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# AFFINE COMPLETE STONE AND POST ALGEBRAS OF ORDER N

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Abstract. In this paper we characterize affine complete Stone algebras of order $\leq n$ ($n \geq 3$) and we show that the variety of Stone algebras of order $\leq 3$ is affine complete. We also prove that each variety of Stone algebras of order $\leq n$ ($n \geq 2$) is locally affine complete. Finally, we show that the Post algebras (in the sense of [Ka-Mt 1972]) of order $n$ ($n \geq 2$) are affine complete.

## 1. Introduction.

G. Grätzer in [G 1962] showed that every compatible function on a Boolean algebra $\boldsymbol{B}$ (i.e. function preserving the congruences of $\boldsymbol{B}$) can be represented by a polynomial of $\boldsymbol{B}$. Later on, in [G 1964] he characterized those bounded distributive lattices of which all compatible functions are polynomials. These were the first results leading to the study of affine complete algebras. By H. Werner [W 1971], an algebra $\boldsymbol{A}$ is called *affine complete* if all $n$-ary compatible functions on $\boldsymbol{A}$ are polynomials ($n \geq 1$). Further, an algebra $\boldsymbol{A}$ is said to be *locally affine complete* if any finite partial function in $A^n \to A$ (i.e. function whose domain is a finite subset of $A^n$) which is compatible (where defined) can be interpolated by a polynomial of $A$ (see e.g. [P 1972] or [Kaa-P 1987]; in [Sz 1986] or [Kaa-Ma-S 1985] the notion 'locally affine complete' has another meaning).

G. Grätzer in [G 1968] (Problem 6) posed the problem of characterizing affine complete algebras. It seems to be very hard to answer such a question in general. A list of particular varieties in which all affine complete members were characterized was published in [C-W 1981], and probably the most recent list of such varieties can be found in [Ha-Pl 1994].

Much can be said about affine completeness if one is interested in varieties of algebras of which all members are affine complete, i.e. *affine complete varieties.* Affine complete varieties have been examined in [Kaa-P 1987] (see also [P 1972], [P 1979]). For a survey of the most recent results concerning affine complete varieties see [P 1993].

In this paper we deal with a special class of Stone algebras - Stone algebras of order $n$. We mainly deal with equationally definable Stone algebras of order $\leq n$ ($n \geq 2$) introduced by T. Katriňák and A. Mitschke [Ka-Mt 1972]. Stone algebras of order n represent one of the best-known generalizations of the Post algebras of order n, which are the algebras corresponding to $n$-valued propositional logic for $n \geq 2$.

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We first note that the Stone algebras of order $\leq 2$ are affine complete. Then we prove that a Stone algebra $\boldsymbol{L}$ of order $\leq n$ $(n \geq 3)$ is affine complete iff its dense filter $D(\boldsymbol{L})$ is an affine complete Stone algebra of order $\leq n-1$ and some extension property holds for partial compatible functions of $D(L)$. We show that the Stone algebras of order $\leq 3$ are affine complete, hence this variety can be added, as a new member, to the list in [Ha-Pl 1994]. Afterwards we show that each variety of Stone algebras of order $\leq n$ $(n \geq 2)$ is locally affine complete. In the last part of this paper we prove that the subdirectly irreducible Post algebra (in the sense of [Ka-Mt 1972]) of order $n$ $(n \geq 2)$ is primal. By [Hu 1971] this yields that each variety of Post algebras of order $n$ $(n \geq 1)$ is affine complete.

**2. Preliminaries.** A *(distributive) p-algebra* is an algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$ such that $(L; \vee, \wedge, 0, 1)$ is a bounded (distributive) lattice and $^\star$ is the unary operation of pseudocomplementation defined by

$$a^\star = \max\{x \in L \mid x \wedge a = 0\} \quad (a \in L).$$

A *Stone algebra* is a distributive p-algebra satisfying the identity $x^\star \vee x^{\star\star} = 1$.

In any Stone algebra $\boldsymbol{L}$, two subsets of $L$ play an important role. The subset $D(\boldsymbol{L}) = \{x \in L \mid x^\star = 0\} = \{x \vee x^\star \mid x \in L\}$ of all *dense* elements of $\boldsymbol{L}$ which forms a filter in $\boldsymbol{L}$, and the subset $B(\boldsymbol{L}) = \{x \in L \mid x = x^{\star\star}\} = \{x^\star \mid x \in L\}$ of all *closed* elements of $\boldsymbol{L}$ which is a Boolean subalgebra of $\boldsymbol{L}$.

We first recall the definition of Stone algebras of order $n$ $(n \geq 1)$ as a subclass of the Stone algebras (see [Ba-D 1975; p. 206]):

Let $\boldsymbol{L}$ be a Stone algebra. $\boldsymbol{L}$ is a *Stone algebra of order 1*, if $\boldsymbol{L} = \mathbf{1}$. $\boldsymbol{L}$ is a *Stone algebra of order* $n$ $(n \geq 2)$, if $L \neq \mathbf{1}$ and $D(\boldsymbol{L})$ is a Stone algebra of order $n-1$.

Let $\boldsymbol{L}$ be a Stone algebra of order $n$ $(n \geq 2)$. We define
$D^0(\boldsymbol{L}) = \boldsymbol{L}$,
$D^i(\boldsymbol{L}) = D(D^{i-1}(\boldsymbol{L}))$ for $i = 1, \ldots, n-1$ .
Hence
$\boldsymbol{L} = D^0(\boldsymbol{L}) \supseteq D^1(\boldsymbol{L}) \supseteq \cdots \supseteq D^{n-1}(\boldsymbol{L}) = \mathbf{1}$
are Stone algebras. We shall denote the smallest element of $D^i(\boldsymbol{L})$ by $d_i$ for $i = 1, \ldots, n-1$ , hence $D^i(\boldsymbol{L}) = [d_i)$. The chain $d_1 < \cdots < d_{n-1} = 1$ is said to be the *chain of smallest dense elements* of $\boldsymbol{L}$.

Before giving the definition of the Stone algebras of order $n$ $(n \geq 2)$ as an equational class, we recall some other necessary notions and results.

A *Brouwerian algebra* is an algebra $\boldsymbol{L} = (L; \vee, \wedge, *)$ where $(L; \vee, \wedge)$ is a lattice, and $*$ is the binary operation of relative pseudocomplementation defined by the rule
$x \leq y * z$ iff $x \wedge y \leq z$ for all $x, y, z \in L$.
One can show that $\boldsymbol{L}$ is distributive and has the greatest element $x * x$ denoted by 1. The class of all Brouwerian algebras is equational (see [Bi 1967] or [Ka-Mt 1972]).

A Brouwerian algebra satisfying the identity $x*y \vee y*x = 1$ is called a *relative Stone algebra.* If a Brouwerian algebra $\boldsymbol{L}$ has the smallest element 0, then the algebra $(L; \vee, \wedge, *, 0, 1)$ is called a *Heyting algebra.* Such algebra is also pseudocomplemented if one puts $x^\star = x*0$ . The following rules of computation in Brouwerian (Heyting) algebras will be useful (see [Ba-D 1975; p. 174]):

(a) $x \leq y$ iff $x * y = 1$
(b) $y \leq x * y$
(c) $(x \vee y) * z = (x * z) \wedge (y * z)$
(d) $x * (y \wedge z) = (x * y) \wedge (x * z)$
(e) $(x * y)^\star = x^{\star\star} \wedge y^\star$.

Let $\boldsymbol{L}$ be a Brouwerian algebra and $\theta$ be a congruence relation on $\boldsymbol{L}$. The set $F_\theta = \{x \in L \mid x \equiv 1\ (\theta)\}$ is a filter of $\boldsymbol{L}$. Congruence relations of Brouwerian algebras can be characterized as follows (see [Ne 1965]):

**2.1 Proposition.** *Let $\boldsymbol{L}$ be a Brouwerian algebra. If $\theta$ is a congruence relation on $\boldsymbol{L}$, then $x \equiv y\ (\theta)$ iff $x \wedge d = y \wedge d$ for a suitable $d \in F_\theta$. If $F$ is a filter on $\boldsymbol{L}$, then the binary relation $\theta(F)$ defined by*
*$x \equiv y\ (\theta(F))$ iff $x \wedge d = y \wedge d$ for a suitable $d \in F$*
*is a congruence relation on $\boldsymbol{L}$.*

Thus the lattice of congruences of a Brouwerian algebra $\boldsymbol{L}$ is isomorphic to the lattice of all filters of $\boldsymbol{L}$, hence it is distributive. Further, it is well-known that Brouwerian (Heyting) algebras have a Congruence Extension Property (CEP).

In [Ka-Mt 1972], Stone algebras of order $\leq n$ $(n \geq 2)$ are characterized as follows:

**2.2 Proposition** ([Ka-Mt 1972; 5.2]). *An algebra $\boldsymbol{L} = (L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ is a Stone algebra of order $\leq n$ $(n \geq 2)$ with a chain $e_0 \leq \cdots \leq e_{n-1}$ of smallest dense elements if and only if it satisfies the lattice identities and the following list of identities:*
*(1)* $x \wedge [(x \wedge y) * z] = x \wedge (y * z)$
*(2)* $x \wedge [(y \wedge z) * z] = x$
*(3)* $x \wedge (x * y) = x \wedge y$
*(4)* $x * y \vee y * x = e_{n-1}$
*(5)* $e_{i+1} \wedge e_i = e_i$
*(6)* $e_{i+1} * e_i = e_i$
*(7)* $x \wedge e_{n-1} = x$
*(8)* $x \wedge e_0 = e_0$
*(9)* $e_{i+1} \wedge (x * e_i) * e_i = (x \wedge e_{i+1}) \vee e_i \quad (i \in \{0, \dots, n-2\})$.

The lattice identities and (1)-(3) above characterize Brouwerian algebras. The identity (4) guarantees that $\boldsymbol{L}$ is a relative Stone algebra. The identities (5), (7) and (8) establish the chain $0 = e_0 \leq \cdots \leq e_{n-1} = 1$ of smallest dense elements, while (6) and (9) state that $[e_{i+1})$ is the filter of all dense elements of $[e_i)$.

It is known that the identity (4) is equivalent to the identity $x * y \vee (x * y) * y = e_{n-1}$. Thus putting $x^\star = x * e_0$ we immediately get $x^\star \vee x^{\star\star} = e_{n-1}$ which means that a Stone algebra of order $\leq n$ can be considered as a Stone algebra as well. Hence in any Stone algebra of order $\leq n$, the equation
*(f)* $x = x^{\star\star} \wedge (x \vee e_1)$
holds. Further, the subsets $B(\boldsymbol{L})$ of all 'closed' elements of $\boldsymbol{L}$ and $D(\boldsymbol{L}) = \{x \vee x^\star;\ x \in L\}$ of all 'dense' elements of $\boldsymbol{L}$ can be defined as above, and the formulas
*(g)* $(x \wedge y)^\star = x^\star \vee y^\star$
and
*(h)* $(x \vee y)^\star = x^\star \wedge y^\star$

are true in $\boldsymbol{L}$. If y is a closed element of a Stone algebra $\boldsymbol{L}$ of order $\leq n$ then the element $x * y$ is also closed and

$(j)\quad x * y = x^\star \vee y$

holds (see [Ne 1965; Lemma 4.2]).

Finally, we mention the description of subdirectly irreducible Stone algebras of order $\leq n$ $(n \geq 2)$ given in [Ka-Mt 1972]. Let $\boldsymbol{L} = \{0 = a_0 < \cdots < a_{m-1} = 1\}$ $(1 \leq m \leq n)$ be an m-element chain. Obviously, $\boldsymbol{L}$ is a Stone algebra of order $\leq n$ if one puts $e_i = a_i$ for $i = 0, \ldots, m-1$ and $e_i = 1$ for $i = m, \ldots, n-1$. This algebra is usually denoted by $\boldsymbol{S_n(m)}$.

**2.3 Proposition** ([Ka-Mt 1972; 5.10]). *The only subdirectly irreducible Stone algebras of order $\leq n$ $(n \geq 2)$ are the algebras $\boldsymbol{S_n(m)}$ where $1 \leq m \leq n$.*

There are several ways to define Post algebras of order $n$ (see [Ba-D 1975]). In this paper we use the equational definition of Post algebas of order $n$ $(n \geq 2)$ presented in [Ka-Mt 1972; 5.3]:

**2.4 Proposition.** *An abstract algebra $\boldsymbol{L} = (L, \vee, \wedge, *, +, e_0, \ldots, e_{n-1})$ is a Post algebra of order $n$ $(n \geq 2)$ if and only if it satisfies the lattice identities, the identities (1)-(9) of Proposition 2.2 and*

(10) $\quad x \vee [(x \vee y) + z] = x \vee (y + z)$

(11) $\quad x \vee [(y \vee z) + z] = x$

(12) $\quad x \vee (x + y) = x \vee y$

(13) $\quad x + y \wedge y + x = e_0$

(14) $\quad e_{n-2} + e_{n-1} = e_{n-1}$.

Note that the identities (10)-(13) state that $\boldsymbol{L}$ is a dual relative Stone algebra, while (14) guarantees that the order of $\boldsymbol{L}$ is just $n$.

The subdirectly irreducible Post algebras of order $n$ are characterized as follows:

**2.5 Proposition** ([Ka-Mt 1972; 5.13]). *Let $\boldsymbol{L} = (L, \vee, \wedge, *, +, e_0, \ldots, e_{n-1})$ be a non-trivial Post algebra of order $n$ $(n \geq 2)$. The following conditions are equivalent:*

(1) *$\boldsymbol{L}$ is subdirectly irreducible;*

(2) *$L$ consists of $n$ elements;*

(3) *$\boldsymbol{L}$ is a chain.*

For these and other facts concerning Post algebras of order $n$ see [Ka-Mt 1972] or [Ba-D 1975].

Finally, recall that an algebra $\boldsymbol{A}$ is said to be *primal*, if it is finite and every function on $A$ is a term function of $\boldsymbol{A}$. Further, an algebra $\boldsymbol{A}$ is called *functionally complete* if every function on $A$ is a polynomial of $\boldsymbol{A}$. H. Werner in ([W 1970] showed that a finite algebra $\boldsymbol{A}$ is functionally complete iff the discriminator is a polynomial of $\boldsymbol{A}$. Recall that the discriminator of an algebra $\boldsymbol{A}$ is the ternary function defined on $A$ by the rule

$$d(x,y,z) = \begin{cases} z, & \text{if } x = y \\ x, & \text{if } x \neq y. \end{cases}$$

## 3. (Local) affine completeness.

We start with Stone algebras of order $n$ as a subclass of Stone algebras having a smallest dense element. We recall a result from [Be 1982]:

**3.1. Theorem** ([Be 1982; Theorem 4]). *Let $\boldsymbol{L}$ be a Stone algebra having a smallest dense element. Then the following are equivalent:*

(1) *$\boldsymbol{L}$ is affine complete;*
(2) *$D(\boldsymbol{L})$ is an affine complete lattice;*
(3) *no proper interval of $D(\boldsymbol{L})$ is Boolean.*

Now let $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$ be a Stone algebra of order $n$ ($n \geq 3$). Then by its definition, $D^{n-2}(\boldsymbol{L})$ is a Boolean algebra, thus $D(\boldsymbol{L})$ contains a Boolean interval. By Beazer's characterization above we get the following:

**3.2 Proposition.** *A Stone algebra $(L; \vee, \wedge, ^\star, 0, 1)$ of order $n$ is not an affine complete Stone algebra for $n \geq 3$.*

**3.3 Example.** Let $L = \{0, d, 1\}$ be a 3-element chain. Considering $L$ to be a p-algebra, $\boldsymbol{L}$ is a Stone algebra of order 3 with $D(\boldsymbol{L}) = \{d, 1\}$, $D^2(\boldsymbol{L}) = \{1\}$. By Proposition 3.2, $\boldsymbol{L}$ is not affine complete. We shall find a unary function on $\boldsymbol{L}$ which is compatible but not polynomial.

Take a function $f' : D(\boldsymbol{L}) \to D(\boldsymbol{L})$ defined by $f'(d) = 1$, $f'(1) = d$. Define $f : L \to L$ by the rule $f(x) = f'(x \vee d)$, i.e. $f(0) = f(d) = 1$, $f(1) = d$. Obviously, $f$ is compatible. Suppose that $f$ is a polynomial function of $\boldsymbol{L}$. Using the formulas (g) and (h), and the fact that $x^\star = 0$, $x^{\star\star} = 1$ for $x \in D(\boldsymbol{L})$, we get that $f' = f \restriction D(\boldsymbol{L})$ is a polynomial function of the lattice $D(\boldsymbol{L})$, thus an order-preserving function. This is, of course, a contradiction, hence $f$ cannot be a polynomial function of the algebra $\boldsymbol{L}$. $\square$

Next we shall deal with equationally definable Stone algebras of order $\leq n$ ($n \geq 2$) (see Proposition 2.2). First we present some preliminary lemmas.

**3.4 Lemma.** *Let $\boldsymbol{L} = (L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ and let $\boldsymbol{L}' = (L; \vee, \wedge, *)$ be its Brouwerian reduct. Then $\boldsymbol{L}$ and $\boldsymbol{L}'$ have the same congruences.*

The proof is straightforward.

**3.5 Lemma.** *Let $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ ($n \geq 2$) and let $\theta$ be a congruence of $\boldsymbol{L}$. Then $\theta \restriction D(\boldsymbol{L})$ is a congruence of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$.*

*Proof.* Evidently $\theta \restriction D(\boldsymbol{L})$ is a Brouwerian congruence as $D(\boldsymbol{L})$ is a Brouwerian subalgebra of $\boldsymbol{L}$. The statement now follows from Lemma 3.4. $\square$

**3.6 Lemma.** *Let $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ ($n \geq 2$) and let $s(\tilde{x})$ be a polynomial of $\boldsymbol{L}$. Then the function $s_1(\tilde{x}) : D(\boldsymbol{L})^n \to D(\boldsymbol{L})$ defined by $s_1(\tilde{x}) = s(\tilde{x}) \vee e_1$ is a polynomial function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$.*

*Proof.* We use induction on the length of the polynomial $s(\tilde{x})$. If $s(\tilde{x})$ is a variable or a constant symbol, then the statement is obvious. Now suppose that the statement holds for all polynomials $s(\tilde{x})$ of a length less than $k$ ($k > 1$) and let $s(\tilde{x})$ be of the length $k$. If

$s(\tilde{x}) = p(\tilde{x}) \vee q(\tilde{x})$ $(s(\tilde{x}) = p(\tilde{x}) \wedge q(\tilde{x}))$ for some polynomials $p(\tilde{x}), q(\tilde{x})$, then by induction hypothesis the statement is obviously true. Now let $s(\tilde{x}) = p(\tilde{x}) * q(\tilde{x})$. Gradually applying (f), (d), (j), (c), and (a) from Section 2, and the distributivity of $L$, we get

$$
\begin{aligned}
(k) \quad (p(\tilde{x}) * q(\tilde{x})) \vee e_1 &= (p(\tilde{x}) * [q(\tilde{x})^{\star\star} \wedge (q(\tilde{x}) \vee e_1)]) \vee e_1 \\
&= [p(\tilde{x}) * q(\tilde{x})^{\star\star} \wedge p(\tilde{x}) * (q(\tilde{x}) \vee e_1)] \vee e_1 \\
&= [(p(\tilde{x})^{\star} \vee q(\tilde{x})^{\star\star}) \wedge (p(\tilde{x}) \vee e_1) * (q(\tilde{x}) \vee e_1)] \vee e_1 \\
&= [(p(\tilde{x})^{\star} \vee e_1) \vee (q(\tilde{x})^{\star\star} \vee e_1)] \wedge [(p(\tilde{x}) \vee e_1) * (q(\tilde{x}) \vee e_1)]
\end{aligned}
$$

.

By induction hypothesis, $p(\tilde{x}) \vee e_1$ and $q(\tilde{x}) \vee e_1$ are polynomials of the algebra $D(\boldsymbol{L})$. Using (g), (h) and (e) from Section 2, we can transform $p(\tilde{x})^{\star}$ and $q(\tilde{x})^{\star\star}$ into polynomials of the Boolean algebra $(B(\boldsymbol{L}), \vee, \wedge, ^{\star}, e_0, e_{n-1})$, i.e. into forms

$$(l) \quad \bigvee_{(i_1,\dots,i_n)\in\{1,2\}^n} (\alpha(i_1, \dots, i_n) \wedge x_1^{i_1} \wedge \dots \wedge x_n^{i_n})$$

(see e.g. [Ba-D 1975; p. 92]) where $x_i^1$ and $x_i^2$ denote $x_i^{\star}$ and $x_i^{\star\star}$, respectively, $\alpha(i_1, \dots, i_n) \in B(\boldsymbol{L})$ and the join $\bigvee$ is taken over all n-tuples $(i_1, \dots, i_n) \in \{1,2\}^n$. Since the function $s_1(\tilde{x})$ is defined on $\mathrm{D}(\mathbf{L})$, we have in $(l)$ $x_i^{\star} = e_0$, $x_i^{\star\star} = e_{n-1}$ for any $x_i \in D(\boldsymbol{L})$, $i = 1, \dots, n$. Thus $p(\tilde{x})^{\star}, q(\tilde{x})^{\star\star}$ can be represented by some constants $p, q \in B(\boldsymbol{L})$. This means that in (k) we get a polynomial of the algebra $D(\boldsymbol{L})$. The proof is complete. □

**3.7 Lemma.** *Let $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ $(n \geq 2)$. Let $\theta_D$ be a congruence of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$. Then the equivalence $\theta$ on $\boldsymbol{L}$ defined by*

*(m) $x \equiv y\ (\theta)$ iff $x \vee e_1 \equiv y \vee e_1\ (\theta_D)$ and $x^{\star} \vee e_1 \equiv y^{\star} \vee e_1\ (\theta_D)$ and $x^{\star\star} \vee e_1 \equiv y^{\star\star} \vee e_1\ (\theta_D)$ is a congruence of the algebra $L$.*

*Proof.* Using the formulas (g) and (h) one can easily verify that $\theta$ is a congruence of the lattice $L$. Now let $x_i \equiv y_i\ (\theta)$, $x_i, y_i \in L$, $i = 1, 2$. Similarly as in (k) of Lemma 3.6 we get

$$
\begin{aligned}
(x_1 * x_2) \vee e_1 &= [(x_1^{\star} \vee e_1) \vee (x_2^{\star\star} \vee e_1)] \wedge [(x_1 \vee e_1) * (x_2 \vee e_1)] \\
&\equiv [(y_1^{\star} \vee e_1) \vee (y_2^{\star\star} \vee e_1)] \wedge [(y_1 \vee e_1) * (y_2 \vee e_1)] \\
&= (y_1 * y_2) \vee e_1\ (\theta_D).
\end{aligned}
$$

Using (e) we have

$$
\begin{aligned}
(x_1 * x_2)^{\star} \vee e_1 &= (x_1^{\star\star} \wedge x_2^{\star}) \vee e_1 = (x_1^{\star\star} \vee e_1) \wedge (x_2^{\star} \vee e_1) \\
&\equiv (y_1^{\star\star} \vee e_1) \wedge (y_2^{\star} \vee e_1) = (y_1 * y_2)^{\star} \vee e_1\ (\theta_D),
\end{aligned}
$$

and applying (g), (h) to this, we finally get

$$
\begin{aligned}
(x_1 * x_2)^{\star\star} \vee e_1 &= [(x_1^{\star} \vee x_2^{\star\star}) \wedge e_{n-1}] \vee e_1 = (x_1^{\star} \vee e_1) \vee (x_2^{\star\star} \vee e_1) \\
&\equiv (y_1^{\star} \vee e_1) \vee (y_2^{\star\star} \vee e_1) = (y_1 * y_2)^{\star\star} \vee e_1\ (\theta_D).
\end{aligned}
$$

Hence $x_1 * x_2 \equiv y_1 * y_2\ (\theta)$ which completes the proof of Lemma 3.7. □

In the following, $\tilde{x}$, $\tilde{x}^\star$ and $\tilde{x} \vee e_1$ will be abbreviations for $(x_1, \dots, x_n)$, $(x_1^\star, \dots, x_n^\star)$ and $(x_1 \vee e_1, \dots, x_n \vee e_n)$, respectively. Analogical meanings will have $\tilde{x}^\star \vee e_1$, $\tilde{x}^{\star\star}$ and $\tilde{x}^{\star\star} \vee e_1$. Futher, if we write e.g. $\tilde{x}^\star \vee e_1 \equiv \tilde{y}^\star \vee e_1\ (\theta)$, we mean that $x_i^\star \vee e_1 \equiv y_i^\star \vee e_1\ (\theta)$, $i = 1, \dots, n$.

The following condition defines an extension property for certain partial compatible functions of $D(\boldsymbol{L})$:

**3.8 Definition.** *Let $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ $(n \geq 2)$. We shall say that $\boldsymbol{L}$ satisfies a condition*

(D) *if for any compatible function $f : L^n \to L$ of the algebra $\boldsymbol{L}$, the partial function $f' : D(\boldsymbol{L})^{3n} \to D(\boldsymbol{L})$ such that*
$$f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = f(\tilde{x}) \vee e_1 \quad (\tilde{x} \in L^n)$$
*and $f'$ is undefined elsewhere can be extended to a total compatible function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$.*

**3.9 Remark.** Let us verify that $f'$ is a well-defined partial function which preserves the congruences of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ where defined. If $\theta_D$ is a congruence of the algebra $D(\boldsymbol{L})$ and $(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) \equiv (\tilde{y} \vee e_1, \tilde{y}^\star \vee e_1, \tilde{y}^{\star\star} \vee e_1)\ (\theta_D)$, then $\tilde{x} \equiv \tilde{y}\ (\theta)$ where $\theta$ is the congruence associated to $\theta_D$ in Lemma 3.7. Now $f(\tilde{x}) \equiv f(\tilde{y})\ (\theta)$ since $f$ is compatible, thus $f(\tilde{x}) \vee e_1 \equiv f(\tilde{y}) \vee e_1\ (\theta_D)$. Hence $f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) \equiv f'(\tilde{y} \vee e_1, \tilde{y}^\star \vee e_1, \tilde{y}^{\star\star} \vee e_1)\ (\theta_D)$ what was to be proved. Using the same procedure with $\theta_D = \triangle_{D(\boldsymbol{L})}$, the smallest congruence on $D(\boldsymbol{L})$, one can show that $f'$ is a well-defined partial function on $D(\boldsymbol{L})$. □

**3.10 Proposition.** *A Stone algebra $\boldsymbol{L} = (L; \vee, \wedge, *, e_0, e_1)$ of order $\leq 2$ is affine complete.*

*Proof.* The Boolean algebra $\boldsymbol{L_1} = (L; \vee, \wedge, ^\star, e_0, e_1)$ is affine complete ([G 1962]). We shall show that $\boldsymbol{L}$ and $\boldsymbol{L_1}$ have the same congruences. Obviously, every congruence of the Heyting algebra $\boldsymbol{L}$ is also a congruence of the p-algebra $\boldsymbol{L_1}$. Conversely, let $\theta$ be a congruence of the p-algebra $\boldsymbol{L_1}$ and $x_i \equiv y_i(\theta)$, $x_i, y_i \in L$, $i = 1, 2$. Since all elements of $\boldsymbol{L_1}$ are closed, we get $x_1 * x_2 = x_1^\star \vee x_2 \equiv y_1^\star \vee y_2 = y_1 * y_2(\theta)$, thus $\theta$ is a congruence of the algebra $\boldsymbol{L}$ too. Hence a function $f : L^n \to L$ preserving the congruences of the algebra $\boldsymbol{L}$ also preserves the congruences of $\boldsymbol{L_1}$, thus $f$ can be represented by a polynomial of the algebra $\boldsymbol{L_1}$. This polynomial, of course, can easily be rewritten as a polynomial of the algebra $\boldsymbol{L}$ replacing each $x^\star$ $(a^\star)$ by $x * e_0$ $(a * e_0)$. □

We get a characterization of affine complete Stone algebras of order $\leq n$ $(n \geq 3)$:

**3.11 Theorem.** *Let $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ be a Stone algebra of order $\leq n$ $(n \geq 3)$. Then $\boldsymbol{L}$ is affine complete if and only if $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ is an affine complete Stone algebra of order $n - 1$ and (D) holds.*

*Proof.* Let $\boldsymbol{L}$ be affine complete and $f' : D(\boldsymbol{L})^n \to D(\boldsymbol{L})$ be a compatible function on the algebra D(**L**). We define a function $f : L^n \to L$ as follows:
$$f(x_1, \dots, x_n) = f'(x_1 \vee e_1, \dots, x_n \vee e_1).$$
Clearly, $f \restriction D(\boldsymbol{L})^n = f'$. Using Lemma 3.5 one can show that $f$ is compatible on $\boldsymbol{L}$. So by the assumption, the function $f$ can be represented by a polynomial $s(\tilde{x}) =$

$s(x_1,\dots,x_n)$ of the algebra $\boldsymbol{L}$. Since in fact $f: L^n \to D(\boldsymbol{L})$, we have for any $\tilde{x} \in D(\boldsymbol{L})^n$, $f'(\tilde{x}) = f(\tilde{x}) = f(\tilde{x}) \vee e_1 = s(\tilde{x}) \vee e_1$, i.e. $f'$ is a polynomial function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ by Lemma 3.6.

To show (D), let $f: L^n \to L$ be a compatible function of the algebra $\boldsymbol{L}$ and let $f': D(\boldsymbol{L})^{3n} \to D(\boldsymbol{L})$ be the associated partial compatible function from Definition 3.8. Since $\boldsymbol{L}$ is affine complete, $f(x_1,\dots,x_n)$ can be represented by a polynomial $p(x_1,\dots,x_n)$ of $\boldsymbol{L}$. Define a function $f_1: L^n \to L$ by

$f_1(\tilde{x}) = f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = p(\tilde{x}) \vee e_1.$

Let $a_1,\dots,a_m$ be all constant symbols appearing in $p(\tilde{x})$. Then $p(\tilde{x})$ can be meant as a term $t(\tilde{x},\tilde{a}) = t(x_1,\dots,x_n,a_1,\dots,a_m)$ of the algebra $\boldsymbol{L_1} = (L; \vee, \wedge, *, a_1, \dots, a_m)$ where $t(x_1,\dots,x_{n+m})$ is a term of the Brouwerian algebra $(L; \vee, \wedge, *)$. Using the distributivity of $L$ and the formula $(x*y) \vee e_1 = [(x^\star \vee e_1) \vee (y^{\star\star} \vee e_1)] \wedge [(x \vee e_1) * (y \vee e_1)]$ (see the proof of Lemma 3.6), one can transform $t(\tilde{x},\tilde{a}) \vee e_1$ to a term $t_1(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1, \tilde{a} \vee e_1)$ of the algebra $(L; \vee, \wedge, *, ^\star, a_1, \dots, a_m, e_1)$ where $t_1(x_1,\dots,x_{3n+m})$ is a term of the algebra $(L; \vee, \wedge, *)$. Hence for any $\tilde{x} \in L^n$ we have

$f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = p(\tilde{x}) \vee e_1 = t(\tilde{x},\tilde{a}) \vee e_1 = t_1(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1, \tilde{a} \vee e_1)$ $= p_1(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1)$ for some polynomial $p_1(x_1,\dots,x_{3n})$ of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$. This polynomial obviously represents the required total compatible function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ extending the partial function $f'$.

Conversely, let $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \dots, e_{n-1})$ be affine complete and (D) holds. Let $f: L^n \to L$ be a compatible function of the algebra $\boldsymbol{L}$. Using (f) we can write

$(m) \qquad f(\tilde{x}) = f(\tilde{x})^{\star\star} \wedge (f(\tilde{x}) \vee e_1).$

We replace '$f(\tilde{x})^{\star\star}$' in (m) by a polynomial of $\boldsymbol{L}$. First we show that for any $\tilde{x} \in L^n$, $f(\tilde{x})^{\star\star} = f(\tilde{x}^{\star\star})^{\star\star}$. For any variable $x_i$ we have

$x_i \wedge (x_i \vee x_i^\star) = x_i^{\star\star} \wedge (x_i \vee x_i^\star),$

whence by Proposition 2.1, $x_i \equiv x_i^{\star\star}$ $(\theta(D(\boldsymbol{L}))$ where $\theta(D(\boldsymbol{L}))$ denotes the (Brouwerian) congruence associated to the filter D(**L**). Since $f$ is compatible, we have $f(\tilde{x}) \equiv f(\tilde{x}^{\star\star})$ $(\theta(D(\boldsymbol{L})))$, thus by Proposition 2.1 again, there exists an element $d \in D(\boldsymbol{L})$ such that $f(\tilde{x}) \wedge d = f(\tilde{x}^{\star\star}) \wedge d$. So we get $f(\tilde{x})^{\star\star} = (f(\tilde{x}) \wedge d)^{\star\star} = (f(\tilde{x}^{\star\star}) \wedge d)^{\star\star} = f(\tilde{x}^{\star\star})^{\star\star}$ what was to be proved.

Now we define a function $f_1: B(\boldsymbol{L})^n \to B(\boldsymbol{L})$ by the rule $f_1(\tilde{x}) = f(\tilde{x})^{\star\star}$. To show that $f_1$ is compatible on the algebra $(B(\boldsymbol{L}); \vee, \wedge, *, e_0, e_{n-1})$, let $\theta_B$ be a congruence of $B(\boldsymbol{L})$ and $x_i \equiv y_i$ $(\theta_B)$, $x_i, y_i \in B(\boldsymbol{L})$, $i = 1,\dots,n$. Obviously, $(B(\boldsymbol{L}); \vee, \wedge, *, e_0, e_{n-1})$ is a Heyting subalgebra of the Heyting algebra $(L; \vee, \wedge, *, e_0, e_{n-1})$. Since Heyting algebras have CEP, there exists an extension $\theta_L$ of the congruence $\theta_B$ to the Heyting algebra $\boldsymbol{L}$. Obviously, $\theta_L$ is a congruence of the algebra $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$ too. Hence $x_i \equiv y_i$ $(\theta_L)$, $i = 1,\dots,n$, therefore $f(\tilde{x}) \equiv f(\tilde{y})(\theta_L)$ since $f$ preserves the congruences of the algebra $\boldsymbol{L}$. It follows $f(\tilde{x})^{\star\star} \equiv f(\tilde{y})^{\star\star}(\theta_B)$, thus $f_1$ is compatible on $B(\boldsymbol{L})$. By Proposition 3.10, $(B(\boldsymbol{L}); \vee, \wedge, *, e_0, e_{n-1})$ is affine complete, thus $f_1$ can be represented by a polynomial $p(x_1,\dots,x_n)$ of the algebra $B(\boldsymbol{L})$. Hence in (m), $f(\tilde{x})^{\star\star} = f(\tilde{x}^{\star\star})^{\star\star} = f_1(\tilde{x}^{\star\star}) = p(\tilde{x}^{\star\star})$ for any $\tilde{x} \in L^n$. Finally, in the polynomial $p(\tilde{x}^{\star\star})$ we can put $x_i^{\star\star} = (x_i * e_0) * e_0$, $i = 1,\dots,n$, in order to get a polynomial of the algebra $(L; \vee, \wedge, *, e_0, \dots, e_{n-1})$.

To replace '$f(\tilde{x}) \vee e_1$' in (m) by a polynomial of $\boldsymbol{L}$, take the partial function $f'$ from

Definition 3.8. By (D), $f'$ can be extended to a total compatible function of the algebra $D(\boldsymbol{L})$, which can be represented by a polynomial $q(x_1,\ldots,x_{3n})$ of $D(\boldsymbol{L})$ since $D(\boldsymbol{L})$ is affine complete. Hence in (m) $f(\tilde{x})\vee e_1 = f'(\tilde{x}\vee e_1,\tilde{x}^\star\vee e_1,\tilde{x}^{\star\star}\vee e_1) = q(\tilde{x}\vee e_1,\tilde{x}^\star\vee e_1,\tilde{x}^{\star\star}\vee e_1)$ for any $\tilde{x}\in L^n$. Putting in $q(\tilde{x}\vee e_1,\tilde{x}^\star\vee e_1,\tilde{x}^{\star\star}\vee e_1)$ again $x_i^\star = x_i * e_0,\ x_i^{\star\star} = (x_i * e_0) * e_0,\ i = 1,\ldots,n,$ we get the required polynomial of the algebra $\boldsymbol{L}$. The proof is complete. □

We have shown that the Stone algebras of order $\leq 2$ are affine complete (Heyting) algebras. Now we prove that the Stone algebras of order $\leq 3$ are affine complete too.

**3.12 Proposition.** *Any Stone algebra $(L;\vee,\wedge,*,e_0,e_1,e_2)$ of order $\leq 3$ satisfies the condition (D).*

*Proof.* Let $D(\boldsymbol{L})\neq \mathbf{1}$ and let $S$ denote the domain of the partial compatible function $f'$, i.e.

$$S = \{(\tilde{x}\vee e_1,\tilde{x}^\star\vee e_1,\tilde{x}^{\star\star}\vee e_1);\ \tilde{x}\in L^n\}\subseteq D(\boldsymbol{L})^{3n}.$$

We define a polynomial function $p: D(\boldsymbol{L})^{3n}\to D(\boldsymbol{L})$ as follows:

$$(n)\quad p(x_1,\ldots,x_{3n}) = \bigvee_{\tilde{a}\in S\cap\{e_1,e_2\}^{3n}} (f'(a_1,\ldots,a_{3n})\wedge y_1\wedge\cdots\wedge y_{3n}),$$

where $y_i = \begin{cases} x_i * e_1, & \text{if } a_i = e_1\\ x_i, & \text{if } a_i = e_2.\end{cases}$

We show that $f'\equiv p$ on $S\cap\{e_1,e_2\}^{3n}$. Let $\tilde{x}\in S\cap\{e_1,e_2\}^{3n}$. For $\tilde{a}=\tilde{x}$ we have $f'(\tilde{a})\wedge y_1\wedge\cdots\wedge y_{3n} = f'(\tilde{x})\wedge e_2 = f'(\tilde{x})$. Now take $\tilde{a}\in S\cap\{e_1,e_2\}^{3n},\ \tilde{a}\neq\tilde{x}$ and $a_j\neq x_j$ for some j, $1\leq j\leq 3n$. Then

$$y_j = \begin{cases} x_j * e_1 = e_1, & \text{if } a_j = e_1\\ x_j = e_1, & \text{if } a_j = e_2,\end{cases}$$

thus $f'(\tilde{a})\wedge y_1\wedge\cdots\wedge y_{3n} = f'(\tilde{a})\wedge e_1 = e_1$. Hence in (n) we get $p(x_1,\ldots,x_{3n}) = f'(x_1,\ldots,x_{3n})$ what was to be proved. We assert that $f'\equiv p$ identically on the whole set $S$, thus that $p(x_1,\ldots,x_{3n})$ is the required total compatible extension of the partial function $f'$. To show this, suppose on the contrary that there exists a $3n$-tuple $(d_1,\ldots,d_{3n})\in S$ such that $f'(d_1,\ldots,d_{3n}) = a\neq b = p(d_1,\ldots,d_{3n})$. Since $a,b\in D(\boldsymbol{L})$ and $D(\boldsymbol{L})$ is a subdirect product of copies of $\mathbf{2}=\{0,1\}$, there exists a 'projection map' $h: D(\boldsymbol{L})\to\{0,1\}$, which is a $0,1$-homomorphism between the algebra $D(\boldsymbol{L})$ and some algebra $\mathbf{2}=\{0,1\}$, such that $h(a)\neq h(b)$. Denote $h(S) = \{(h(x_1),\ldots,h(x_{3n}))\in\{0,1\}^{3n}\mid (x_1,\ldots,x_{3n})\in S\}$. Now define functions $f_2',\ p_2': h(S)\cap\{0,1\}^{3n}\to\{0,1\}$ by the following rules:

$$f_2'(h(x_1),\ldots,h(x_{3n})) = h(f'(x_1,\ldots,x_{3n})),$$
$$p_2'(h(x_1),\ldots,h(x_{3n})) = h(p'(x_1,\ldots,x_{3n}))$$

where $(x_1,\ldots,x_{3n})\in S$. To show that $f_2',\ p_2'$ are well-defined, suppose that $h(x_1) = h(y_1),\ldots,h(x_{3n}) = h(y_{3n})$ for some $(x_1,\ldots,x_{3n}),\ (y_1,\ldots,y_{3n})\in S$. Since $f'$ ($p'$) preserves the kernel Ker h $=\theta_D$ of the homomorphism $h$ where defined, we get $f'(x_1,\ldots,x_{3n})\equiv f'(y_1,\ldots,y_{3n})\ (\theta_D)$, thus $f_2'(h(x_1),\ldots,h(x_{3n})) = f_2'(h(y_1),\ldots,h(y_{3n}))$ (analogously for $p_2'$). Obviously, $f_2'\equiv p_2'$ identically on $h(S)\cap\{0,1\}^{3n}$, because $f'\equiv p'$ identically on $S\cap\{e_1,e_2\}^{3n}$ and $h(e_1)=0,\ h(e_2)=1$. Therefore

$h(a) = h(f'(d_1,\ldots,d_{3n})) = f_2'(h(d_1),\ldots,h(d_{3n})) = p_2'(h(d_1),\ldots,h(d_{3n})) = h(p'(d_1,\ldots,d_{3n})) = h(b)$, a contradiction. Hence $f'\equiv p'$ identically on $S$ and the proof is complete. □

**3.13 Theorem.** *A Stone algebra* $(L; \vee, \wedge, *, e_0, e_1, e_2)$ *of order* $\leq 3$ *is affine complete.*

*Proof.* It follows from Theorem 3.11 and Propositions 3.10 and 3.12. $\square$

**3.14 Remark.** In Example 3.3 we illustrated the fact that a 3-element chain considered as a p-algebra is not an affine complete Stone algebra of order 3. This means that it has too little polynomials 'to cover' all its compatible functions. Of course, if we replace in a Stone algebra of order $n$ the operation $^\star$ by $*$ , the number of its polynomials will increase considerably. Theorem 3.13 says that for $n \leq 3$ they already 'cover' all compatible functions, i.e. the 3-element chain is an affine complete Heyting algebra. Note that the function $f$ from Example 3.3 can be represented simply by the polynomial $p(\tilde{x}) = x * d$.

By Theorems 3.11 and 3.13, a Stone algebra of order $\leq 4$ is affine complete iff $\boldsymbol{L}$ satisfies the condition (D). In general, we get the following result:

**3.15 Corollary.** *A Stone algebra* $\boldsymbol{L}$ *of order* $\leq n$ $(n \geq 4)$ *is affine complete if and only if* $D^i(\boldsymbol{L})$ *satisfies (D) for all* $i = 0, \ldots, n-4$.

The condition (D) above might actually be superfluous - we still do not know an example of a Stone algebra of order $\leq n$ in which (D) is not satisfied. Therefore we pose the following problem:

**3.16 Problem.** Find an example of a Stone algebra of order $\leq n$ in which the condition (D) is not satisfied or show that (D) in the characterizations above is superfluous.

The latter case would, of course, automatically mean that the variety of Stone algebras of order $\leq n$ is affine complete for $n > 3$ too.

However, we can show that each variety of Stone algebras of order $\leq n$ is locally affine complete $(n \geq 2)$:

**3.17 Theorem.** *Every Stone algebra* $(L; \vee, \wedge, *, e_0, \ldots, e_{n-1})$ *of order* $\leq n$ $(n \geq 2)$ *is locally affine complete.*

*Proof.* For $n = 2$ the result is well-known (see e.g. [P 1972]). Now let $n \geq 3$. Let $\boldsymbol{L}$ be locally affine complete and $f' : S \to D(\boldsymbol{L})$ be a partial compatible function of the algebra $D(\boldsymbol{L})$ where $S \subset D(\boldsymbol{L})^n$ is finite. One can easily verify that $f \equiv f'$ is a finite partial compatible function of the algebra $\boldsymbol{L}$ too. So by hypothesis, $f$ can be interpolated in all elements of $S$ by a polynomial $s(\tilde{x}) = s(x_1, \ldots, x_n)$ of the algebra $\boldsymbol{L}$. For any $\tilde{x} \in S$ we consequently have $f'(\tilde{x}) = f(\tilde{x}) = f(\tilde{x}) \vee e_1 = s(\tilde{x}) \vee e_1$, thus by Lemma 3.6, $f'$ can be interpolated on $S$ by polynomial function of the algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$.

Conversely, let $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$ be locally affine complete and let $f : S \to L$ be a finite partial compatible function of the algebra $\boldsymbol{L}$. Using (f) we can write again

$$(m) \qquad f(\tilde{x}) = f(\tilde{x})^{\star\star} \wedge (f(\tilde{x}) \vee e_1) \quad (\tilde{x} \in S).$$

Analogously as in the proof of Theorem 3.11 one can show that $f(\tilde{x})^{\star\star}$ can be interpolated in all $\tilde{x} \in S$ by a polynomial of the algebra $(L; \vee, \wedge, *, e_0, \ldots, e_{n-1})$.

To replace '$f(\tilde{x}) \vee e_1$' in (m) by a polynomial of $\boldsymbol{L}$ for all $\tilde{x} \in S$, take the partial function $f'$ defined by $f'(\tilde{x} \vee e_1, \tilde{x}^\star \vee e_1, \tilde{x}^{\star\star} \vee e_1) = f(\tilde{x}) \vee e_1$ for all $\tilde{x} \in S$. Since $f'$ is a finite partial compatible function of the locally affine complete algebra $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$, it can be interpolated by a polynomial of $(D(\boldsymbol{L}); \vee, \wedge, *, e_1, \ldots, e_{n-1})$. The proof is complete. $\square$

In the last part of this paper we show that for each $n \geq 2$, the variety of Post algebras of order $n$ is affine complete. (As there are various definitions of Post algebras of order $n$ in the literature, for some varieties of Post algebras of order $n$ such result might already be known). Here we make use of the result of T.K. Hu [Hu 1971] which yields that a variety generated by a primal algebra is affine complete. First we show that the discriminator function on the subdirectly irreducible Post algebra $\boldsymbol{L}$ of order $n$ is a polynomial of $\boldsymbol{L}$:

**3.18 Proposition.** *Let* $(L, \vee, \wedge, *, +, e_0, \ldots, e_{n-1})$ *be the subdirectly irreducible Post algebra of order* $n$ *(*$n \geq 2$*). Then the discriminator is a polynomial of* $\boldsymbol{L}$*.*

*Proof.* Define a binary function $b(x, y)$ on $L$ as follows:

$$b(x, y) = (x * y \wedge y * x) + e_{n-1}.$$

Obviously,

$$b(x, y) = \begin{cases} e_0, & \text{if } x = y \\ e_{n-1}, & \text{if } x \neq y. \end{cases}$$

One can easily verify that

$$d(x, y, z) = [b(x, y) \wedge x] \vee [b(b(x, y), e_{n-1}) \wedge z].$$

Hence the discriminator is a polynomial of $\boldsymbol{L}$. □

**3.19 Theorem.** *The variety of Post algebras of order* $n$ *(in the sense of* [Ka-Mt 1972]*) is affine complete.*

*Proof.* From Proposition 3.18 it follows that the subdirectly irreducible Post algebra $\boldsymbol{L}$ of order $n$ ($n \geq 2$) is functionally complete and from the fact that all constants $e_0, \ldots, e_{n-1}$ are the nullary operations of $\boldsymbol{L}$ we get that $\boldsymbol{L}$ is primal. Hence by Hu's result, the variety of Post algebras of order $n$ is affine complete. □

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