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THE DISTANCE POSET OF POSETS

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Abstract. In [3] a distance between isomorphism classes of ordered sets was introduced. Let $\mathcal{F}$ be a set of all (non-isomorphic) posets on a finite set P. For $(P,R),(P,S) \in \mathcal{F}$, we define $(P,R) \leq (P,S)$ if and only if there exists a bijective isotone map f of P onto itself. We will study the distance poset $(\mathcal{F}, \leq)$.

1. Introduction

In [4], [5] and [6] some properties of the distance graphs for some type of a metric for graphs and posets were investigated. In this paper some analogous results will be derived for a metric introduced in [3].

Throughout this paper all partially ordered sets are assumed to be finite. Let (P,R) be a partially ordered set (shortly poset). If $a,b \in P$, b covers a, then we will write $a \prec_R b$.

In [3] a metric on a system of isomorphism classes of posets, which have the same cardinality, is defined. Without loss of generality we can suppose that all posets are defined on the same (finite) set P. We will often write a poset R instead of a poset (P,R).

Let $B(P)$ be the set of all bijective maps of P onto itself. For any $f \in B(P)$ and posets $(P,R),(P,S)$ we denote by $d_f(R,S)$ the number defined by

$$d_f(R,S) = |f(R) \setminus S| + |S \setminus f(R)|, \tag{1}$$

where $f(R) = \{[f(a), f(b)];\ [a,b] \in R\}$ (cf. [3]). Since the posets (P,R) and $(P,f(R))$ are isomorphic, then

$$d_f(R,S) = |R| + |S| - 2|f(R) \cap S|. \tag{2}$$

The *distance* of the posets $(P,R),(P,S)$ is defined by

$$d(R,S) = \min\{d_f(R,S);\ f \in B(P)\}. \tag{3}$$

If we identify isomorphic posets, then (3) defines a *metric* on the set of all (finite, non-isomorphic) posets defined on the same set P.

If a map $f \in B(P)$ is an isotone map of a poset (P,R) onto a poset (P,S), then $f(R) \subseteq S$ and $d(R,S) = d_f(R,S) = |S| - |R|$ (cf. Remark 2 in [3]).

The following lemma is easy to verify (cf. Lemma 1 in [3]).

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Lemma 1.1. *For any posets $(P,R),(P,S)$ and any maps $f,g \in B(P)$ the following properties are satisfied:*

$$\begin{array}{lcc}
\text{(i)} & d_f(R,S) = d_g(R,S) \quad iff \quad |f(R)\cap S| = |g(R)\cap S|, \\
\text{(ii)} & d_f(R,S) < d_g(R,S) \quad iff \quad |f(R)\cap S| > |g(R)\cap S|, \\
\text{(iii)} & |f(R)\cap S| = |R \cap f^{-1}(S)|.
\end{array}$$

The following three lemmas are obvious.

Lemma 1.2. *Let $(P,R),(P,S)$ be posets and $f \in B(P)$.*

a) *If $S \setminus f(R) \neq \emptyset$, then there exists $[a,b] \in S \setminus f(R)$ such that $a \prec_S b$.*

b) *If $f(R) \setminus S \neq \emptyset$, then there exists $[u,v] \in R$ such that $[f(u),f(v)] \in f(R) \setminus S$ and $u \prec_R v$.*

Lemma 1.3. *Let (P,R) be a poset. If for $a,b \in P$, $a \prec_R b$, then $(P, R \setminus \{[a,b]\})$ is also a poset.*

Lemma 1.4. *Let (P,R) be a poset. If $a,b \in P$, $a \prec_R b$, then $d(R, R \setminus \{[a,b]\}) = 1$.*

Let $(P,R),(P,S)$ be posets and let $f \in B(P)$. If $d_f(R,S) = d(R,S)$, f is said to be an *optimal map* of (P,R) onto (P,S) (cf. Definition in [3]). From Lemma 1.1 it follows that f is an optimal map if and only if $|f(R)\cap S|$ is maximal. Any isotone map $f \in B(P)$ is optimal (cf. Remark 2 in [3]).

Lemma 1.5. *Let $(P,R),(P,S)$ be posets and let $f \in B(P)$ be an optimal map of (P,R) onto (P,S). If $a \prec_S b$, $[a,b] \notin f(R)$, then f is an optimal map of (P,R) onto $(P, S\setminus\{[a,b]\})$ and $d(R, S \setminus \{[a,b]\}) = d(R,S) - 1$.*

Proof. Since $[a,b] \in S \setminus f(R)$, then

$$d_f(R, S \setminus \{[a,b]\}) = d_f(R,S) - 1 = d(R,S) - 1.$$

Now it is sufficient to prove that $d(R, S\setminus\{[a,b]\}) \geq d(R,S) - 1$. Suppose on the contrary that there exists a map $g \in B(P)$ with $d_g(R, S\setminus\{[a,b]\}) \leq d(R,S) - 2$. We distinguish two cases:

a) If $[a,b] \notin g(R)$, then $d_g(R, S\setminus\{[a,b]\}) = d_g(R,S) - 1$ and so $d_g(R,S) \leq d(R,S) - 1$, a contradiction.

b) If $[a,b] \in g(R)$, then $d_g(R, S\setminus\{[a,b]\}) = d_g(R,S) + 1$ and so $d_g(R,S) \leq d(R,S) - 3$, a contradiction. □

A map $f \in B(P)$ is an optimal map of (P,R) onto (P,S) if and only if f^{-1} is an optimal map of (P,S) onto (P,R) (cf. Lemma 1.1, (iii)). From this the next lemma follows (see Lemma 4 in [3]).

Lemma 1.6. *Let $(P,R),(P,S)$ be posets and let $f \in B(P)$ be an optimal map of (P,R) onto (P,S). If $a \prec_R b$ and $[f(a),f(b)] \notin S$, then f is an optimal map of $(P, R\setminus\{[a,b]\})$ onto (P,S) and $d(R\setminus\{[a,b]\}, S) = d(R,S) - 1$.*

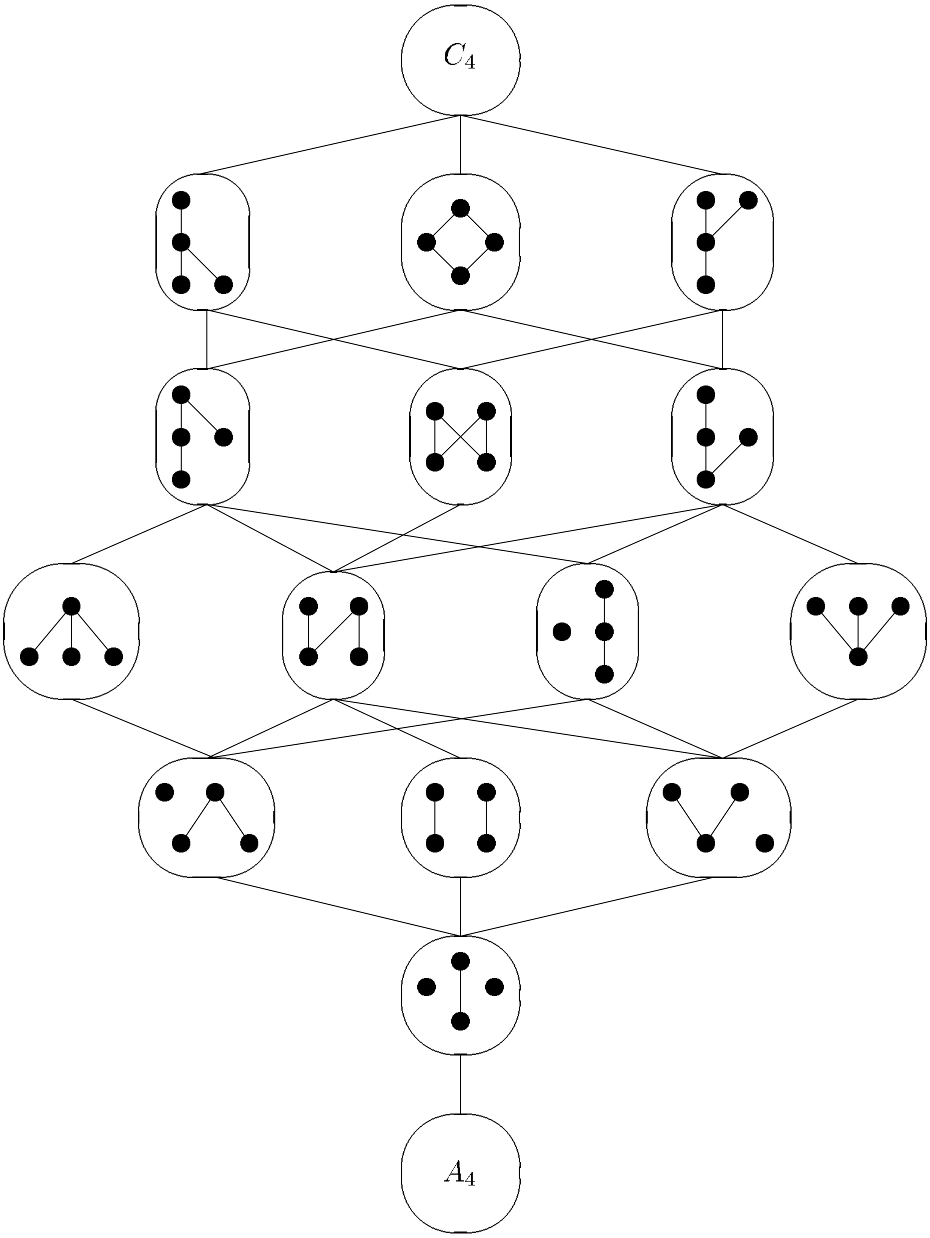
2. The distance poset

Let $\mathcal{F}_n$, $n \in N$, be a set of all (non-isomorphic) posets on a set P of cardinality n. For $(P,R),(P,S) \in \mathcal{F}_n$, we define

$$(P,R) \leq (P,S) \text{ if and only if there exists an isotone map } f \in B(P).$$

A binary relation $\leq$ is a partial order on $\mathcal{F}_n$. The poset $(\mathcal{F}_n, \leq)$ will be called the *distance poset* (of n-element posets). We shall study this poset.

Example. The following figure depicts the poset $(\mathcal{F}_4, \leq)$. C_4 is a four-element chain and A_4 is a four-element antichain.



Lemma 2.1. *Let $(\mathcal{F}_n, \leq)$ be the distance poset and let $(P,R),(P,S) \in \mathcal{F}_n$, $n \in N$. Then $d(R,S) = 1$ if and only if $(P,R) \prec (P,S)$ or $(P,S) \prec (P,R)$.*

The proof is simple; it will be omitted.

Lemma 2.2. *Let $(P,R),(P,S) \in \mathcal{F}_n$, $n \in N$, $(P,R) \leq (P,S)$ and $d(R,S) = m$, $m \in N$. Then all maximal chains from (P,R) to (P,S) have the length m.*

Proof. Let $f \in B(P)$ be an isotone map of (P,R) onto (P,S). Since the posets (P,R) and $(P,f(R))$ are isomorphic, then $d(R,S) = m = |S| - |R| = |S \setminus f(R)|$. Thus, by Lemma 2.1, for each maximal chain from $(P,R) = (P,f(R))$ to (P,S) there exists a sequence of ordered pairs $[a_1,b_1], \dots [a_m,b_m] \in S$ such that

$$C = ((P,S),(P,S\setminus\{[a_1,b_1]\}),(P,S\setminus\{[a_1,b_1],[a_2,b_2]\}),\dots,(P,S\setminus\{[a_1,b_1],\dots,[a_m,b_m]\})).$$

□

Now we recall some further notions and facts concerning posets and graphs.

If a poset (P,R) has the least element 0_P, then we define the *height* $h(a)$ of an element $a \in P$ as the length of the longest chain from 0_P to a. If a poset (P,R) has the least element and all maximal chains between the same endpoints have the same length, then we say that (P,R) is a *graded* poset. A poset (P,R) is said to have length n, denoted by $l(P) = n$, if the length of the longest chain in (P,R) is n. If (P,R) has the greatest element 1_P, then $l(P) = h(1_P)$.

A *graph* $G = (V,E)$ consists of a nonempty finite vertex set V together with a prescribed edge set E of unordered pairs of distinct vertices of V. Every edge can be written in the form ab, where $a,b \in V$.

Let $\delta(a,b)$ denote the distance from a to b (i.e. the length of the shortest path from a to b in a connected graph $G = (V,E)$), and let $\operatorname{diam} G = \max\{\delta(a,b);\, a,b \in V\}$ denote the diameter of G. The function δ is a metric. The *covering graph* $C(P)$ of a poset (P,R) is the graph whose vertices are the elements of P and whose edges are those pairs ab, $a,b \in P$, for which a covers b or b covers a. For elements a,b of a poset (P,R), $\delta(a,b)$ shall denote the distance from a to b in the covering graph $C(P)$ of (P,R).

From Lemma 2.2 we immediatelly obtain

Theorem 2.1. *The distance poset $(\mathcal{F}_n, \leq)$ is a graded poset with the least element $0_{\mathcal{F}_n} = A_n$ (an n-element antichain) and the greatest element $1_{\mathcal{F}_n} = C_n$ (an n-element chain).*

The following lemma is a part of Lemma 2.1 in [2].

Lemma 2.3. *Let (P,R) be a graded poset and let $a,b \in P$. Then*

$$\delta(a,b) = h(a) - h(b) \quad \text{if and only if} \quad [b,a] \in R.$$

Clearly, if a poset $(P,R) \in \mathcal{F}_n$, $n \in N$, then $h(P,R) = |R| - n$. From Lemma 2.3 we have

Lemma 2.4. *Let $(P,R),(P,S)$ be posets from $\mathcal{F}_n$, $n \in N$. If $(P,R) \leq (P,S)$, then $\delta(R,S) = d(R,S)$.*

The following theorem was motivated by the similar results of Zelinka in [5] and [6].

Theorem 2.2. *Let $(\mathcal{F}_n, \leq)$, $n \in N$ be the distance poset. If $(P,R),(P,S) \in \mathcal{F}_n$, then*

$$d(R,S) = \delta(R,S),$$

where $\delta(R,S)$ is the distance of vertices (P,R), (P,S) of the graph $C(\mathcal{F}_n)$.

Proof. Let $(P,R),(P,S) \in \mathcal{F}_n$. Then there exists a map $f \in B(P)$ such that

$$d(R,S) = d_f(R,S) = |f(R) \setminus S| + |S \setminus f(R)|.$$

Since $f(R) \cap S \subseteq f(R)$, $f(R) \cap S \subseteq S$, then

$$(P, f(R) \cap S) \leq (P, f(R)) = (P,R) \quad \text{and} \quad (P, f(R) \cap S) \leq (P,S).$$

From the triangle inequality for δ and from Lemma 2.4 it follows that

$$\begin{aligned}\delta(R,S) \leq &\delta(R, f(R) \cap S) + \delta(f(R) \cap S, S) = d(R, f(R) \cap S) + d(f(R) \cap S, S) = \\ =&d(f(R), f(R) \cap S) + d(f(R) \cap S, S) = |f(R) \setminus (f(R) \cap S)| + |S \setminus (f(R) \cap S)| = \\ =&|f(R) \setminus S| + |S \setminus f(R)| = d(R,S).\end{aligned}$$

Thus

(a) $$\delta(R,S) \leq d(R,S).$$

Let

$$R = R_{11}, R_{12}, \ldots, R_{1i_1} = R_{21}, R_{22}, \ldots, R_{2i_2}, \ldots, R_{j1}, R_{j2}, \ldots, R_{ji_j} = S$$

be a shortest path from (P,R) to (P,S) in $C(\mathcal{F}_n)$, where $\{R_{k1}, R_{k2}, \ldots, R_{ki_k}\}$ is a chain in $(\mathcal{F}_n, \leq)$ for all $k \in \{1, 2, \ldots, j\}$. From Lemma 2.4 and from the triangle inequality for d it follows that

$$\begin{aligned}\delta(R,S) =&\delta(R, R_{1i_1}) + \delta(R_{21}, R_{2i_2}) + \cdots + \delta(R_{j1}, S) = \\ =&d(R, R_{1i_1}) + d(R_{21}, R_{2i_2}) + \cdots + d(R_{j1}, S) \geq d(R,S).\end{aligned}$$

Thus

(b) $$\delta(R,S) \geq d(R,S).$$

From (a) and (b) we have $\delta(R,S) = d(R,S)$. □

A simple induction yields the following corollary.

Corollary 2.1. *Let $(\mathcal{F}_n, \leq)$, $n \in N$ be the distance poset. If $(P,R),(P,S) \in \mathcal{F}_n$, then*

$$h(P,R) - h(P,S) \equiv d(R,S) \pmod 2$$

The next lemma is implicit in Alvarez [1].

Lemma 2.4. *Let (P,R) be a graded poset with the greatest element. Then*

$$\operatorname{diam} C(P) = \delta(0_P, 1_P) = l(P).$$

From Lemma 2.4, Lemma 2.3 and Theorem 2.1 we immediately get

Corollary 2.2. *Let $(\mathcal{F}_n, \leq)$, $n \in N$ be the distance poset. Then*

$$\operatorname{diam} C(\mathcal{F}_n) = \frac{n(n-1)}{2}.$$

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