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THE EDGE DISTANCE IN SOME FAMILIES OF GRAPHS

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Abstract. The edge distance between graphs is defined by the equality $d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + ||V_1| - |V_2||$ where $|A|$ is the cardinality of A and $E_{1,2}$ is an edge set of the maximal common subgraph of G_1 and G_2. Further, $\operatorname{diam} F_{p,q} = max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$ where $F_{p,q}$ denotes the set of all graphs with p vertices and q edges. In this paper we prove that for $p \geqq 10$ $\operatorname{diam} F_{p,p+1} = 2p - 8$ and $\operatorname{diam} F_{p,p+2} = 2p - 6$. At the end of the paper we give the answer to a problem recently posed by M. Šabo.

1. Preliminaries

A graph $G = (V, E)$ consists of a non-empty finite vertex set V and an edge set E. In this paper we consider undirected graphs without loops and multiple edges. A subgraph H of the graph G is a graph obtained from G by deleting some edges and vertices; notation: $H \subseteq G$. By $\triangle(G)$ we denote the maximal degree of vertices of the graph G. A graph G is a common subgraph of graphs G_1, G_2 if there exist graphs H_1, H_2 such that $H_1 \subseteq G_1$, $H_2 \subseteq G_2$ and $H_1 \cong G$, $H_2 \cong G$. A maximal common subgraph is a common subgraph which contains the maximal number of edges.

A distance of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined (see [2]) by

$$d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + ||V_1| - |V_2|| \tag{1}$$

where $|E_1|$, $|E_2|$, $|V_1|$, $|V_2|$ are the cardinalities of the edge sets and the vertex sets respectively, and $|E_{1,2}|$ is the number of edges of a maximal common subgraph of G_1 and G_2.

Throughout this paper, by $F_{p,q}$ we denote the set of all graphs with p vertices and q edges, $q \geq 1$. Further, $\operatorname{diam} F_{p,q} := \max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$. If $\operatorname{diam} F_{p,q} = d(G, H)$ and $c_{p,q}$ is the number of edges of the maximal common subgraph of the graphs G, H, then

$$\operatorname{diam} F_{p,q} = 2q - 2c_{p,q}. \tag{2}$$

We denote by v a firmly chosen vertex of the maximal degree in the considered graph G and by $v_1, v_2, \ldots, v_k$ the vertices adjacent to v (if $\triangle(G) = k$). We denote $U :=$

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$\{v_1, v_2, \ldots, v_k\}$ and $U' := V - \{v, v_1, \ldots, v_k\}$. The subgraph of the graph G induced by the vertex set X ($X \subset V$) we denote by $G(X)$ and the set of its edges by $E(G(X))$ or briefly by $E(X)$. The subgraph of the graph G which contains all edges with one vertex in the set U and the other in the set U' we denote by $G(U, U')$ and the set of its edges by $E(U, U')$.

2. Diameters of $F_{p,p+1}$ and $F_{p,p+2}$

Lemma 1. *If $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 3$ then G contains at least two of the graphs H_1, H_2, H_3 (Fig.1).*

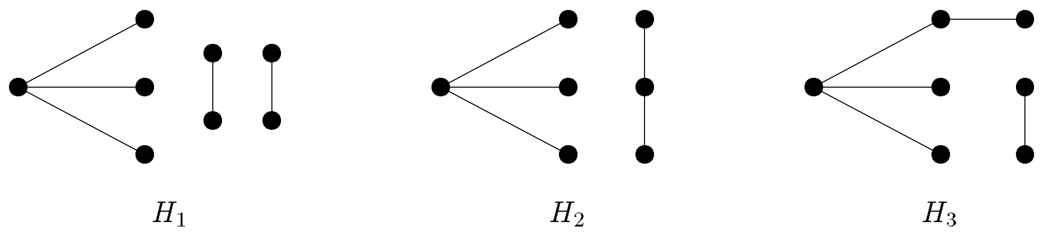


Fig.1

Proof. Since G has at least 11 edges and $\triangle(G) = 3$ then $|E(U')| \geqq 2$ from which it follows that G contains at least one of the graphs H_1 and H_2. Further we show that if G contains exactly one of the graphs H_1 and H_2 then G contains also the graph H_3. In fact, if G does not contain the graph H_1 then $G(U')$ has at most 3 edges. If G does not contain the graph H_2 then $G(U')$ has at most $[\frac{p-4}{2}]$ edges. In both cases $|E(U, U')| \geqq 2$ holds. Since $|E(U')| \geqq 2$ then from these facts it follows easily that G contains H_3.

Lemma 2. *If $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 3$ then G contains at least one of the graphs H_2 and H_4 (Figs.1,2).*

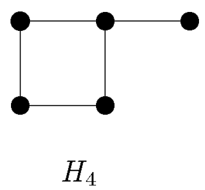
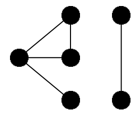


Fig.2

Proof. If G does not contain H_2 then $G(U')$ has the vertices of degree at most 1. It follows $|E(U)| + |E(U, U')| \geqq 5$ and this is possible only if $|E(U)| = 1$ or $|E(U)| = 0$ (since $\triangle(G) = 3$). The considered statement is easy to verify in both cases. In fact, if some vertex from U' has the degree at least 2 in $G(U, U')$ then G contains H_4. Otherwise G contains H_2.

Lemma 3. *If $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 3$ then G contains at least one of the graphs H_3 and H_5 (Figs.1,3).*



H_5

Fig.3

Proof. Obviously, G has a component H in which there are less vertices than edges. So, in H there is a vertex of degree 3 and we choose it as the vertex of degree 3 in the desired subgraph H_3 resp. H_5. Now it is sufficient to realize that H has at least 5 edges and G has at least 11 edges.

Lemma 4. *If $G \in F_{p,p+1}$ and $\triangle(G) = 3$ then G contains at least one of the graphs in Fig. 4.*

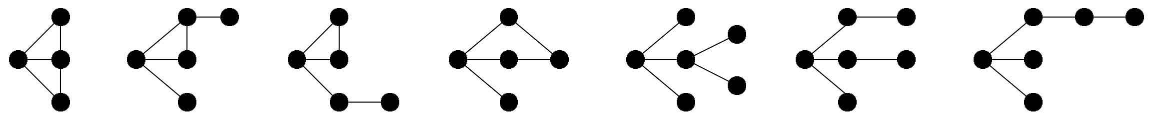


Fig.4

Proof. The graph G must have a component which has more edges than vertices. Obviously, this component contains a connected subgraph with five edges and with a vertex of degree 3. Since all such possibilities are listed in Fig. 4 the proof is finished.

Lemma 5. *Let $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 4$. If G contains neither the subgraph H_6 nor the subgraph H_7 (Fig. 5) then G contains at least one of the graphs in Fig. 4.*

H_6 H_7

Fig. 5

Proof. If G contains neither the subgraph H_6 nor the subgraph H_7 then $|E(U')| = |U'|+2$. So, the graph $G(U')$ must have a component having more edges than vertices. This component cannot have any vertex of degree 4, since otherwise the graph G would contain the subgraph H_6 or H_7, a contradiction. Then, by the same argument as in the proof of Lemma 4, the considered component contains at least one of the graphs in Fig. 4.

Lemma 6. *Each subgraph G of the graph K_6 with at least 11 edges contains the graphs H_6 and H_7 as well as each of the graphs in Fig. 4.*

Proof. It is sufficient to take into account that any vertex of a minimal degree in G has the degree at least 1 and the subgraph of the graph G induced by the set of the remaining vertices is the graph K_5 without at most two edges.

Lemma 7. *If $G_1, G_2 \in F_{p,p+1}, p \geqq 10$ and at least one of the graphs G_1, G_2 has a single non-trivial component which is a subgraph of the graph K_6 then $|E_{1,2}| \geqq 5$.*

Proof. Let, say, G_1 has the property that after removing its isolated vertices we get a subgraph of K_6. Let H be a component of the graph G_2 with more edges than vertices. If $\triangle(H) \geqq 5$ then H contains H_6 and it is sufficient to use Lemma 6. If $\triangle(H) \leqq 4$ then the statement is a consequence of Lemmas 4,5,6.

Theorem 8. *If $G_1, G_2 \in F_{p,p+1}, p \geqq 10$ and $\triangle(G_1) = \triangle(G_2) = 3$ then $|E_{1,2}| \geqq 5$.*

Proof. According to Lemma 1, the statement is obvious.

Theorem 9. *Let $G_1, G_2 \in F_{p,p+1}, p \geqq 10$, $\triangle(G_1) = 3$ and $\triangle(G_2) \geqq 4$. Then $|E_{1,2}| \geqq 5$.*

Proof. If $U' = \emptyset$ in the graph G_2 then $|E(U)| = 2$ and the considered statement is a consequence of Lemmas 1 and 2. So, in the sequel we suppose that $U' \neq \emptyset$. We distinguish three cases.

I. Let $E(U) \neq \emptyset$ and $E(U') \neq \emptyset$. We get the considered statement by Lemma 3 (obviously, the graph G_2 contains the graphs H_3 and H_5).

II. Let $E(U) = \emptyset$ and $E(U') \neq \emptyset$. Further we distinguish two subcases.

a) If $E(U, U') = \emptyset$ then $|E(U')| = |U'| + 2$ and $G_2(U')$ has a vertex of degree at least 3 and also two independent edges. Then the considered statement follows from Lemma 1.

b) Let $E(U, U') \neq \emptyset$. First of all we show that G_2 contains the subgraph H_2. If $\triangle(G_2) \geqq 5$ the statement follows from the fact that there is a vertex in U' which has the degree at least 2 (since $|E(U, U')| + |E(U')| = |U'| + 2$). If $\triangle(G_2) = 4$ the statement, obviously, holds if there is a vertex of U having the degree at least 3. In the opposite case it holds $|E(U')| \geqq |U'| - 2$ and since $|U'| - 2 > \frac{|U'|}{2}$, the graph $G_2(U')$ has a vertex of degree at least 2. So, G_2 really contains the graph H_2. Further it is possible to verify that G_2 contains at least one of the graphs H_3 and H_4. The considered statement follows from Lemmas 1 and 2.

III. Let $E(U') = \emptyset$.

a) Let $\triangle(G_2) \geqq 5$. If there is a vertex of U' having the degree at least 2 then G_2 contains H_2 and H_4, so the statement follows from Lemma 2. Further we can suppose that each vertex of U' has the degree at most 1. We get that $|E(U)| \geqq 2$.

a_1) First we suppose that $\triangle(G_2) \geqq 6$. If there are two adjacent edges in $G_2(U)$ then the considered statement follows from Lemma 2. In the opposite case it is sufficient to use Lemma 3.

a_2) Now, we suppose that $\triangle(G_2) = 5$. If $E(U, U') = \emptyset$ then the considered statement follows from Lemma 7. If $E(U, U') \neq \emptyset$ then there is an edge from $E(U)$ and an edge from $E(U, U')$, which are not adjacent. We get the considered statement by Lemma 3.

b) Let $\triangle(G_2) = 4$. According to Lemma 7 we can assume that there are at least two vertices from U' having non-zero degrees. If $|E(U)| \geqq 2$ then G_2 contains the subgraphs H_3 and H_5 and the considered statement is a consequence of Lemma 3. If $|E(U)| \leqq 1$ then some vertex of U has the degree at least 2 in the graph $G_2(U, U')$ and some vertex of U' has the degree at least 2 in the graph $G_2(U, U')$. So, G_2 contains the graphs H_2 and H_4 and according to Lemma 2 the proof is finished.

Lemma 10. *If $G \in F_{p,p+1}, p \geqq 10$ and $\triangle(G) = 4$ then G contains at least one of the graphs H_6 and H_8 (Figs. 5,6).*

H_8

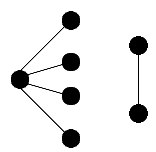


Fig. 6

Proof. It is sufficient to take into account that G has at least 11 edges and a graph with 5 vertices can have at most 10 edges.

Lemma 11. *If $G \in F_{p,p+1}, p \geqq 10$, $\triangle(G) = 4$ and G without its isolated vertices is not a subgraph of the graph K_6 then G contains at least one of the graphs H_2 and H_8 (Figs. 1,6).*

Proof. If G does not contain the subgraph H_8 then $E(U') = \emptyset$. Further, if there is a vertex from U having the degree at least 2 in $G(U, U')$ then G contains H_2. In the opposite case each vertex from U has the degree at most 1 in $G(U, U')$, i.e., $|E(U, U'| \leqq 4$. To finish the proof it suffices to consider all possible numbers of edges in $E(U, U')$.

Theorem 12. *If $G_1, G_2 \in F_{p,p+1}, p \geqq 10$ and $\triangle(G_1) = \triangle(G_2) = 4$ then $|E_{1,2}| \geqq 5$.*

Proof. The statement holds if both the graphs G_1, G_2 have the subgraph H_6 and also if they have the subgraph H_8. Then, without loss of generality we can assume that G_1 does not contain H_6 and G_2 does not contain H_8. Let us consider the graph H_7 in Fig. 5. If both the graphs G_1, G_2 contain H_7, the statement holds again. There are two possibilities in the opposite case:

a) G_1 contains H_7 and G_2 does not contain H_7. According to Lemma 10, the graph G_1 has the subgraph H_3 which is also contained in the graph G_2 because $|E(U,U')| = |U'| + 2$ and so $G_2(U,U')$ contains two independent edges.

b) G_1 does not contain H_7. According to Lemma 10, G_1 has the subgraph H_8. Further, the graph G_1 contains H_2, because $|E(U')| = |U'| + 2$ and so $G_1(U')$ contains a vertex of degree at least 3. Now it is sufficient to use Lemmas 7,11.

The proof is finished.

Theorem 13. *If $G_1, G_2 \in F_{p,p+1}, p \geqq 10$, $\triangle(G_1) = 4$ and $\triangle(G_2) \geqq 5$ then $|E_{1,2}| \geqq 5$.*

Proof. We can assume (according to Lemma 7) that neither the graph G_1 without its isolated vertices nor the graph G_2 without its isolated vertices is a subgraph of the graph K_6. Obviously, G_2 contains the graph H_8. If G_2 does not contain the graph H_6 then $|E(U)| = 0$ and also $|E(U,U')| = 0$. We get $|E(U')| = |U'| + 2$ and, obviously, $G_2(U')$ has a vertex of degree at least 3. This means that G_2 contains H_2. To finish the proof it is sufficient to use Lemmas 10 and 11.

Theorem 14. $\operatorname{diam} F_{p,p+1} = 2p - 8$ *for* $p \geqq 10$.

Proof. By Theorems 8, 9, 12 and 13 it suffices to find two graphs $G_1, G_2 \in F_{p,p+1}$ with $|E_{1,2}| = 5$. Such graphs are depicted in Fig. 7.

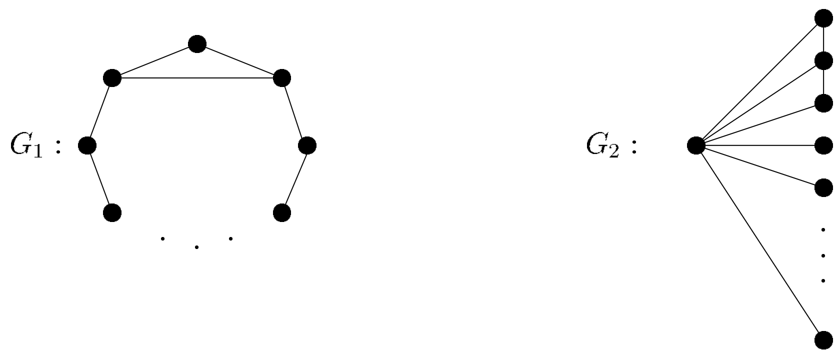


Fig. 7

Theorem 15. $\operatorname{diam} F_{p,p+2} = 2p - 6$ *for* $p \geqq 10$.

Proof. By Theorem 14 for any two graphs $G_1, G_2 \in F_{p,p+2}$ it holds $|E_{1,2}| \geqq 5$ (if $p \geqq 10$). Now it is sufficient to find two graphs $G_1, G_2 \in F_{p,p+2}$ for which $|E_{1,2}| = 5$ and the proof will be finished. Such graphs are depicted in Fig. 8.

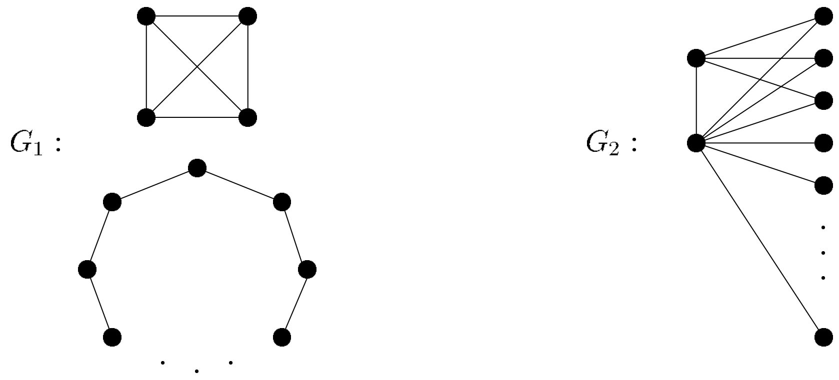


Fig. 8

Remark. By [1] $\operatorname{diam} F_{p,p+1} = 2p-6$, if $5 \leqq p \leqq 9$. Further, obviously, $\operatorname{diam} F_{4,5} = 0$. These facts together with Theorem 14 mean that we know $\operatorname{diam} F_{p,p+1}$ for every p (there are graphs with p vertices and $p+1$ edges only for $p \geqq 4$). If we consider Theorem 15 and the following three values: $\operatorname{diam} F_{4,6} = 0$, $\operatorname{diam} F_{5,7} = \operatorname{diam} F_{5,3} = 2$, $\operatorname{diam} F_{6,8} = \operatorname{diam} F_{6,7} = 6$ (see [1] and [2]) then there are unknown $\operatorname{diam} F_{p,p+2}$ only for $p \in \{7,8,9\}$. In these cases $\operatorname{diam} F_{p,p+2} \in \{2p-6, 2p-4\}$.

3. $|E_{1,2}| = 2$

The following theorem gives the answer to the problem 6b which is listed in [2]. We denote the star with n edges by S_n (Fig. 9), the path with n edges by P_n and the circle with n edges by C_n.

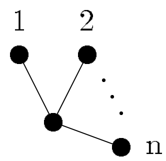


Fig. 9

Theorem 16. *If $G_1 \in F_{p_1,q_1}$ and $G_2 \in F_{p_2,q_2}$ then*

$$d(G_1, G_2) = q_1 + q_2 + |p_1 - p_2| - 4$$

if and only if the graphs G_1, G_2 satisfy one of the following three conditions (we do not pay attention to the isolated vertices; there can be any finite number of them):

1. $\triangle(G_1) = \triangle(G_2) = 1$
and one of the graphs G_1, G_2 has exactly two edges and the other one has at least two edges;

2. $\triangle(G_1) > 1$, $\triangle(G_2) > 1$
and at least one of the following conditions holds:
a) *one of the graphs G_1, G_2 is S_2,*
b) *one of the graphs G_1, G_2 is S_n for $n \geqq 3$ and the other one is a graph G for which $\triangle(G) = 2$,*
c) *one of the graphs G_1, G_2 is the graph K_3 and the other one is a graph which does not contain C_3,*
d) *one of the graphs G_1, G_2 is P_3 or C_4 and the other one is a graph which does not contain P_3 (i.e., each of its non-trivial components is K_3 or S_n),*
e) *one of the graphs G_1, G_2 is one of the graphs in Fig. 10 and the other one is a graph having only components of type S_n for $n \leqq 2$;*

3. $\triangle(G_i) = 1$, $\triangle(G_j) > 1$ $(\{i, j\} = \{1, 2\})$
and one of the following conditions holds:
a) *$|E(G_i)| = 2$ and G_j has at least two independent edges (i.e., G_j has at least two non-trivial components or has a component which is different from K_3 and S_n),*
b) *$|E(G_i)| \geqq 3$ and G_j has exactly two non-trivial components and each of them is K_3 or S_n,*
c) *$|E(G_i)| \geqq 3$ and G_j has only one non-trivial component which is a subgraph of K_5 containing P_3 or G_j is one of the graphs in Fig. 11 (the graphs in Fig. 11 contain all line edges and an arbitrary subset of pointed edges).*

Proof. The case 1 is trivial. In Case 2 it is sufficient to take into account that at least one of the graphs G_1, G_2 does not contain the graph in Fig. 12 (and so, this graph is S_n or a subgraph of K_4). In case 3 the graph G_j, obviously, can have at most two non-trivial components if $|E(G_i)| \geqq 3$. If G_j has exactly one non-trivial component (denote it by H), we will distinguish two cases. The case $|V(H)| < 6$ is trivial. In the opposite case ($|V(H)| \geqq 6$) it is sufficient to distinguish whether H contains P_4 or not (obviously, H must contain P_3 and cannot contain P_5).

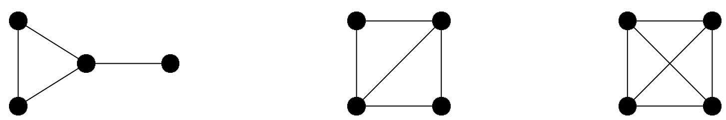


Fig. 10

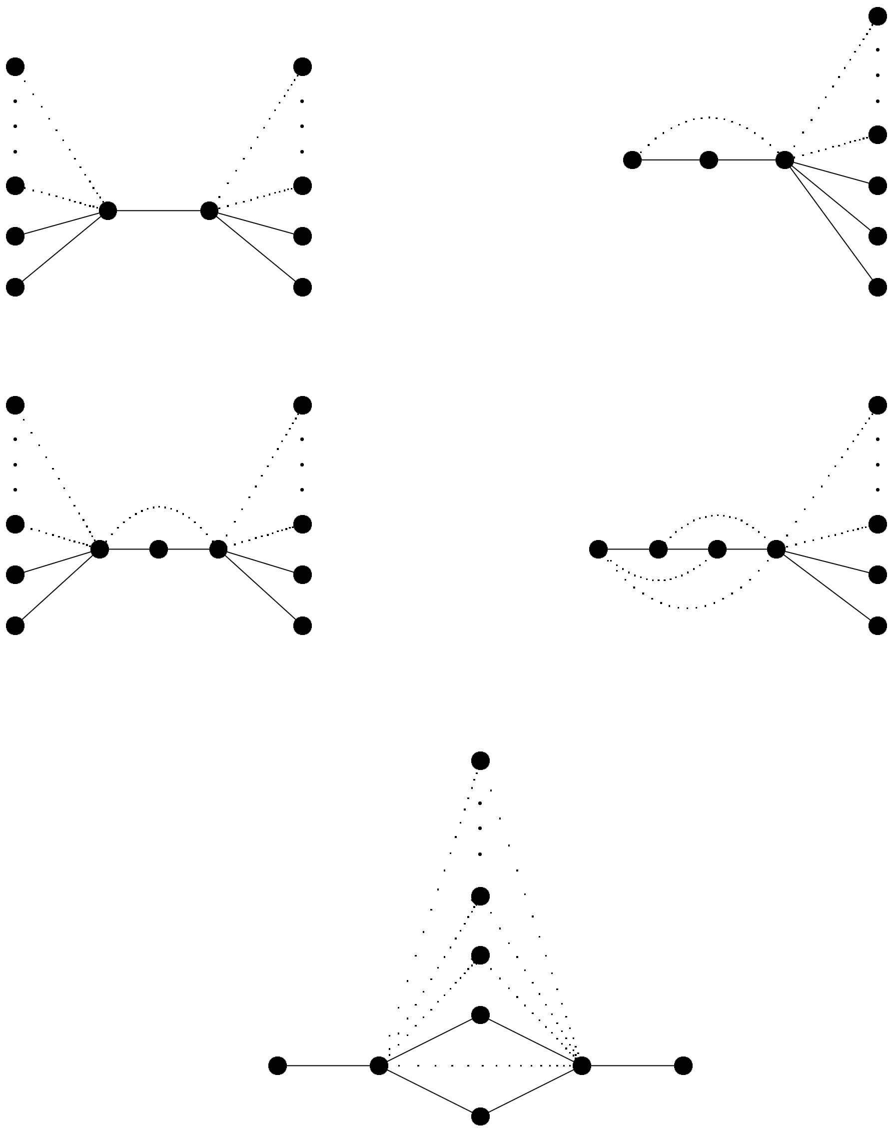


Fig. 11

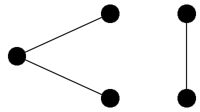


Fig. 12

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