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PRODUCTS OF STATES ON SOME KINDS OF TENSOR PRODUCTS

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Abstract. We study states on tensor products $B \otimes P$, where B is a horizontal sum of an arbitrary set of Boolean algebras and P is a bounded orthocomplemented poset. Such tensor products exist and they are (in a slightly more general case) constructed in [2]. It is shown that each pair of states on B and P generates a state (a product state) on $B \otimes P$ which in a certain way corresponds to these states.

It has been shown in [2] that for a horizontal sum ($\{0;1\}$-pasting) B of an arbitrary set of Boolean algebras and for a poset $(P, 0, 1, \perp)$ there exists a tensor product $B \otimes P$ and its construction has been given as well. We study states in such tensor products.

We will present the result of the above mentioned construction after a few neccessary definitions. By the abbreviation OCP we will understand a bounded orthocomplemented poset. In fact all the results of this work remain valid also in a little bit more general case, when P is only a bounded quasi-orthocomplemented poset. Since the differences are not substantial, we restrict ourselves to the more usual case of an OCP.

We recall, that if P is an OCP, $a, b \in P$, then a is orthogonal to b (we denote $a \perp b$) iff $a \leq b^{\perp}$.

Definition 1. Let P, Q, R be bounded OCPs. A mapping $\beta : P \times Q \to R$ is said to be a *bimorphism* if the following conditions are satisfied:

(1) for each orthogonal pair $a, b \in P$ and for any $c \in Q$ there is

$$\beta(a, c) \text{ and } \beta(b, c) \text{ are orthogonal,}$$

and

$$\beta(a \vee b, c) = \beta(a, b) \vee \beta(a, c),$$

(2) for each orthogonal pair $c, d \in Q$ and for any $a \in P$ there is

$$\beta(a, c) \text{ and } \beta(a, d) \text{ are orthogonal,}$$

and

$$\beta(a, c \vee d) = \beta(a, c) \vee \beta(b, d),$$

(3) $\beta(1, 1) = 1$.

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A tensor product for a horizontal sum (a $\{0,1\}$-pasting) of Boolean algebras and a OCP will be defined in a similar way to the tensor product of orthoalgebras used in [1].

Definition 2. Let B be a horizontal sum of an arbitrary set of Boolean algebras and let P be an OCP. Then a pair (T,τ) consisting of an OCP T and a bimorphism $\tau : B \times P \to T$ is said to be a *tensor product* of B and P iff the following conditions are satisfied:

(1) Each element of T is a finite join of mutually orthogonal elements of the form $\tau(a,b)$, where $a \in B, b \in P$,

(2) If L is an OCP fulfilling the previous property with the bimorphism $\beta : B \times P \to L$ then there is a morphism $\phi : T \to L$ such that $\beta = \phi \circ \tau$.

A morphism in the previous definition is understood in the usual way, i.e. it maps joins on joins and a unity element on a unity element. If no misunderstanding could occur, we will use T as a notation for the tensor product instead of (T,τ).

The construction of the tensor product $B \otimes P$ is based on the same idea as the construction of a sum for a Boolean algebra and a quantum logic which was introduced in [5].

Let S be a set consisting of all the elements of the type

$$\{(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)\},$$

where n is a natural number, $a_i \in B, b_i \in P$ and for each a_i the set of all a_j compatible with a_i (i.e. those from the same block of the horizontal sum) is an orthogonal partition of unity in B. We define a binary relation $\leq$ and an operation $\perp$ on S in the following way:

$$\{(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)\} \leq \{(c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)\}$$

iff $b_i \leq d_j$ whenever $a_i \wedge c_j \neq 0$, and

$$\{(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)\}^{\perp} = \{(a_1,b_1^{\perp}),(a_2,b_2^{\perp}),\ldots,(a_n,b_n^{\perp})\}.$$

As the next step we identify those $p,q \in S$ for which both $p \leq q$ and $q \leq p$ hold. The set of all the equivalence classes obtained by this identification will be denoted by T and its elements will be writen in square brackets. A routine verification shows that T is an OCP. The properties of this structure in case when B is a single Boolean algebra and P an orthomodular lattice is studied in [3] where there are results concerning mostly its completeness and in [4], where states and homomorphisms are studied.

Later we will make use of the following: If the elements

$$[(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)] \text{ and } [(c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)]$$

in T are mutually orthogonal, then without a loss of generality we may suppose that $a_1,a_2,\ldots,a_p,c_1,c_2,\ldots,c_q$ are from the same block of B, while all the remaining pairs a_i,c_j are from different blocks. If we assume this, then their least upper bound is the element

$$[(a_i \wedge c_j, b_i \vee d_j),(a_{p+1},b_{p+1}),\ldots,(a_n,b_n),(c_{q+1},d_{q+1}),\ldots,(c_m,d_m)],$$

where $i = 1,2,\ldots,p$ and $j = 1,2,\ldots,q$.

The following propositions are proved in [2]:

Proposition 3. *The mapping $\tau : B \times P \to T$ such that $\tau(a,b) = [(a,b),(a^{\perp},0)]$ is a bimorphism.*

Proposition 4. *Let B, P, T and τ have the same meaning as above. Then (T,τ) is a tensor product of B and P.*

The tensor product of B and P will be denoted by $B \otimes P$. Our aim is to study states on it. A state is understood in the usual way, i.e. a state on a bounded poset K is a mapping $s : K \to [0;1]$ such that $s(1) = 1$ and for each orthogonal pair $a, b \in K$ there is $s(a \vee b) = s(a) + s(b)$.

Proposition 5. *Let B, P, T and τ have the same meaning as above, let s_1 and s_2 be states on B and P respectively. Then there is a state s on T such that $s(\tau(a,1)) = s_1(a)$ and $s(\tau(1,b)) = s_2(b)$ for each $a \in B, b \in P$.*

Proof. Similarly to [5] we define the state S by the following rule:

$$s([(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)]) = \sum_{i=1}^{n} s_1(a_i)s_2(b_i).$$

As the unity element in T is $[(1,1)]$, we immediately have $s([(1,1)]) = 1$. Let now $[(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)]$ and $[(c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)]$ be orthogonal elements of T. Again we will suppose that $a_1, a_2, \ldots, a_p, c_1, c_2, \ldots, c_q$ are from the same block of B, while all the remaining pairs a_i, b_j are from different blocks. Then (see the remark before Proposition 3) we have

$$\begin{aligned} & s([(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)] \vee [(c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)]) = \\ &= s([(a_i \wedge c_j, b_i \vee d_j),(a_{p+1},b_{p+1}),\ldots,(a_n,b_n),(c_{q+1},d_{q+1}),\ldots,(c_m,d_m)]) = \\ &= \sum_{i=1}^{p}\sum_{j=1}^{q} s_1(a_i \wedge c_j)s_2(b_i \vee d_j) + \sum_{i=p+1}^{n} s_1(a_i)s_2(b_i) + \sum_{i=q+1}^{m} s_1(c_i)s_2(d_i). \end{aligned}$$

Making use of the additivity of s_2 and the fact that the sets $\{a_1, a_2, \ldots, a_p\}$ and $\{c_1, c_2, \ldots, c_q\}$ are partitions of unity we obtain that the term on the right-hand side is further equal to

$$\begin{aligned} & \sum_{i=1}^{p} s_1(a_1)s_2(b_i) + \sum_{i=1}^{q} s_1(c_i)s_2(d_i) + \sum_{i=p+1}^{n} s_1(a_i)s_2(b_i) + \sum_{i=q+1}^{m} s_1(c_i)s_2(d_i) = \\ &= s([(a_1,b_1),(a_2,b_2),\ldots,(a_n,b_n)]) + s([(c_1,d_1),(c_2,d_2),\ldots,(c_m,d_m)]). \end{aligned}$$

Hence s is a state on T.

Moreover, if $a \in B$, then due to Proposition 3 we have $\tau(a,1) = [(a,1),(a^{\perp},0)]$ and evidently $s(\tau(a,1)) = s_1(a)$. If $b \in P$, then $\tau(1,b) = [(1,b)]$ and also in this case we have $s(\tau(1,b)) = s_2(b)$.

Therefore s is the state on $B \otimes P$ with the required properties and the proof is completed.

We have shown that each pair of states on B and P generates a corresponding state (a product state) on their tensor product $B \otimes P$. In fact there exist also states on $B \otimes P$ that are not generated by any such pair of states, even in the case when B is a Boolean algebra. For more details about states and homomorphisms on that structure see [4].

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