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ON A CONGRUENCE LATTICE REPRESENTATIONS

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Abstract. In [2] the following problem is formulated: Is it true that for every n-unary algebra $\mathcal{A}$ (n finite), there exists a 2-unary algebra $\mathcal{B}$ with $Con\mathcal{A} \cong Con\mathcal{B}$? This paper contains contributions to the solution of this problem. Certain results concerning the mentioned problem can be found in [3]. In this paper some results of [3] are generalized. Moreover, results for the lattice of subuniverses and the authomorphism group are presented.

Representations of congruence lattice have been considered by many authors. The survay of basic results on this topic may be found in [1], [2] and [4].

In this paper the set of all positive integers will be denoted by N. Further, $Con\mathcal{A}$, $Sub\mathcal{A}$ and $Aut\mathcal{A}$ denote the congruence lattice, the lattice of subuniverses and the automorphism group of an algebra $\mathcal{A}$, respectively.

Let $(A, f_1, ..., f_n)$ be a unary algebra and let a be any (but fixed) element of $\mathcal{A}$. We define unary operations f, g on the set $B = A \times \{1, 2, ..., n+3\}$ as follows:

(1) $f(x,k) = (x, k+1)$ for $k \in \{1,2\}$, $f(x,3) = (x,1)$,
(2) $f(x,k) = (f_{k-3}(x), k)$ for $k \in \{4, ..., n+3\}$,
(3) $g(x,1) = (a,1)$, $g(x, n+3) = (x,1)$,
(4) $g(a,2) = (a,2)$, $g(a,k) = (a,k+1)$ for $k \in \{3, ..., n+2\}$,
(5) $g(x,k) = (x,k+1)$ for $x \neq a$, $k \in \{2, ..., n+2\}$

(see Fig.1).

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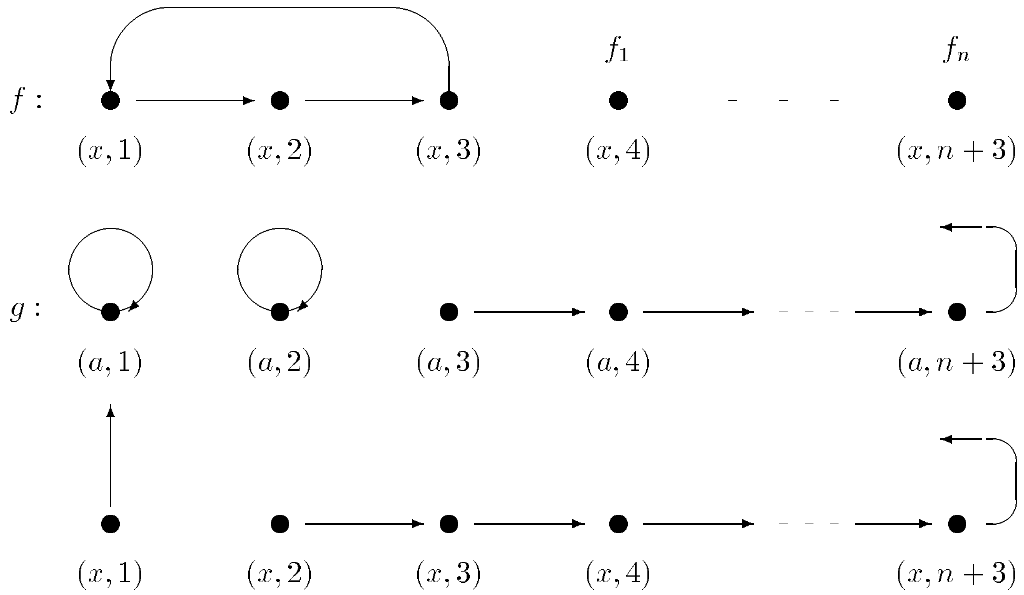

Fig. 1

Lemma 1. *Let* $(A, f_1, ..., f_n)$ *be a unary algebra,* $a \in A$ *and* (B, f, g) *be the unary algebra defined above, i.e., the operations* f, g *satisfy* $(1) - (5)$. *For any congruence relation* Φ *of the algebra* (B, f, g) *the following holds:*

(i) $(x,p)\,\Phi\,(y,p)$ *iff* $(x,q)\,\Phi\,(y,q)$ *for any* $p, q \in \{1, ..., n+3\}$,

(ii) $(a,p)\,\Phi\,(y,q)$ *and* $(a,p) \neq (y,q)$ *imply* $(a,i)\,\Phi\,(y,j)$ *for all* $i, j \in \{1, ..., n+3\}$,

(iii) $(x,p)\,\Phi\,(y,q)$ *and* $p \neq q$ *imply* $(a,1)\,\Phi\,(y,1)$.

Proof. (i). Let $(x,p)\Phi(y,p)$. Then

(a) $$f^k(x,p)\Phi f^k(y,p) \quad \text{and}$$

(b) $$g^k(x,p)\Phi g^k(y,p) \quad \text{for any } k \in N.$$

If $p \in \{1,2,3\}$ then from (a) we get

(c) $$(x,s)\Phi(y,s) \quad \text{for every} \quad s \in \{1,2,3\}.$$

Using (b), from $(x,3)\Phi(y,3)$ we get (c) for all $s \in \{4, ..., n+3\}$. If $p \in \{4, ..., n+3\}$ then from (b) we get $(x,1)\Phi(y,1)$ and then we proceed in the same way as in the previous case.

(ii). The assumption $(a,p)\Phi(y,q)$ implies

(d) $$f^k(a,p)\Phi f^k(y,q) \qquad \text{and}$$

(e) $$g^k(a,p)\Phi g^k(y,q) \qquad \text{for any} \quad k \in N.$$

Now we consider the following cases.

a) Let $y = a$ (i.e., $p \neq q$), $p, q \in \{1,2,3\}$. Then we get $(a,2)\Phi(a,3)$ from (d); thus, $g^k(a,2)\Phi g^k(a,3)$, $k \in N$, and, consequently, $(a,2)\Phi(a,s)$ for all $s = 3,4,...,n+3,1,2$.

b) Let $y = a$ and $p \leq 3 < q$. If $p = 2$ then from (e) we get $(a,2)\Phi(a,1)$, i.e., the case a). If $p = 1$ then from $f(a,1)\Phi f(a,q)$ we get $(a,2)\Phi(f_{q-3}(a),q)$, $g^k(a,2)\Phi g^k(f_{q-3}(a),q)$ and $(a,2)\Phi(a,1)$, respectively, i.e., the case a) again. For $p = 3$ we use $f^2(a,3)\Phi f^2(a,q)$, and then we proceed as in the previous case.

c) Let $y = a$ and $p, q \in \{4,...,n+3\}$. Then we get the case b) using (e) since Φ is symmetric.

d) Let $y \neq a$, $p,q \in \{1,2,3\}$. Then (d) implies $(a,2)\Phi(y,s)$ for some $s \in \{1,2,3\}$. Further, from $g^k(a,2)\Phi g^k(y,s)$, $k \in N$, we obtain $(a,2)\Phi(a,1)$ and then by a)

(f) $$(a,i)\Phi(a,j) \quad \text{for all} \quad i,j \in \{1,...,n+3\}.$$

The assumption $(a,p)\Phi(y,q)$ together with (f) and (i) implies

(g) $$(a,i)\Phi(y,j) \quad \text{for} \quad i,j \in \{1,...,n+3\}.$$

e) Let $y \neq a$, $p \leq 3 < q$. From (d) we get $(a,2)\Phi(z,q)$ for some $z \in A$. Thus, we obtain $g^k(a,2)\Phi g^k(z,q)$ for each $k \in N$ and so $(a,2)\Phi(a,1)$, and then we proceed as in the case d).

f) Let $y \neq a$ and $q \leq 3 < p$. We obtain $(z,p)\Phi(y,2)$ from (d) for some element $z \in A$. Thus, $g^k(z,p)\Phi g^k(y,2)$ for each $k \in N$ and, consequently, $(a,1)\Phi(y,s)$ for some $s \in \{4,...,n+3\}$, hence we have the case e).

g) Let $y \neq a$ and $p,q \in \{4,...,n+3\}$. Using (e) we obtain $(a,s)\Phi(y,1)$ or $(a,1)\Phi(y,s)$ for some s, which are again the previous cases.

(iii). Let $(x,p)\Phi(y,q)$ and $p \neq q$. For $y = a$ the statement is evident; for $x = a$ we use (ii).

Now let $x \neq a$, $y \neq a$. For $p = 1$ the assumption and the operation g imply $(a,1)\Phi(y,s)$, where $s = q+1$ or $s = 1$. Then by (ii) we get

(h) $$(a,1)\Phi(y,1).$$

If $1 < p < q$ then using the operation g we get $(x,s)\Phi(a,1)$ for some s. We obtain (h) by using (ii) and the hypothesis. If $q < p$ then analogously we obtain (h) by using (ii).

Theorem 1. *Let* $(A, f_1, ..., f_n)$ *be a unary algebra. Let there exist* $a \in A$ *such that*

$$a\Theta x \text{ and } a \neq x \quad \text{imply} \quad a\Theta f_i(x),\ i = 1, ..., n \tag{6}$$

for any congruence relation $\Theta \in Con\mathcal{A}$ *and any element* $x \in A$.
Then there exists a unary algebra (D, f, g) *such that* $Con\mathcal{A} \cong Con\mathcal{D}$.

Remark. An algebra $(A, f_1, ..., f_n)$ satisfies the assumptions of Theorem 1 if one of the following conditions holds:

(i) a is a fixed point of every operation f_i, $i = 1, 2, ..., n$.

(ii) If $x \neq a$ then $\Theta(x, a) = \nabla_A$
(∇_A means the largest element of $Con\mathcal{A}$).

(iii) If $i \in \{1, ..., n\}$ and $f_i(a) \neq a$ then $\Theta(a, f_i(a)) \leq \Theta(a, x)$
for every element $x \in A$, $x \neq a$.

Proof. Let (B, f, g) be an algebra whose operations are defined by $(1) - (5)$. We define a mapping $F : Con\mathcal{B} \to Con\mathcal{A}$ as follows

$$xF(\Phi)y \qquad \text{iff} \qquad (x, 1)\Phi(y, 1) \tag{7}$$

for any congruence relation $\Phi \in Con\mathcal{B}$.

a) We prove that the mapping F is well-defined, i.e., that $F(\Phi) \in Con\mathcal{A}$. Obviously, $F(\Phi)$ is an equivalence on A. Let $xF(\Phi)y$. Then by (7) and (i) we get $(x, p)\Phi(y, p)$ for all $p \in \{4, ..., n + 3\}$. So, we get
$f(x, p)\Phi f(y, p)$, $(f_{p-3}(x), p)\Phi(f_{p-3}(y), p)$, $(f_{p-3}(x), 1)\Phi(f_{p-3}(y), 1)$ (by (i))
and $f_{p-3}(x)F(\Phi)f_{p-3}(y)$, respectively, for $p = 4, ..., n + 3$.
Hence, $F(\Phi) \in Con\mathcal{A}$.

b) Using Lemma 1 we have
$\Phi_1 \leq \Phi_2$ iff $F(\Phi_1) \leq F(\Phi_2)$
for any congruences $\Phi_1, \Phi_2 \in Con\mathcal{B}$.

Now we consider two cases.

c1) Let a be the fixed point of all operations $f_1, ..., f_n$, i.e., $f(a, p) = (a, p)$ holds for all $p = 4, ..., n + 3$. We define the relation Ω on B as follows:

$$(x, p)\Omega(y, q) \quad \text{iff} \quad (x, p) = (y, q) \quad \text{or} \quad x = y = a. \tag{8}$$

Obviously, Ω is a congruence relation of the algebra (B, f, g).

Now we prove that F is a mapping of the interval $[\Omega, \nabla_B]$ of the lattice $Con\mathcal{B}$ onto $Con\mathcal{A}$. Let $\Theta \in Con\mathcal{A}$. We define a relation Φ on B by the rule:
$(x, p)\Phi(y, q)$ iff $x\Theta y$ and ($y\Theta a$ or $p = q$).
It is easy to show that Φ is an equivalence relation on B.

Let $x\Theta y\Theta a$. Then by (6) we have
$f_i(x)\Theta f_j(y)\Theta a$ for all $i, j \in \{1, 2, ..., n\}$,
and, consequently, $f(x, p)\Phi f(y, q)$. Obviously, we also get $g(x, p)\Phi g(y, q)$.

Let $p=q$ and $x\Theta y$. Then, $f_i(x)\Theta f_i(y)$ for all $i\in\{1,...,n\}$ which implies

(k) $$f(x,p)\Phi f(y,p) \qquad \text{for all } p\in\{4,...,n+3\}.$$

Obviously, (k) also holds for $p\in\{1,2,3\}$. Further, it is clear that $g(x,p)\Phi g(y,p)$. Thus, $\Phi\in Con\mathcal{B}$ and evidently $\Phi\geq\Omega$, $F(\Phi)=\Theta$.

Assume that $\Phi_1,\Phi_2\in[\Omega,\nabla_B]$, $\Phi_1\neq\Phi_2$. Then there are $(x,p),(y,q)\in B$ such that $(x,p)\Phi_1(y,q)$ but $(x,p)\,(B^2-\Phi_2)\,(y,q)$. We can suppose $x\neq a\neq y$. If $p=q$ then $xF(\Phi_1)y$ but $x\,(A^2-F(\Phi_2))\,y$ by Lemma 1. If $p\neq q$ then
$(x,1)\Phi_1(a,1)\Phi_1(y,1)$
by Lemma1 and hence $xF(\Phi_1)aF(\Phi_1)y$. If $xF(\Phi_2)aF(\Phi_2)y$ i.e., $(x,1)\Phi_2(a,1)\Phi_2(y,1)$, then $(x,p)\Phi_2(y,q)$, in view of Lemma 1(ii), since $x\neq a\neq y$, a cotradiction. Thus, the restriction of the mapping F to the interval $[\Omega,\nabla_B]$ is an isomorphism, and, consequently, $Con\mathcal{B}/\Omega\cong Con\mathcal{A}$. Hence, the statement holds for the algebra $(D,f,g)=(B/\Omega,f,g)$.

c2) Let f_i, $1\leq i\leq n$, be such an operation that $f_i(a)=b\neq a$. Now we prove that we can take $(D,f,g)=(B,f,g)$. If $a\Theta x$ for some element $x\in A$, $x\neq a$, then by (6) $a\Theta f_i(x)$, thus, $a\Theta f_i(a)$, i.e., $a\Theta b$. Hence, $\Theta(a,b)\leq\Theta(a,x)$ for every element $x\in A$, $x\neq a$.

If $(a,1)\Phi(b,1)$, $\Phi\in Con\mathcal{B}$, then using Lemma 1 (ii) we get $(a,p)\Phi(b,q)$ for all $p,q\in\{1,...,n+3\}$. Conversely, from $(a,p)\Phi(a,q)$, $p\neq q$ we have $(a,1)\Phi(a,i+3)$ using Lemma 1 (ii). Hence we get $f(a,1)\Phi f(a,i+3)$ and then $(a,2)\Phi(f_i(a),i+3)$. Then by Lemma 1 (iii) we obtain $(a,1)\Phi(f_i(a),1)$, i.e., $(a,1)\Phi(b,1)$. Using Lemma 1 again we conclude that F is an injection.

To prove that F is onto, suppose that $\Theta\in Con\mathcal{A}$. If $a\Theta b$ we define a relation Φ on B as follows:
$(x,p)\Phi(y,q)$ iff $x\Theta y$ and ($y\Theta a$ or $p=q$).
In the other case, the relation Φ is defined on B by the rule:
$(x,p)\Phi(y,q)$ iff $p=q$ and $x\Theta y$.
One can prove in a similar way as in c1) that in both cases $\Phi\in Con\mathcal{B}$. Obviously, $F(\Phi)=\Theta$ by Lemma 1. Hence, F is the isomorphism between $Con\mathcal{B}$ and $Con\mathcal{A}$.

Lemma 2. *Let $(A,f_1,...,f_n)$ be a unary algebra, $a\in A$ and (B,f,g) be the unary algebra whose operations are defined by (1) - (5). If ϕ is an automorphism of the algebra (B,f,g), then*

(m) $$\phi(a,i)=(a,i) \qquad \textit{for all} \qquad i\in\{1,...,n+3\},$$

(n) $$\phi(x,i)=(y,j) \quad \textit{yields} \quad i=j \textit{ and } \phi(x,s)=(y,s)$$

for all $s\in\{1,...,n+3\}$.

Proof. (m). If $\phi\in Aut\mathcal{B}$, then there exist elements (x,i), (y,j) such that $\phi(x,i)=(y,j)$, $(x,i)\neq(a,2)$ and $(y,j)\neq(a,2)$. Then

(p) $$\phi g^k(x,i)=g^k(y,j) \qquad \text{for any} \quad k\in N,$$

which implies $\phi(a,1)=(a,1)$. Thus $\phi f(a,1)=f(a,1)$, i.e., $\phi(a,2)=(a,2)$ and similarly $\phi(a,3)=(a,3)$. From $\phi g^k(a,3)=g^k(a,3)$ we obtain $\phi(a,i)=(a,i)$ for all $i\in\{4,...,n+3\}$.

(n). Let $\phi(x,i)=(y,j)$ and $i\neq j$. Then $x\neq a$ and $y\neq a$ by (m). Then (p) is valid and from (p) we conclude the existence of positive integer p such that

$\phi(x,p)=(a,1)$ or $\phi(a,1)=(y,p)$

which contradicts (m).

If $\phi(x,i)=(y,i)$, $x\neq a$ then $y\neq a$ and $\phi f^k(x,i)=f^k(y,i)$ and $\phi g^k(x,i)=g^k(y,i)$ imply $\phi(x,s)=(y,s)$ for every $s\in\{1,...,n+3\}$.

Theorem 2. *Let $(A,f_1,...,f_n)$ be a unary algebra, and $a\in A$ and (B,f,g) be the unary algebra whose operations are defined by (1) - (5). Then*

1) the group $Aut\mathcal{B}$ is isomorphic to the subgroup of those automorphisms $\phi\in Aut\mathcal{A}$ having a as a fixed point and

2) the lattice $Sub\mathcal{B}$ is isomorphic to the sublattice L of all subuniverses of the algebra $\mathcal{A}$ containing the element a.

Proof. 1) Define a mapping $F: Aut\mathcal{B}\to Aut\mathcal{A}$ as follows:

$F(\phi)(x)=y$ iff $\phi(x,1)=(y,1)$

for any $\phi\in Aut\mathcal{B}$ and any $x,y\in A$. Since ϕ is a bijection on B such that (n) holds, we conclude that $F(\phi)$ is a bijection taking A onto A. By (m) we get $F(\phi)(a)=a$. If f_i is an operation on A and $F(\phi)(x)=y$, then $\phi(x,i+3)=(y,i+3)$ by (n), which implies

$$\phi f(x,i+3)=f(y,i+3),\qquad \phi(f_i(x),i+3)=(f_i(y),i+3),$$
$$\phi(f_i(x),1)=(f_i(y),1),\qquad F(\phi)f_i(x)=f_i(y)=f_iF(\phi)(x),$$

thus $F(\phi)\in Aut\mathcal{A}$.

If $\phi_1\neq\phi_2$, then $F(\phi_1)\neq F(\phi_2)$ by (n). It is easy to prove that $F(\phi_1\circ\phi_2)=F(\phi_1)\circ F(\phi_2)$ for any $\phi_1,\phi_2\in Aut\mathcal{B}$.

If $\psi\in Aut\mathcal{A}$ such that $\psi(a)=a$ then one can easily verify that the mapping $\phi: B\to B$ satisfying the condition

$\phi(x,i)=(y,i)$ for all $i\in\{1,...,n+3\}$ iff $\psi(x)=y$

is an automorphism of the algebra (B,f,g), and obviously $F(\phi)=\psi$.

2) Let $F: Sub\mathcal{B}\to Sub\mathcal{A}$ be a mapping defined as follows:

$F(S)=\{s\in A;(s,1)\in S\}$ for any subuniverse $S\in Sub\mathcal{B}$.

a) We prove that F is well-defined, i.e., $F(S)\in Sub\mathcal{A}$. Take $s\in F(S)$. From $(s,1)\in S$ we get $f^k(s,1)\in S$ and $g^k(s,1)\in S$, $k\in N$; therefore,

$(s,i)\in S$ and $(a,i)\in S$

for all $i\in\{1,...,n+3\}$. From $(s,j+3)\in S$, $1\leq j\leq n$ we get $f(s,j+3)=(f_j(s),j+3)\in S$; thus, $(f_j(s),1)\in S$, i.e., $f_j(s)\in F(S)$.

b) If $\mathcal{A}_1$ is a subalgebra of the algebra $\mathcal{A}$ such that $a\in A_1$, then

$B_1=\{(s,i);s\in A_1,\quad i\in\{1,...,n+3\}\}$

is obviously a subuniversum of the algebra $\mathcal{B}$ and $F(\mathcal{B}_1)=\mathcal{A}_1$.

c) Clearly, F is one-to-one and

$$\mathcal{S}_1 \subseteq \mathcal{S}_2 \quad \text{iff} \quad F(\mathcal{S}_1) \subseteq F(\mathcal{S}_2)$$

for any subalgebras $\mathcal{S}_1, \mathcal{S}_2 \in Sub\mathcal{B}$, which completes the proof.

Remark 1. *If $a \in A$ is a fixed point of the operations $f_1, ..., f_n$ then in Theorem 2 one can replace the algebra (B, f, g) by the factor algebra $(B/\Omega, f, g)$ where Ω means the congruence relation given by (8).*

Corollary 1. *Let $(A, f_1, ..., f_n)$ be a unary algebra. If there exists an element $a \in A$ such that (6) holds and*

(i) $\phi(a) = a$ *for every automorphism* $\phi \in Aut\mathcal{A}$ *and*

(ii) *every subalgebra of the algebra $\mathcal{A}$ contains the element a*

then there exists a unary algebra (D, f, g) such that

$$Con\mathcal{A} \cong Con\mathcal{D}, \qquad Aut\mathcal{A} \cong Aut\mathcal{D} \qquad \text{and} \qquad Sub\mathcal{A} \cong Sub\mathcal{D}.$$

Although the propositions are stated for algebras with finitely many operations, it is also true for algebras with countably many operations.

Theorem 3. *Let $(A, f_1, ..., f_n, ...)$ be a unary algebra and $a \in A$ an element such that*

(9) $$a\Theta x \quad \text{and} \quad a \neq x \qquad \text{imply} \qquad a\Theta f_n(x)$$

for any congruence $\Theta \in Con\mathcal{A}$, any element $x \in A$ and any operation f_n, $n \in N$. Then there exists an algebra (D, f, g) such that

(j) $$Con\mathcal{D} \cong Con\mathcal{A},$$

(jj) $$Aut\mathcal{D} \cong Aut\mathcal{A}'$$

where $Aut\mathcal{A}'$ is a subgroup of all automorphisms $\phi \in Aut\mathcal{A}$ having the element a as a fixed point.

(jjj) $$Sub\mathcal{D} \cong Sub\mathcal{A}''$$

where $Sub\mathcal{A}''$ is a sublattice of all subuniverses of the lattice $Sub\mathcal{A}$ containing the element a.

Proof. We define operations f, g on the set $B = A \times N$ as follows (see Fig. 2) :

(1') $f(x, k) = (x, k+1)$ for $k \in \{1, 2\}$ or k odd, $k \geq 5$, and $f(x, 3) = (x, 1)$

(2') $f(x, 2k) = (f_{k-1}(x), 2k)$ for $k > 1$,

(3') $g(x, 1) = (a, 1)$,

(4') $g(x, 2) = (x, 1)$ for $x \neq a$ and $g(a, 2) = (a, 2)$

(5') $g(x, 2k+1) = (x, 2k+3)$, $g(x, 2k+2) = (x, 2k)$ for $k \geq 1$.

Analogously, as in previous parts, one can show that $f_n(a) = a$ for any $n \in N$ yields the crucial algebra (D, f, g) to be the algebra $(B/\Omega, f, g)$, where Ω means the congruence of the algebra (B, f, g) given by (8). If there exists $n \in N$ such that $f_n(a) \neq a$ then $(D, f, g) = (B, f, g)$.

Corollary 2. *(Kogalovskij, Soldatova). For any unary algebra $\mathcal{A}$ with a countable system of operations and a fixed point there exists a 2-unary algebra $\mathcal{B}$ for which*

1. *$Con\mathcal{B} \cong Con\mathcal{A}$ and*
2. *if $\mathcal{A}$ is finite then so is $\mathcal{B}$.*

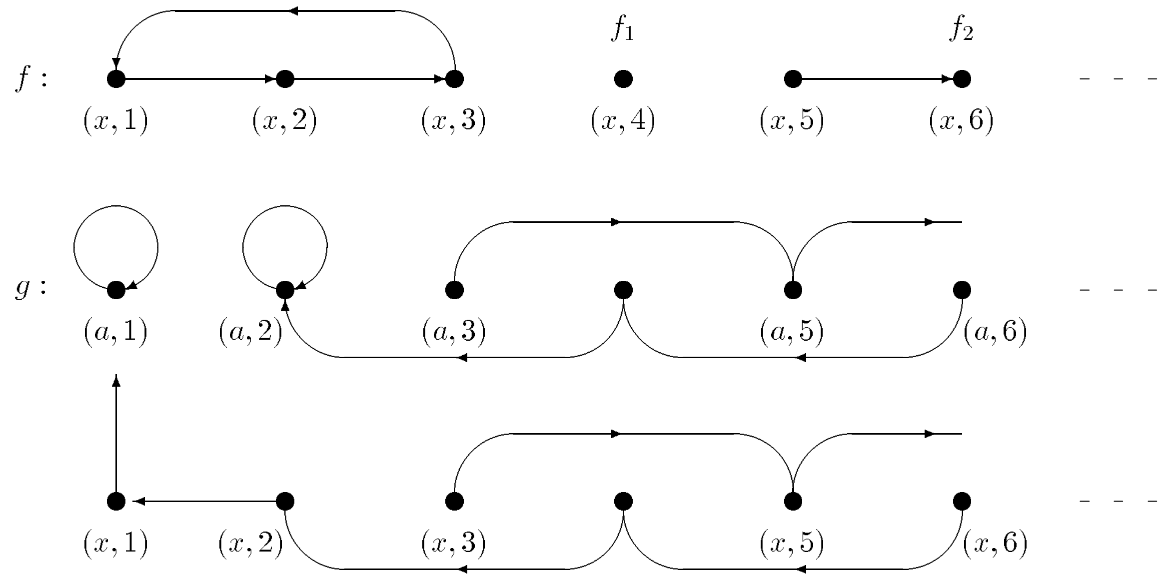

Fig. 2

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