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FUNCTIONALLY COMPLETE ALGEBRAS

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Abstract. It is known that every funtionally complete algebra has the k-interpolation property for $k \geq 1$. We will prove that for each integer $k > 2$, every k-element algebra with the k-interpolation property is functionally complete.

1. Introduction

The polynomial function of an algebra (A, F) is represented by some "correctly arranged" string containing (possibly) the variables, the symbols "(" , ")", the constants (from A), and the symbols of the operations (from F). The algebra (A, F) is **functionally complete** iff every function $A^n \to A$ is polynomial.

The algebra (A, F) has the k-interpolation property iff for every integer $n > 0$, every k-tuple

$$(\overrightarrow{a_1}, \overrightarrow{a_2}, \dots, \overrightarrow{a_k}) \in (A^n)^k$$

of pairwise distinct vectors and every k-tuple

$$(b_1, b_2, \dots, b_k) \in A^k$$

there exists a n-ary polynomial function F such that

$$F(\overrightarrow{a_1}) = b_1,\ F(\overrightarrow{a_2}) = b_2,\ \dots,\ F(\overrightarrow{a_k}) = b_k.$$

A lot of interesting information on the functional completness and the interpolation properties can be found in [1] or [2]. Further we will use the Composition theorem (Wille + Werner). By this theorem, if there exist two different elements $0, 1 \in A$, two binary polynomial operations "+" and "$\cdot$" satisfying the identities

$$x + 0 = x,\ 0 + x = x,\ x \cdot 1 = x,\ x \cdot 0 = 0$$

and several unary polynomial functions (the unary functions g for which one value of g is equal to 1 and all other values of g are equal to 0), then (A, F) is functionally complete. (It is assumed that A is finite.) Further we will study only k-element algebras with the k-interpolation property, consequently, all unary functions will be polynomial. Moreover, each k-element algebra with the $2k$-interpolation property is functionally complete because the identities for "+" and "$\cdot$" determine at most $2k$ values of each operation.

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Lemma 1.1. *Assume that (A, F) has the $(m+1)$-interpolation property. Then (A, F) has the m-interpolation property, too.*

Example 1.1. Let $(A, F) = (Z_2, +)$ be the additive group of the 2-element field. This algebra is not functionally complete (every unary function is polynomial but it is easy to prove that among 16 binary functions there are only 8 polynomial functions). It can be proved that this algebra has the 3-interpolation property (we leave it to the reader).

2. Three-element algebras

Assume that $A = \{0, 1, 2\}$ and the algebra (A, F) of any type has the 3-interpolation property. Every function $M : A^2 \to A$ can be represented by the following 3×3 matrix:

$$M = \begin{vmatrix} M(0,0) & M(0,1) & M(0,2) \\ M(1,0) & M(1,1) & M(1,2) \\ M(2,0) & M(2,1) & M(2,2) \end{vmatrix}.$$

Using such matrices, we can simply write the "interpolation polynomials". For example, the (possibly not existing) binary polynomial function G satisfying the equalities

$$G(0,0) = 0,\ G(0,1) = 1,\ G(1,2) = 2,\ G(2,0) = 1$$

will be represented by the following matrix ("$*$" denotes the non-determined value):

$$G = \begin{vmatrix} 0 & 1 & * \\ * & * & 2 \\ 1 & * & * \end{vmatrix}$$

and we will say that the matrix G is **polynomial** iff it represents some binary polynomial function G.

Lemma 2.1. *Every unary function on (A, F) is polynomial.*

Lemma 2.2. *Every 3×3 matrix over the set $\{0, 1, 2, *\}$ containing at most 3 numbers is polynomial.*

Lemma 2.3. *For 3×3 matrices K, L, M over the $\{0, 1, 2, *\}$, let us define the matrix KLM by the following way:*

$$KLM(x, y) = K(L(x, y), M(x, y)).$$

(The value $KLM(x, y)$ is not determined in the following three cases:
- *the value $L(x, y)$ is not determined,*
- *the value $M(x, y)$ is not determined,*
- *the values $L(x, y) = a$, $M(x, y) = b$ are determined, but $K(a, b)$ is not.) If the matrices K, L, M are polynomial, then the matrix KLM is polynomial, too.*

Lemma 2.4. *Let M be a polynomial matrix. Then the matrix K defined by*

$$K(x, y) = M(y, x)$$

is polynomial, too.

Lemma 2.5. *Every matrix of the form*

$$G = \begin{vmatrix} a & a & a \\ b & b & b \\ c & c & c \end{vmatrix}$$

(resp. the transposed matrix) is polynomial.

Proof. Let us define the unary function

$$f(0) = a, \ f(1) = b, \ f(2) = c.$$

By Lemma 2.1, the function f is polynomial. Moreover, $G(x, y) = f(x)$.

Lemma 2.6. *Assume that M is a polynomial matrix and that the matrix K can be obtained from M by one of the following operations:*

1) *some permutation of the rows (columns),*
2) *replacing of some row (column) by any other row (column).*

Then the matrix K is polynomial, too.

Proof. For example, we wish to exchange the first two rows of the matrix M. By Lemma 2.1, the unary function

$$f(0) = 1, \ f(1) = 0, \ f(2) = 2$$

is polynomial. Moreover, it holds $M(x, y) = M(f(x), y)$.
Another example - we wish to replace the column 0 by the column 2. It suffices to use the function

$$f(0) = 2, \ f(1) = 1, \ f(2) = 2$$

and the equality $M(x, y) = M(x, f(y))$.

Lemma 2.7. *Let M be a polynomial matrix and let f be an unary function. Then the matrix fM defined by*

$$fM(x, y) = f(M(x, y))$$

is polynomial, too.

Lemma 2.8. *The algebra (A, F) is functionally complete iff the following two matrices are polynomial:*

$$S = \begin{vmatrix} 0 & 1 & 2 \\ 1 & * & * \\ 2 & * & * \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 & * \\ 0 & 1 & * \\ 0 & 2 & * \end{vmatrix}.$$

Proof. Apply Lemma 2.1 and the Composition theorem. (The matrix S corresponds to "+" and the matrix P corresponds to "·".) Here we assume that (A, F) has the 3-interpolation property.

Remark. In the following theorem, we will explicitly write all assumptions.

Theorem 2.1. *Assume that $A = \{0, 1, 2\}$ and that the algebra (A, F) has the 3-interpolation property. Then (A, F) is functionally complete.*

Proof. We are going to prove that the matrices S, P (Lemma 2.8) are polynomial. By Lemma 2.2, the matrices

$$S_1 = \begin{vmatrix} 0 & 1 & * \\ 1 & * & * \\ * & * & * \end{vmatrix}, \quad Q = \begin{vmatrix} 0 & 1 & * \\ 2 & * & * \\ * & * & * \end{vmatrix}$$

are polynomial. Applying Lemma 2.6, we succesively obtain the following polynomial matrices:

$$S_2 = \begin{vmatrix} 0 & 1 & * \\ 1 & * & * \\ 0 & 1 & * \end{vmatrix}, \quad S_3 = \begin{vmatrix} 0 & 1 & 0 \\ 1 & * & 1 \\ 0 & 1 & 0 \end{vmatrix}, \quad S_4 = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ 1 & * & 1 \end{vmatrix}.$$

Applying Lemma 2.4 and Lemma 2.6, we obtain the polynomial matrices

$$S_5 = \begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & * \\ 0 & 0 & 1 \end{vmatrix}, \quad S_6 = \begin{vmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & * \end{vmatrix}.$$

Direct calculations give (see Lemma 2.3)

$$QS_6S_3 = \begin{vmatrix} 0 & 1 & 2 \\ 1 & * & * \\ 2 & * & * \end{vmatrix} = S.$$

Proposition 2.1. *The matrix* $P' = \begin{vmatrix} 0 & 0 & * \\ 0 & 1 & * \\ * & * & * \end{vmatrix}$ *is polynomial.*

Let us continue in the proof of Theorem 2.1. By Proposition 2.1, Lemma 2.6 and Lemma 2.3, we obtain the polynomial matrices

$$P_1 = \begin{vmatrix} 0 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & * \end{vmatrix}, \quad P_2 = \begin{vmatrix} 0 & 0 & * \\ 0 & 0 & * \\ 0 & 1 & * \end{vmatrix}, \quad P = QP_2P_1.$$

It remains only to prove Proposition 2.1. We know that the matrix Q is polynomial. The value $Q(1, 1)$ is not determined and it is easy to see that at least one of the following three matrices is polynomial:

$$P_3 = \begin{vmatrix} 0 & 1 & * \\ 2 & 0 & * \\ * & * & * \end{vmatrix}, \quad P_4 = \begin{vmatrix} 0 & 1 & * \\ 2 & 1 & * \\ * & * & * \end{vmatrix}, \quad P_5 = \begin{vmatrix} 0 & 1 & * \\ 2 & 2 & * \\ * & * & * \end{vmatrix}.$$

By Lemma 2.7, at least one of the following two matrices is polynomial:

$$P_6 = \begin{vmatrix} 0 & 1 & * \\ 0 & 0 & * \\ * & * & * \end{vmatrix} \;, \quad P_7 = \begin{vmatrix} 1 & 0 & * \\ 0 & 0 & * \\ * & * & * \end{vmatrix} \;.$$

It suffices to apply Lemma 2.6.

Theorem 2.2. *Assume that $A = \{0, 1, 2\}$ and that the algebra (A, F) has the following properties:*

1) *every unary function is polynomial,*
2) *there exists a binary polynomial function Q such that*

$$Q(0,0) = 0,\; Q(0,1) = 1,\; Q(1,0) = 2.$$

Then (A, F) is functionally complete.

Proof. The binary function Q corresponds to the matrix Q in the proof of Theorem 2.1. Starting from Q, it is possible to derive S_1 and P_2 (apply Lemma 2.6 and Lemma 2.7).

Corollary. *Assume that $A = \{0, 1, 2\}$ and that the algebra (A, F) has the following properties:*

1) *among unary polynomial functions there exist at least one transposition, at least one 3-cycle and at least one function with exactly 2 values,*
2) *among binary polynomial functions there exists a function G and there exist $a, b, c, d \in A$ such that*

$$\{G(a,c),\; G(a,d),\; G(b,c),\; G(b,d)\} = A.$$

Then (A, F) is functionally complete.

3. Four-element and "big" algebras

In the case of 4-element algebras we can assume that $A = \{0, 1, 2, 3\}$ and the binary functions will be represented by 4×4 matrices. The results above (Lemma 2.1 – Lemma 2.8) must be modified (it is easy). The fundamental matrices in the modification of Lemma 2.8 will be

$$S = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & * & * & * \\ 2 & * & * & * \\ 3 & * & * & * \end{vmatrix} \;, \quad P = \begin{vmatrix} 0 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 2 & * & * \\ 0 & 3 & * & * \end{vmatrix} \;.$$

Theorem 3.1. *Assume that $A = \{0, 1, 2, 3\}$ and that the algebra (A, F) has the 4-interpolation property. Then (A, F) is functionally complete.*

Proof. Starting from the "fundamental" polynomial matrix

$$Q = \begin{vmatrix} 0 & 1 & * & * \\ 2 & 3 & * & * \\ * & * & * & * \\ * & * & * & * \end{vmatrix},$$

we successively obtain the polynomial matrices

$$P_1 = \begin{vmatrix} 0 & 0 & * & * \\ 0 & 1 & * & * \\ * & * & * & * \\ * & * & * & * \end{vmatrix}, \quad P_2 = \begin{vmatrix} 0 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & * & * \\ 0 & 1 & * & * \end{vmatrix}, \quad P_3 = \begin{vmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 1 & * & * \end{vmatrix},$$

$P = QP_3P_2$,

$$S_1 = \begin{vmatrix} 0 & 1 & * & * \\ 1 & * & * & * \\ * & * & * & * \\ * & * & * & * \end{vmatrix}, \quad S_2 = \begin{vmatrix} 0 & 1 & * & * \\ 0 & 1 & * & * \\ 1 & * & * & * \\ 1 & * & * & * \end{vmatrix}, \quad S_3 = \begin{vmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & * & * \\ 1 & 1 & * & * \end{vmatrix},$$

$$S_4 = \begin{vmatrix} 0 & 1 & 0 & 1 \\ 1 & * & 1 & * \\ 0 & 1 & 0 & 1 \\ 1 & * & 1 & * \end{vmatrix}, \quad S_5 = QS_3S_4 = \begin{vmatrix} 0 & 1 & 2 & 3 \\ 1 & * & 3 & * \\ 2 & 3 & * & * \\ 3 & * & * & * \end{vmatrix}.$$

Trivially, S is a special case of S_5.

Theorem 3.2. *Assume that $A = \{0, 1, 2, 3\}$ and that the algebra (A, F) has the following properties:*

1) *every unary function is polynomial,*
2) *there exists a binary polynomial function Q such that* $Q(0,0) = 0$, $Q(0,1) = 1$, $Q(1,0) = 2$, $Q(1,1) = 3$.

Then (A, F) is functionally complete.

Corollary. *Assume that $A = \{0, 1, 2, 3\}$ and that the algebra (A, F) has the following properties:*

1) *among unary polynomial functions there exist at least one 4-cycle, at least one 3-cycle and at least one function with exactly 3 values,*
2) *among binary polynomial functions there exists a function G and there exist $a, b, c, d \in A$ such that $\{G(a,c),\ G(a,d),\ G(b,c),\ G(b,d)\} = A$.*

Then (A, F) is functionally complete.

Remark. The method of the proof of Theorem 3.1 can be applied more generally.

Theorem 3.3. *Assume that $A = \{0, 1, 2, \ldots, k-1\}, k > 5$ and that the algebra (A, F) has the k-interpolation property. Then (A, F) is functionally complete.*

The idea of the proof. First modify the results above (Lemma 2.1 – Lemma 2.8) and prove that there exists an integer m such that the following inequalities are satisfied:

$$m^2 \geq k,\ 2m \leq k.$$

The fundamental polynomial matrix Q can be defined by the following way:

$$Q(x, y) = mx + y, \ x < m, \ y < m, \ mx + y < k.$$

(All other values are not determined.) We leave the details to the reader. This method is not convenient in the case $k = 5$ because there exists no integer m satisfying the inequalities

$$m^2 \geq 5, \ 2m \leq 5.$$

Example 3.1. Put $k = 7$. The inequalities

$$m^2 \geq 7, \ 2m \leq 7$$

have the solution $m = 3$. Here the matrices S, P are:

$$S = \begin{vmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & * & * & * & * & * & * \\ 2 & * & * & * & * & * & * \\ 3 & * & * & * & * & * & * \\ 4 & * & * & * & * & * & * \\ 5 & * & * & * & * & * & * \\ 6 & * & * & * & * & * & * \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 & * & * & * & * & * \\ 0 & 1 & * & * & * & * & * \\ 0 & 2 & * & * & * & * & * \\ 0 & 3 & * & * & * & * & * \\ 0 & 4 & * & * & * & * & * \\ 0 & 5 & * & * & * & * & * \\ 0 & 6 & * & * & * & * & * \end{vmatrix}$$

and the fundamental polynomial matrix is:

$$Q = \begin{vmatrix} 0 & 1 & 2 & * & * & * & * \\ 3 & 4 & 5 & * & * & * & * \\ 6 & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \\ * & * & * & * & * & * & * \end{vmatrix}.$$

The derivations of S, P from Q we leave to the reader. (Use the same method as in the case $k = 4$.)

4. Five-element algebras

Theorem 4.1. *Assume that $A = \{0, 1, 2, 3, 4\}$ and that the algebra (A, F) has 5-interpolation property. The (A, F) is functionally complete.*

Proof. The fundamental matrices in the modification of Lemma2.8 will be

$$S = \begin{vmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & * & * & * & * \\ 2 & * & * & * & * \\ 3 & * & * & * & * \\ 4 & * & * & * & * \end{vmatrix}, \quad P = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 4 & * & * & * \end{vmatrix}.$$

By the 5-interpolation property the following two matrices are polynomial:

$$Q = \begin{vmatrix} 0 & 1 & 2 & * & * \\ 3 & 4 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}, \quad R = \begin{vmatrix} 0 & 1 & 2 & * & * \\ 1 & * & * & * & * \\ 2 & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}.$$

From Q we derive polynomial matrices

$$S_1 = \begin{vmatrix} 0 & 1 & * & * & * \\ 1 & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}, \quad S_2 = \begin{vmatrix} 0 & 1 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 1 & * & * & * \\ 1 & * & * & * & * \\ 1 & * & * & * & * \end{vmatrix}, \quad S_3 = \begin{vmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & * & * \\ 1 & 1 & 1 & * & * \end{vmatrix}$$

and from R we derive polynomial matrices

$$S_4 = \begin{vmatrix} 0 & 1 & 2 & * & * \\ 1 & * & * & * & * \\ 2 & * & * & * & * \\ 0 & 1 & 2 & * & * \\ 1 & * & * & * & * \end{vmatrix}, \quad S_5 = \begin{vmatrix} 0 & 1 & 2 & 0 & 1 \\ 1 & * & * & 1 & * \\ 2 & * & * & 2 & * \\ 0 & 1 & 2 & 0 & 1 \\ 1 & * & * & 1 & * \end{vmatrix}.$$

The matrix QS_3S_5 is a special case of S. It is more difficult to derive the matrix P. By the 5-interpolation property, the matrices

$$P_1 = \begin{vmatrix} * & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}, \quad Q' = \begin{vmatrix} 0 & * & * & * & * \\ * & 1 & 3 & * & * \\ * & 4 & 2 & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}$$

are polynomial. From P_1 we derive polynomial matrices

$$P_2 = \begin{vmatrix} * & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \end{vmatrix}, \quad P_3 = \begin{vmatrix} * & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 1 & * & * & * \end{vmatrix}.$$

Direct computations give

$$P_4 = Q' P_2 P_3 = \begin{vmatrix} * & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 4 & * & * & * \end{vmatrix}.$$

In the case $P_4(0,0) = 0$, the proof is finished ($P_4 = P$). All other cases are equivalent and we can assume that $P_4(0,0) = 1$. Then we have the polynomial matrix

$$P_5 = \begin{vmatrix} 1 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 4 & * & * & * \end{vmatrix}.$$

By the 4-interpolation property, the matrices

$$P_6 = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}, \quad U = \begin{vmatrix} 0 & 2 & * & * & * \\ 3 & 1 & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{vmatrix}$$

are polynomial. From P_6 we derive the polynomial matrices

$$P_7 = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \end{vmatrix}, \quad P_8 = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \end{vmatrix}.$$

Direct computations give

$$P_9 = U P_7 P_8 = \begin{vmatrix} 0 & 0 & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 2 & * & * & * \\ 0 & 3 & * & * & * \\ 0 & 0 & * & * & * \end{vmatrix}.$$

By the 5-interpolation property, the matrix

$$P_{10} = \begin{vmatrix} * & * & * & * & * \\ 0 & 1 & * & * & * \\ * & * & 2 & * & * \\ * & * & * & 3 & * \\ 4 & * & * & * & * \end{vmatrix}$$

is polynomial. From P_{10} we derive the polynomial matrix

$$P_{11} = \begin{vmatrix} 0 & 1 & * & * & * \\ 0 & 1 & * & * & * \\ * & * & 2 & * & * \\ * & * & * & 3 & * \\ 4 & * & * & * & * \end{vmatrix}$$

and direct computations give: $P = P_{11}P_5P_9$.

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