}^\xi_\eta C^\eta).$$

For $c^q_p = C^\beta$ we have

$$\Phi^i_j = \int \psi^i_{j\beta}(a^u_v, h^u_v, C^\beta - A^\beta_\xi A^\xi_\eta C^\eta) dC^\beta.$$

Then $\Phi^i_j = \varphi^i_{j\beta}(a,h)C^\beta + \varphi^i_j(a,h,C^\eta)$ if $\psi^i_{j\beta}$ are independent of $u^\beta = C^\beta - A^\beta_\xi \tilde{A}^\xi_\eta C^\eta$ and
$\Phi^i_j = G^i_j(a,h,u)$ if $\psi^i_{j\beta}$ are dependent of u^β. As $\frac{\partial \Phi^i_j}{\partial C^\xi}$ are dependent of a, h, u only, there is $\varphi^i_j = \varphi^i_{j\xi}(a,h)C^\xi$. We conclude $\Phi^i_j = \varphi^{iq}_{jp}(a,h)c^p_q + G^i_j(a,h,u)$.

Now, the Ker π^2_1-equivariancy leads to the equations

$$F^i_j = \varphi^{iu}_{jv}(a,h)[\delta^v_s a^k_u - h^v_s \delta^u_k] F^s_k$$

from which it follows that κ must be regular. We have proved the following:

Proposition 2. *If $J\alpha J = 0$ and κ is singular then there is no connection on TM which is determined by the natural operator of zero order from the space of (1,1)-forms α that $J\alpha J = 0$ and κ is singular into the space of connections on TM.*

Example. In the case of a Grifone form $a^i_j = \delta^i_j,\ h^i_j = -\delta^1_j,\ J\alpha J = 0,\ \kappa = 2id$ and so $\Gamma^i_j = \frac{1}{2}c^i_j$ are the Christoffel's functions of the connection Γ_α.

2) Let $J\alpha J \neq 0$. There are two cases:

2a) Let $J\alpha J$ be regular, $\det(b^i_j) \neq 0$. Consider $J\alpha = (a^i_j dx^j + b^i_j dx^j_1) \otimes \partial/\partial x^i_1$. Denote by Γ_α the connection the horizontal distribution of which is spanned on vectors $X,\ J\alpha(X) = 0$, i.e. $\Gamma^i_j = -\tilde{b}^i_k a^k_j$ are the Christoffel's functions of Γ_α. Then any conection Γ on TM is of the form $\Gamma_\alpha + \varphi$, where $\varphi = \varphi^i_j dx^j \otimes \partial/\partial x^i_1$ is a semibasic (1,1)-form on TM with values in VTM. Now we find all natural operators φ of zero order form $C^\infty(T^*TM \otimes TTM)$ into $C^\infty(T^*M \otimes VTM)$, i.e. we find all L^2_m-equivariant maps from $(T^*TR^m \otimes T(TR^m))_0 = S_1$ into $(T^*R^m \otimes VTR^m)_0 = S_3$. In coordinates, we find all functions $\varphi^i_j(a,b,c,h)$ which satisfy:

$$f^i_k \varphi^k_s(a,b,c,d)\tilde{f}^s_j = \varphi^i_j(\overline{a},\overline{b},\overline{c},\overline{d}), \tag{7}$$

where $\overline{a},\overline{b},\overline{c},\overline{d}$ are determined by (1). In the case of the Ker π^2_1-equivariancy the equations (7) are of the form $\varphi^i_j(a,b,c,h) = \varphi^i_j(\overline{a},\overline{b},\overline{c},\overline{h})$. Therefore functions φ^i_j are constant on the Ker π^2_1-orbits, the equations of which are

$$\begin{aligned} &\overline{a}^i_j = a^i_j - b^i_k F^k_j,\ \overline{c}^i_j = F^i_k a^k_j + c^i_j - (F^i_k b^k_u + h^i_u)F^u_j \\ &\overline{h}^i_j = F^i_s b^s_j + h^i_j,\ \overline{b}^i_j = b^i_j. \end{aligned} \tag{8}$$

Calculating $F^u_v = \tilde{b}^u_k(a^k_v - \overline{a}^k_v)$ we get the equations

$$\begin{aligned} &b^s_k \overline{h}^k_p + \overline{a}^s_k b^k_p = b^s_k h^k_p + a^s_k b^k_p \\ &\overline{c}^q_p - \overline{h}^q_s \tilde{b}^s_v \overline{a}^v_p = c^q_p - h^q_s \tilde{b}^s_k a^k_p \\ &\overline{b}^i_j = b^i_j \end{aligned}$$

of the Ker π^2_1-orbits. This means that the functions φ^i_j are constant on the Ker π^2_1-orbits iff they are of the form

$$\varphi^i_j = \varphi^i_j(b^q_p, b^q_t h^t_p + a^q_t b^t_p,\ c^q_p - h^q_s \tilde{b}^s_k a^k_p).$$

It is easy to see that these functions are L^2_m-equivariant iff they are L^1_m-equivariant. Calculating

$$\begin{aligned} &J\alpha^2 J = (a^i_s b^s_j + b^i_s h^s_j)dx^j \otimes \partial/\partial x^i_1, \\ &\alpha - \alpha b^{-1} T_p \alpha = (c^i_j - h^i_s \tilde{b}^s_k a^k_j)dx^j \otimes \partial/\partial x^i_1 \end{aligned}$$

we can conclude:

Lemma 1. *All natural operators of zero order from the space of the (1,1)-forms on TM such that $J\alpha J$ is regular into the space of the semibasic (1,1)-forms with values in VTM are of the form $\varphi(J\alpha J, J\alpha^2 J, \alpha - \alpha\beta^{-1}T\pi\alpha)$ where φ is an operator of zero order from $C^\infty((T^*M \otimes VTM)x_{TM}T^*M \otimes VTMx_{TM}T^*M \otimes VTM))$ into $C^\infty(T^*M \otimes VTM)$.*

It immediately gives:

Proposition 3. *All natural operators of zero order from the space of the (1,1)-forms on TM such that $J\alpha J$ is regular into the space of connections on TM are of the form*

$$\Gamma_\alpha + \varphi(J\alpha J, J\alpha^2 J, \alpha - \alpha\beta^{-1}T\pi\varphi)$$

where φ is the operator described in Lemma 1.

2b) Let $J\alpha J$ be singular, $\det(b^i_j) = 0$. We find all L^2_m-equivariant maps $\Gamma^i_j = \Phi^i_j(x^k_1, a^q_p, b^q_p, c^q_p, h^q_p)$ from S_1 into S_2. It follows from the homothety subgroup equivariancy that Φ^i_j are independent of x^k_1. The condition of the subgroup Ker π^2_1-equivariancy is of the form

$$F^i_j + \Phi^i_j(a^q_p, b^q_p, c^q_p, h^q_p) = \Phi^i_j(\overline{a}^q_p, \overline{b}^q_p, \overline{c}^q_p, \overline{h}^q_p)$$

where $\overline{a}^q_p, \overline{b}^q_p, \overline{c}^q_p, \overline{h}^q_p$ are determined by the equations (8) which are the equations of Ker π^2_1-orbits on S_1. It is easy to see that the functions $\frac{\partial \Phi^i_j}{\partial c^q_p}$ are constant on the Ker π^2_1-orbits. The equations (8) can be arranged in the form

$$\begin{aligned}
&\overline{a}^q_p = a^q_p - b^q_r F^r_p \\
&\overline{b}^q_p = b^q_p,\ \overline{a}^q_k \overline{b}^k_p + \overline{b}^q_s \overline{h}^s_p = a^q_k b^k_p + b^q_s h^s_p :\equiv H^q_p \\
&\overline{b}^q_s(\overline{c}^s_t \overline{b}^t_p + \overline{h}^s_t \overline{h}^t_p) + \overline{a}^q_t \overline{H}^t_p = b^q_s(c^s_t b^t_p + h^s_t h^t_p) + a^q_t H^t_p
\end{aligned}$$

By the analogous procedure as in the case 1b) based on the equations $\overline{a}^q_p = a^q_p - b^q_r F^r_p$ we can deduce

Proposition 4. *If a (1,1)-form is such that $J\alpha J \neq 0$ but singular then there is no connection on TM which is determined by the natural operator of zero order from the space of such (1,1)-forms α on TM that $J\alpha J \neq 0$ but singular, into CTM.*

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Department of Mathematics, TU Zvolen, Masarikova 24,
960 54 Zvolen, SLOVAKIA

E-mail address: dekret@vsld.tuzvo.sk

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