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A VECTOR LATTICE VARIANT OF THE MARTINGAL THEOREM

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Abstract. The purpose of this paper is to give a variant of the martingal theorem for random variables with values in a vector lattice. The following theorem is known as the inverse martingal theorem, see [5,p.360]. The direct martingal theorem we do not use, because its generalization for vector lattice valued random variables is more difficult.

1. Introduction

Theorem 1.1. *(inverse martingal theorem) Let $(\Omega, \mathcal{S}, P)$ be a probability measure space, $\{\mathcal{S}_k\}$ be a decreasing sequence of σ-subalgebras of $\mathcal{S}$ and $\mathcal{S}_\infty$ be the intersection of $\{\mathcal{S}_k\}$. Then for any random variable $\xi : \Omega \to R$ with a finite expectation $E(\xi)$*

(i) $E(\xi|\mathcal{S}_k) \to E(\xi|\mathcal{S}_\infty)$ *almost certainly and*
(ii) $E|E(\xi|\mathcal{S}_k) - E(\xi|\mathcal{S}_\infty)| \to 0$.

The present paper generalizes the inverse martingal theorem for vector lattice valued random variables. The proof of the main result is very similar to the proof of the ergodic theorem published by author in [3] and it might be omitted. However, we give the proof for the sake of completness of results. Similarly as in [3] we use results of [2] about mean value and conditional mean value for vector lattice valued variable. Similar results for mean value were established in [6] under stricter conditions.

2. Vector lattices

More complete information about vector lattices may be found in [1] and [4].

A real vector space V is called a vector lattice if it has a partial ordering $\leq$ such that $(V, \leq)$ is a lattice and:

$\forall x, y, z \in V : x \leq y \implies x + z \leq y + z$

$\forall x, y \in V : \lambda \geq 0 : x \leq y \implies \lambda x \leq \lambda y$. Lattice operations are denoted by symbols $\vee$ and $\wedge$.

If $a \in V$ then the symbol $|a|$ denotes the element $a \vee (-a)$.

A vector lattice V is called σ-complete if every sequence $\{a_n\} \subset V$ bounded from above has a least upper bound which is denoted by the symbol $\bigvee_{n=1}^{\infty} a_n$ (or equivalently, every sequence $\{a_n\}$ bounded from below has a greatest lower bound which is denoted by $\bigwedge_{n=1}^{\infty} a_n$).

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Definition 2.1. Let V be a σ-complete lattice. A sequence $\{a_n\} \subset V$ is called *decreasing* to 0 if:

$\forall n : a_{n+1} \leq a_n$ and $\bigwedge_{n=1}^{\infty} a_n = 0$. We write $a_n \searrow 0$ in this case.

A sequence $\{x_n\} \subset V$ is called *converging* to $x \in V$ if there is a sequence $\{a_n\} \subset V$ decreasing to 0 such that $|x_n - x| \leq a_n$ for all n. We write $x_n \to x$ $(n \to \infty)$ in this case.

Proposition 2.2. *Let V be a σ-complete vector lattice.*

(i) *A sequence $\{x_n\} \subset V$ converges to $x \in V$ if and only if $\{x_n\}$ is bounded and $x = \bigwedge_{n=1}^{\infty} \bigvee_{m=n}^{\infty} x_m = \bigvee_{n=1}^{\infty} \bigwedge_{m=n}^{\infty} x_m$*

(ii) $a_n \searrow 0,\ b_n \searrow 0 \implies (a_n + b_n) \searrow 0$

(iii) $a_n \searrow 0, \lambda \geq 0 \implies \lambda a_n \searrow 0$

(iv) $x_n \to x, y_n \to y \implies (x_n + y_n) \to (x + y)$

(v) $x_n \to x \implies \lambda x_n \to \lambda x$.

The following lemma will be important in the proof of the main result in this paper.

Lemma 2.3. *Let V be a σ-complete vector lattice and $\{a_n\} \subset V, \{b_{n,k}\} \subset V$ be sequences such that:*

$\forall n, k : b_{n,k} \geq 0$

$\forall n : b_{n,k} \to 0\ (k \to \infty)$

$a_n \searrow 0\ (n \to \infty)$.

Put $c_k = \bigwedge_{n=1}^{\infty}(a_n + b_{n,k})$. Then $\forall k : c_k \geq 0$ and $c_k \to 0\ (k \to \infty)$.

Proof. The inequality $c_k \geq 0$ for all k is obvious. The sequence $\{c_k\}$ is bounded because $0 \leq c_k \leq a_1 + b_{1,k}$ for all k and $b_{1,k} \to 0\ (k \to \infty)$. It means that the element $\bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} c_j$ exists. We have:

$$\bigvee_{j=k}^{\infty} c_j = \bigvee_{j=k}^{\infty} \bigwedge_{n=1}^{\infty} (a_n + b_{n,j}) \leq \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j})$$

and

$$\bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} c_j \leq \bigwedge_{k=1}^{\infty} \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) = \bigwedge_{n=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) =$$

$$= \bigwedge_{n=1}^{\infty} (a_n + \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} b_{n,j}) = \bigwedge_{n=1}^{\infty} (a_n + 0) = \bigwedge_{n=1}^{\infty} a_n = 0.$$

Except for assumptions of lemma we used the obvious facts:

$$\bigvee_{j=k}^{\infty} \bigwedge_{n=1}^{\infty} (a_n + b_{n,j}) \leq \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j})$$

and

$$\bigwedge_{k=1}^{\infty} \bigwedge_{n=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}) = \bigwedge_{n=1}^{\infty} \bigwedge_{k=1}^{\infty} \bigvee_{j=k}^{\infty} (a_n + b_{n,j}).$$

3. Integral and conditional mean value of vector lattice valued functions

In this section we give a summary of results of author's paper [2].

Let $(\Omega, \mathcal{S}, P)$ be a probability measure space and V be a σ-complete vector lattice. The symbol $F(\Omega, V)$ denotes the set of all functions $f : \Omega \to V$. Obviously, $F(\Omega, V)$ is a σ-complete vector lattice under natural operations and ordering.

Two functions $f, g \in F(\Omega, V)$ are called equivalent if there exists a set $A \in \mathcal{S}$ such that

$$P(A) = 0 \text{ and } \forall \omega \in \Omega \setminus A : f(\omega) = g(\omega).$$

The set of all equivalence classes is denoted by $\mathcal{F}(\Omega, \mathcal{S}, P, V)$ and it is a σ-complete vector lattice under natural operations and ordering. A function $f \in F(\Omega, V)$ is called simple if $f(\omega) = a_i$ for $\omega \in A_i$, where $\{A_i\}$ is a finite measurable partition of Ω and $a_i \in V$. We put $\displaystyle\int_\Omega f(\omega) dP(\omega) = \sum_{i=1}^{n} P(A_i) a_i$ in this case.

A class $\varphi \in F(\Omega, \mathcal{S}, P, V)$ is called simple if it contains some simple function f. We put $E(\varphi) = \int_\Omega \varphi dP = \int_\Omega f(\omega) dP(\omega)$ in this case.

The set of all simple functions is denoted by $L_0^\infty(\Omega, \mathcal{S}, P, V)$ and the set of all simple classes is denoted by $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$.

Let $\{f_n\} \subset F(\Omega, V)$ and $f \in F(\Omega, V)$. We say that a sequence $\{f_n\}$ converges to the function f uniformly almost everywhere if there exist $A \in \mathcal{S}$, $\{a_n\} \subset V$ such that:

$P(A) = 0$

$\forall \omega \in \Omega \setminus A : \forall n : |f_n(\omega) - f(\omega)| \leq a_n$

$a_n \to 0 \ (n \to \infty)$.

Obviously, the condition $a_n \to 0$ may be replaced by a stronger one $a_n \searrow 0$. We write $f_n \to f$ u.a.e. $(n \to \infty)$ in this case.

Let $\{\varphi_n\} \subset \mathcal{F}(\Omega, \mathcal{S}, P, V)$ and $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$. We say that the sequence $\{\varphi_n\}$ converges to a class φ uniformly almost everywhere if $f_n \to f$ u.a.e. for some $f_n \in \varphi_n$ and $f \in \varphi$. We write $\varphi_n \to \varphi$ u.a.e. $(n \to \infty)$ in this case.

Let $\mathcal{M}$ be a system of all vector subspaces of $\mathcal{F}(\Omega, \mathcal{S}, P, V)$ which contain the set $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$ and are closed with respect to the convergence which was described above. Obviously, $\mathcal{M}$ has the minimal element with respect to inclusion. This vector space is denoted by $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$.

Theorem 3.1.

(i) *$\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ is a vector sublattice of $\mathcal{F}(\Omega, \mathcal{S}, P, V)$, which is closed with respect to u.a.e. convergence.*

(ii) *There exists a unique nonnegative linear extension $\overline{E}$ of the set E onto $\mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$, which is continuous in the following sense: $\varphi_n \to \varphi$ u.a.e. $\Longrightarrow$ $\overline{E}(\varphi_n) \to \overline{E}(\varphi)$.*

Remark. We shall write $E(\varphi)$ or $\int_\Omega \varphi dP$ for $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$ instead of $\overline{E}(\varphi)$.

In a similar way a conditional mean value operator can be constructed. Let $(\Omega, \mathcal{S}, P)$ be a probability measure space, $\mathcal{S}_0$ be a σ-algebra of $\mathcal{S}$ and $E(.|\mathcal{S}_0)$ be a conditional mean value operator for real functions. Take $\varphi \in \mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$; φ is an equivalence class of some simple function f of the form $\sum_{i=1}^{n} \chi_{Ai} a_i$. Denote by ψ the equivalence class of the function $\sum_{i=1}^{n} E(\chi_{Ai}|\mathcal{S}_0)a_i$. In this case $\psi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$.

Putting $E(\varphi|\mathcal{S}_0) = \psi$ we obtain a linear nonnegative operator

$$E(.|\mathcal{S}_0) : \mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V) \to \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V).$$

Theorem 3.2.

(i) *There exists a unique nonnegative linear extension*

$$\overline{E}(.|\mathcal{S}_0) : \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V) \to \mathcal{L}^\infty(\Omega, \mathcal{S}_0, P, V) \text{ of } E(.|\mathcal{S}_0).$$

(ii) *The operator $\overline{E}(.|\mathcal{S}_0)$ is continuous in the following sense: $\varphi_n \to \varphi$ u.a.e. $\implies$ $\overline{E}(\varphi_n|\mathcal{S}_0) \to \overline{E}(\varphi|\mathcal{S}_0)$ u.a.e.*

Remark. We shall write $E(\varphi|\mathcal{S}_0)$ instead of $\overline{E}(\varphi|\mathcal{S}_0)$.

We shall also use the pointwise convergence.

Let $\{f_n\} \subset F(\Omega, V)$ and $f \in F(\Omega, V)$. We say that the sequence $\{f_n\}$ converges to f almost everywhere if there exists a set $A \in \mathcal{S}$ such that $P(A) = 0$ and $\forall \omega \in \Omega \setminus A$: $f_n(\omega) \to f(\omega)$ $(n \to \infty)$. We write $f_n \to f$ a.e. in this case.

If $\{\varphi_n\} \subset \mathcal{F}(\Omega, \mathcal{S}, P, V)$ and $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$ then the notation $\varphi_n \to \varphi$ a.e. $(n \to \infty)$ means that $f_n \to f$ a.e. $(n \to \infty)$ for some $f_n \in \varphi_n$ and $f \in \varphi$.

4. The inverse martingal theorem for vector lattice valued random variables

Theorem 4.1. *Let $(\Omega, \mathcal{S}, P)$ be a probability measure space, V be a σ-complete vector lattice, $\{\mathcal{S}_k\}$ be a decreasing sequence of σ-subalgebras of $\mathcal{S}$ and $\mathcal{S}_\infty$ be the intersection of $\{\mathcal{S}_k\}$. Then for all $\varphi \in \mathcal{L}^\infty(\Omega, \mathcal{S}, P, V)$*

(i) *$E(\varphi|\mathcal{S}_k) \to E(\varphi|\mathcal{S}_\infty)$ a.e. and*

(ii) *$E|E(\varphi|\mathcal{S}_k) - E(\varphi|\mathcal{S}_\infty)| \to 0$.*

Proof. Denote by $\mathcal{M}$ the set of all $\varphi \in L^\infty(\Omega, \mathcal{S}, P, V)$, for which (i) is true. The martingal theorem for real functions implies that $\mathcal{M}$ contains $\mathcal{L}_0^\infty(\Omega, \mathcal{S}, P, V)$. It is sufficient to prove that $\mathcal{M}$ is closed with respect to u.a.e. convergence. Let $\{\varphi_n\} \subset \mathcal{M}$ be a sequence which converges to $\varphi \in \mathcal{F}(\Omega, \mathcal{S}, P, V)$ u.a.e.. Obviously, $\varphi \in \mathcal{L}(\Omega, \mathcal{S}, P, V)$ and $E(\varphi_n|\mathcal{S}_k) \to E(\varphi_n|\mathcal{S}_\infty)$ a.e. when $k \to \infty$ for all n.

Let f_n, f, g_n, g, h_{nk} and h be representants of the following equivalence classes φ_n, φ, $E(\varphi_n|\mathcal{S}_\infty)$, $E(\varphi|\mathcal{S}_\infty)$, $E(\varphi_n|\mathcal{S}_k)$ and $E(\varphi|\mathcal{S}_k)$. There is a set $A \in \mathcal{S}$ with $P(A) = 0$ and a sequence $\{a_n\} \subset V$ with $a_n \searrow 0$ such that $|f_n(\omega) - f(\omega)| \leq a_n$ for all $\omega \in \Omega \setminus A$. Since conditional mean value preserves ordering and constants, there are sets B_k and $B \in \mathcal{S}$ with $P(B_k) = 0$ and $P(B) = 0$ such that

$|g_n(\omega) - g(\omega)| \leq a_n$ for all $\omega \in \Omega \setminus B$ and

$|h_{nk}(\omega) - h(\omega)| \leq a_n$ for all $\omega \in \Omega \setminus B_k$.

By assumptions $h_{nk}(\omega) \to g_n(\omega)$ a. e. when $k \to \infty$ for any n; the exceptional ω form a set C_n with $P(C_n) = 0$. Denote $D = A \bigcup \left(\bigcup_{k=1}^{\infty} B_k \right) \bigcup \left(\bigcup_{n=1}^{\infty} C_n \right)$. For $\omega \in \Omega \setminus D$ denote $b_{nk} = |h_{nk}(\omega) - g_n(\omega)|$. We have

$$|h_k(\omega) - g(\omega)| \le |h_k(\omega) - h_{nk}(\omega)| + |h_{nk}(\omega) - g_n(\omega)| + |g_n(\omega) - g(\omega)| \le$$

$$\le a_n + b_{nk} + a_n = 2a_n + b_{nk}.$$

Put $c_k = \bigwedge_{n=1}^{\infty} (2a_n + b_{n,k})$.

Then $|h_k(\omega) - g(\omega)| \le c_k$ and $c_k \to 0$ by lemma 2.3. The proof of (i) is complete. Part (ii) can be proved by the same idea (it is not necessary to use the representants).

References

[1] Luxemburg,W.A.J. - Zannen,A.C., *Riesz Spaces I.*, North-Holland, 1971.

[2] Maličký,P., *Random variables with values in a vector lattice*, Acta Math.Univ.Com. **52-53** (1987), 249-263.

[3] Maličký,P., *A vector lattice variant of the ergodic theorem,*, Supplemento ai Rendiconti del Circolo Matematico di Palermo Serie II **14** (1987), 391-397.

[4] Peressini,A.L., *Order topological vector spaces*, Harper & Row Publishers, New York.

[5] Štěpán,J, *Teorie pravděpodobnosti*, Academia, Praha, 1987.

[6] Vrábelová,M., *On the conditional expectation in a regular ordered space*, Math.Slovaca **38**, no. 2, 159-166.

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