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ON ALMOST COMPLEX STRUCTURES ON FIBRE BUNDLES

Anton Dekrét

Abstract. If α is an almost complex structure on a manifold M then there is not a connection on M induced by α. In this paper the problem of connections on a fibre bundle $\pi : Y \to M$, $\dim M = \dim$ of fibres, which can be constructed from a given almost complex structure α on M only is explored.

Introduction

Let α be an almost complex structure (ACS) on a manifold M, $\dim M = 2m$, α is a (1,1)-tensor field on M such that $\alpha^2 = -Id_{TM}$. It is known, see [3], [4], that there is no connection on M, linear connection on the tangent bundle $p_M : TM \to M$, which is canonically induced by α. If α is an ACS on a fibre bundle $\pi : Y \to M$, $\dim M$ is the dimension of fibres, then the question of connections on Y entirely determined by α arises. Examples of such fibre bundles are $p_M : TM \to M$ and the cotangent bundle $\pi : T^*M \to M$. In this paper we construct connections from the given (1,1)-tensor field α on Y with emphasis on the ACS-case. If $Y = TM$ then there are some special geometric objects on TM which are in interesting relations to our topic. We have discussed them in [2]. In this paper all maps and manifolds are supossed to be smooth.

Connections and almost complex structures on Y

Let (x^i, y^i) be a local fibre chart on a fibre bundle $\pi : Y \to M$, $\dim M$ is the dimension of fibres.

Let us recall that a connection Γ on TY can be considered as a (1,1)-tensor field h_Γ (horizontal form of Γ), such that $T\pi h_\Gamma = T\pi, h_\Gamma(VY) = 0$, where $T\pi$ is the tangent map of the map π and VY is the vector bundle of all vertical vectors on Y, $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma^i_j(x,y) dx^j \otimes \partial/\partial y^i$. Then $h_\Gamma(TY) = H\Gamma$ is the so-called horizontal subbundle of Γ; $(x^i, y^i, dx^i, dy^i) \in H\Gamma$ if and only if $dy^i = \Gamma^i j dx^j, \Gamma^i_j(x,y)$ are said to be the functions of Γ.

Let $\alpha = (a^i_j(x,y)dx^j + b^i_j(x,y)dy^j) \otimes \partial/\partial x^i + (c^i_j(x,y)dx^j + h^i_j(x,y)dy^j) \otimes \partial/\partial y^i$ be a (1,1)-tensor field on Y. It is called vertical if $\alpha(VY) \subset VY$.

Denote $B : T\pi\alpha|_{VY} = b^i_j dy^j \otimes \partial/\partial x^i$.

It means that B can be considered as a vector bundle morphism $VY \to TM$ over π or $VY \to Yx_M TM$ over $Id|_Y$, i.e. as a section $Y \to V^*Y \otimes_Y TM$.

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Lemma 1. *A (1,1)-tensor field α is vertical iff $B = 0$.*

Proof is evident from the local form of α and B.

Remark 1. If B is regular, i.e. if it is an isomorphism, then we get the inverse vector bundle isomorphism $B^{-1} : Yx_M TM \to VY$ over $Id|_Y$, i.e. a section $B^{-1} : Y \to T^*M \otimes \otimes_Y VY, B^{-1} = \tilde{b}^i_j dx^j \otimes \partial/\partial y_i, \tilde{b}^i_k b^k_j = \delta^i_j$, i.e. a semibasic (1,1)-vector form with values in VY.

We will consider two cases.

1. $B \neq 0$, i.e. $\alpha(VY) \not\subset VY$, i.e. α is not vertical.
Let $\Gamma, dy^i = \Gamma^i_j dx^j$, be a connection on Y. Let $X = \eta^i \partial/\partial y^i$ be an arbitrary vertical vector on Y. Then $\alpha(X) = b^i_j \eta^j \partial/\partial x^i + h^i_j \eta^j \partial/\partial y^i$ is Γ-horizontal, i.e. $\alpha(X) \in H\Gamma$, if and only if $\Gamma^i_k b^k_j \eta^j = h^i_j \eta^j$. It means that $\alpha(VY) \subset H\Gamma$ iff

$$\Gamma^i_k b^k_j = h^i_j. \tag{1}$$

It immediately gives

Proposition 1. If and only if B is regular there is a unique connection Γ^2_α on Y such that $H\Gamma^2_\alpha = \alpha(VY)$.

The relation (1) induces that if B is regular then the functions of the connection Γ^2_α are $\Gamma^i_j = h^i_k \tilde{b}^k_j$.

We will construct another connections on Y when B is regular. Let $X = \xi^i \partial/\partial x^i + \eta^i \partial/\partial y^i$ be a vector on Y. Then $\alpha(X) = (a^i_j \xi^j + b^i_j \eta^j)\partial/\partial x^i + (c^i_j \xi^j + h^i_j \eta^j)\partial/\partial y^i$ is vertical if and only if

$$a^i_j \xi^j + b^i_j \eta^j = 0. \tag{2}$$

This leads

Proposition 2. If and only if B is regular there is a unique connection Γ^1_α on Y such that $\alpha(H\Gamma^1_\alpha) = VY$, i.e. with the functions $\Gamma^i_j = -\tilde{b}^i_k a^k_j$.

Remark 2. Recall that if φ is a semibasic (1,1)-form on Y with values in VY, i.e. if φ is a section $Y \to T^*M \otimes_Y VY$ and h_Γ is the horizontal form of a connection Γ on Y then $h_\Gamma + \varphi$ is the other connection on Y. So if B is regular then $h_{\Gamma^1_\alpha} + cB^{-1}$ and $h_{\Gamma^2_\alpha} + cB^{-1}, c \in \mathbb{R}$, are another connections on Y.

The (1,1)-tensor form α is a vector bundle morphism $TY \to TY$ over $Id|_{TM}$. Then

$$\begin{aligned}\alpha^2 = \alpha\alpha = [(a^i_s a^s_j + b^i_s c^s_j)dx^j + (a^i_s b^s_j + b^i_s h^s_j)dy^j] \otimes \partial/\partial x^i + \\ + [(c^i_s a^s_j + h^i_s c^s_j)dx^j + (c^i_s b^s_j + h^i_s h^s_j)dy^j] \otimes \partial/\partial y^i.\end{aligned}$$

So α is an ACS on Y, i.e. $\alpha^2 = -Id|_{TY}$, iff

$$a^i_s a^s_j + b^i_s c^s_j = -\delta^i_j, \quad a^i_s b^s_j + b^i_s h^s_j = 0, \quad c^i_s a^s_j + h^i_s c^s_j = 0, \quad c^i_s b^s_j + h^i_s h^s_j = -\delta^i_j. \tag{3}$$

It is easy to see that if B is regular then the third and fourth equations of the relations (3) are the consequence of the first and second ones.

Proposition 3. Let α be a (1,1)-tensor field on Y such that B is regular. Then $\Gamma^1_\alpha = \Gamma^2_\alpha$ if and only if α^2 is vertical.

Proof. α^2 is vertical iff the second equation of (3) is satisfied, i.e. iff $a^i_j = b^i_s h^s_k \tilde{b}^k_j$. Then ${}^1\Gamma^i_j = -\tilde{b}^i_s a^s_j = h^i_k \tilde{b}^k_j = {}^2\Gamma^i_j$. Conversally, if $\Gamma^1_\alpha = \Gamma^2_\alpha$ then $-\tilde{b}^i_s a^s_j = h^i_s \tilde{b}^s_j$, i.e. $-a^i_s b^s_j = b^i_s h^s_j$, i.e. α^2 is vertical.

We will focus ourselves to the connections Γ which are invariant according to α, i.e. $\alpha(H\Gamma) \subset H\Gamma$.

Let $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma^i_j dx^j \otimes \partial/\partial y^i$ be an arbitrary connection on Y. Then

$$\alpha h_\Gamma = (a^i_j + b^i_k \Gamma^k_j) dx^j \otimes \partial/\partial x^i + (c^i_j + h^i_k \Gamma^k_j) dx^j \otimes \partial/\partial y^i.$$

Let $\overline{\Gamma}$ be another connection given by the equation $dy^i = \overline{\Gamma}^i_j dx^j$. Then $\alpha(H\Gamma) \subset H\overline{\Gamma}$ if and only if

$$\overline{\Gamma}^i_k (a^k_j + b^k_u \Gamma^u_j) = c^i_j + h^i_k \Gamma^k_j \quad \text{or} \tag{4}$$

$$c^i_j = \Gamma^i_k a^k_j - h^i_k \Gamma^k_j + \Gamma^i_k b^k_u \Gamma^u_j \quad \text{for } \overline{\Gamma} = \Gamma. \tag{5}$$

Consider the space $VY \otimes_Y T^*M$ of all semibasic VY-valued (1,1)-forms on Y. Let $\gamma = \gamma^i_j dx^j \otimes \partial/\partial y^i \in T^*M \otimes_Y VY$. Denote

$$\alpha^- : \gamma \to \alpha\gamma = b^i_t \gamma^t_j dx^j \otimes \partial/\partial x^i + h^i_t \gamma^t_j dx^j \otimes \partial/\partial y^i,\ T^*M \otimes_Y VY \to T^*M \otimes TY,$$

$$\alpha^+ : \gamma \to \gamma\alpha = (\gamma^i_k a^k_j dx^j + \gamma^i_k b^k_j dy^j) \otimes \partial/\partial y^i,\ T^*M \otimes_Y VY \to T^*Y \otimes VY.$$

Note that if B is regular then $\alpha^-(B^{-1}) = h_{\Gamma^2_\alpha}$ and $\alpha^+(B^{-1})$ is the vertical form $v_\Gamma = = Id_{TY} - h_{\Gamma^1_\alpha}$ of the connection Γ^1_α.

Definition 1. Two (1,1)-tensor fields α_1, α_2 on Y will be called $(+,-)$-equivalent if $\alpha_1^- = = \alpha_2^-$, $\alpha_1^+ = \alpha_2^+$.

It is evident that the relations ${}^1a^i_j = {}^2a^i_j$, ${}^1b^i_j = {}^2b^i_j$, ${}^1h^i_j = {}^2h^i_j$ are the coordinate conditions for α_1, α_2 to be $(+,-)$-equivalent.

The relation (4) immediately yields

Proposition 4. Let $\Gamma, \overline{\Gamma}$ be connections on Y. Then in every class of the $(+,-)$-equivalent (1,1)-tensor fields on Y there exists a unique (1,1)-tensor field $\alpha_{\Gamma,\overline{\Gamma}}$ such that $\alpha_{\Gamma,\overline{\Gamma}}(H\Gamma) \subset \subset H\overline{\Gamma}$.

If $\Gamma = \overline{\Gamma}$ then we use the denotation α_Γ instead of $\alpha_{\Gamma,\Gamma}$.

Proposition 5. Let α be such a (1,1)-tensor field on Y that B is regular. Then $\alpha_{\Gamma^1_\alpha} = \alpha_{\Gamma^2_\alpha}$ and $\alpha_{\Gamma^1_\alpha}$ cannot be an almost complex structure on Y.

Proof. By the relation (5) in both cases of Γ^1_α and Γ^2_α we get $c^i_j = h^i_t \tilde{b}^t_s a^s_j$. So $\alpha_{\Gamma^1_\alpha} = = \alpha_{\Gamma^2_\alpha} = (a^i_j dx^j + b^i_j dy^j) \otimes \partial/\partial x^i + (h^i_t \tilde{b}^t_s a^s_j dx^j + h^i_j dy^j) \otimes \partial/\partial y^i$.

If $\alpha_{\Gamma^1_\alpha}$ is an ACS then the first and second equations of (3) read

$$a^i_s a^s_j + b^i_s h^s_t \tilde{b}^t_k a^k_j = -\delta^i_j, \quad \tilde{b}^i_t a^t_j + h^i_t \tilde{b}^t_j = 0.$$

Then $a^i_s a^s_j - b^i_s \tilde{b}^s_t a^t_k a^k_j = -\delta^i_j$. It is not possible. So $\alpha_{\Gamma^1_\alpha}$ cannot be an almost complex structure on Y.

Definition 2. Let $(1,1)_B$ denote the set of all (1,1)-tensor field α on Y such that B is regular. We will say that two (1,1)-tensor field $\alpha_1, \alpha_2 \in (1,1)_B$ are (+)-equivalent if $\alpha_1^+ = \alpha_2^+$.

In coordinates, α_1 and α_2 are (+)-equivalent iff ${}^1a^i_j = {}^2a^i_j$, ${}^1b^i_j = {}^2b^i_j$, $\det {}^1b^i_j \neq 0$, $\det {}^2b^i_j \neq 0$.

Proposition 6. In every class of all (+)-equivalent (1,1)-tensor fields there is a unique almost complex structure on Y.

Proof in coordinates. A class of all (+)-equivalent (1,1)-tensor fields is given by the local functions $a^i_j, b^i_j, \det b^i_j \neq 0$. By the first and second equations of the relation (3) a tensor field of this class is an ACS iff $c^i_j = -\tilde{b}^i_j - \tilde{b}^i_k a^k_s a^s_j$, $h^i_j = -\tilde{b}^i_s a^s_k b^k_j$. It completes our proof.

Remark 3. The same can be said for the class of (−)-equivalent tensor fields.

Remark 4. If α is an ACS on Y then α is vertical and so $\Gamma^1_\alpha = \Gamma^2_\alpha$.

Proposition 7. Let Γ be a connection on Y. Let $B : Y \to V^*Y \otimes_Y TM$, $B = b^i_j dy^j \otimes \partial/\partial x^i$, be a vector bundle isomorphism $VY \to TM$ over π. Then there exists a unique almost complex structure α on Y such that $T\pi\alpha|_{VY} = B$ and $\Gamma^1_\alpha = \Gamma = \Gamma^2_\alpha$.

Proof. Let Γ^i_j be the functions of Γ. Let α be an arbitrary (1,1)-tensor field on Y such that $T\pi\alpha|_{VY} = B$ and $\Gamma^1_\alpha = \Gamma = \Gamma^2_\alpha$. Then $\Gamma^i_j = -\tilde{b}^i_k a^k_j$, $\Gamma^i_j = h^i_k \tilde{b}^k_j$, i.e. $a^i_j = -b^i_s \Gamma^s_j$, $h^i_j = \Gamma^i_s b^s_j$ and the second equation of (3) is satisfied. By the first equation of (3) α is an ACS iff $c^i_j = -\tilde{b}^i_j - \Gamma^t_s b^s_u \Gamma^u_j$. So such an ACS locally exists and is unique.

We are turning to the second case of α.

2. Let $B = T\pi\alpha|_{VY} = 0$, i.e. $\alpha(VY) \subset VY$. We have

$$\begin{aligned} &\alpha = a^i_j dx^j \otimes \partial/\partial x^i + (c^i_j dx^j + h^i_j dy^j) \otimes \partial/\partial y^i, \\ &A := T\pi\alpha = a^i_j dx^j \otimes \partial/\partial x^i \\ &H := \alpha|_{VY} = h^i_j dy^j \otimes \partial/\partial y^i. \end{aligned}$$

So A is a section $Y \to T^*M \otimes_Y TM$ determining a vector bundle morphism $TY \to TM$ over $\pi : Y \to M$ or $Yx_M TM \to Yx_M TM$ over Id_Y and H is a section $Y \to V^*Y \otimes VY$ determining a vector bundle morphism $VY \to VY$ over Id_Y.

Let $\Gamma, \overline{\Gamma}$ be two connections on Y with the local functions $\Gamma^i_j, \overline{\Gamma}^i_j$. When $B = 0$ the equations (4) and (5) read

$$c^i_j = \overline{\Gamma}^i_k a^k_j - h^i_k \Gamma^k_j, \tag{4'}$$

$$c^i_j = \Gamma^i_k a^k_j - h^i_k \Gamma^k_j. \tag{5'}$$

Proposition 4 can be reformulated as follows

Proposition 8. Let $H : Y \to V^*Y \otimes VY,\ A : Y \to T^*M \otimes_Y TM$ be two sections. Let $\Gamma, \overline{\Gamma}$ be two connections on Y. Then there is a unique vertical (1,1)-tensor field $\alpha(A, H, \Gamma, \overline{\Gamma})$ on Y such that $\alpha|_{VY} = H,\ T\pi\alpha = A,\ \alpha(H\Gamma) \subset H\overline{\Gamma}$.

If $\Gamma = \overline{\Gamma}$ we use the denotation $\alpha(A, H, \Gamma)$ instead of $\alpha(A, H, \Gamma, \overline{\Gamma})$.

In the case of a vertical (1,1)-tensor field α the coordinate conditions (3) for α to be an ACS are of the form

$$(3') \qquad a^i_s a^s_j = -\delta^i_j, \quad c^i_s a^s_j + h^i_s c^s_j = 0, \quad h^i_s h^s_h = -\delta^i_j.$$

Preserving the above denotations we have the following vector bundle morphism on $T^*M \otimes_Y VY$ over Id_Y:

$$H^- : \gamma \to H\gamma = h^i_k \gamma^k_j dx^j \otimes \partial/\partial y^i,\ \gamma \in T^*M \otimes_Y VY, \text{ so } H^- = \alpha^-,$$
$$A^+ : \gamma \to \gamma A = \gamma^i_k a^k_j dx^j \otimes \partial/\partial y^i, \text{ so } A^+ = \alpha^+,$$
$$\mathcal{H} : A^+ - H^- : \gamma \to (\gamma^i_k a^k_j - h^i_k \gamma^k_j) dx^j \otimes \partial/\partial y^i,$$
$$\overline{\mathcal{H}} := A^+ + H^- : \gamma \to (\gamma^i_k a^k_j + h^i_k \gamma^k_j) dx^j \otimes \partial/\partial y^i.$$

The relation (5') immediately gives

Proposition 9. Let α be a such vertical (1,1)-tensor field on Y that the map $\mathcal{H}$ is a vector bundle isomorphism on $T^*M \otimes_Y VY$ over Id_Y. Then there is a unique connection Γ on Y such that $\alpha(H\Gamma) \subset H\Gamma$.

Lemma 2. *If a vertical (1,1)-tensor field α is an almost complex structure on Y then the maps $\mathcal{H}$ and $\overline{\mathcal{H}}$ are not isomorphisms on $T^*M \otimes_Y VY$.*

Proof. The map $(H^- + A^+)\mathcal{H} = H^-\mathcal{H} + A^+\mathcal{H} : \gamma \to (h^i_t \gamma^t_k a^k_j - h^i_t h^t_k \gamma^k_j) + (\gamma^i_t a^t_k a^k_j - -h^i_t \gamma^t_k a^k_j)$ is a vector bundle morphism on $T^*Y \otimes_Y VY$. If α is on ACS on Y then by (3') we get $(H^- + A^+)\mathcal{H} = 0$. If $\mathcal{H}$ is regular then $H^- + A^+ = 0$. But the equation $H\gamma = -\gamma A$ is satisfied for all $\gamma \in T^*M \otimes_Y VY$ if and only if $H = k \cdot Id = -A,\ k \in \mathbb{R}$. Then $h^i_k h^k_j = k^2 \delta^i_j$, i.e. $k^2 = -1$. It is contrary with $k \in \mathbb{R}$. Analogously the supposition "$\overline{\mathcal{H}}$ is regular" leads to contradiction.

Remark 5. If α is a vertical ACS on Y then according to (5') such a connection Γ that $\alpha(H\Gamma) \subset H\Gamma$ can but not have to exist. If it exists then it does not need to be unique.

Remark 6. Let $A : Y \to T^*M \otimes_Y TM$ be an ACS on $Yx_M TM$. Let $H : Y \to V^*Y \otimes VY$ be an ACS on VY. In view of the relation (3') there exists a vertical ACS α on Y such that $T\pi\alpha = A,\ \alpha|_{VY} = H$ and is not unique.

Proposition 10. Let $A : Y \to T^*M \otimes_Y TM$ be an ACS on $Yx_M TM$. Let $H : Y \to \to V^*Y \otimes VY$ be an ACS on VY. Let Γ be a connection on Y. Then the vertical (1,1)-tensor field $\alpha(A, H, \Gamma)$ described in Proposition 8 is an almost complex structure.

Proof. By Proposition 8, $\alpha(A, H, \Gamma)$ is the unique vertical (1,1)-tensor field on Y such that $\alpha(A, H, \Gamma)|_{VY} = H,\ T\pi\alpha(A, H, \Gamma) = A$ and $\alpha(A, H, \Gamma)(H\Gamma) \subset H\Gamma$. If $A = a^i_j dx^j \otimes$

$\otimes\partial/\partial x^i$, $H = h^i_j dy^j \otimes \partial/\partial y^i$ and Γ^i_j are the functions of Γ then the coordinates c^i_j of $\alpha(A, H, \Gamma)$ are determined by (5'). So $\alpha(A, H, \Gamma) = a^i_j dx^j \otimes \partial/\partial x^i + [(\Gamma^i_k a^k_j - h^i_k \Gamma^k_j)dx^j + {} + h^i_j dy^j] \otimes \partial/\partial y^i$. The functions a^i_j, h^i_j satisfy the first and third equations of (3'). Then $c^i_s a^s_j + h^i_s c^s_j = (\Gamma^i_k a^k_s - h^i_k \Gamma^k_s)a^s_j + h^i_s(\Gamma^s_k a^k_j - h^s_k \Gamma^k_j) = 0$. So $\alpha(A, H, \Gamma)$ satisfies the relations 3 and is an almost complex structure.

Remark 7. Let $\pi : Y \to M$ be a vector fibre bundle. Let α be a VB-(1,1)-tensor field, i.e. $\alpha(X)$ is a linear projectable vector field on Y for all linear projectable vector fields X on Y. In a local fibre chart $\alpha = a^i_j(x)dx^j \otimes \partial/\partial x^i + [c^i_{jk}(x)y^k dx^j + h^i_j(x)dy^j] \otimes \otimes\partial/\partial y^i$, see [1]. In this case $\alpha|_{VY} = H$ is a vector bundle morphism on Y over Id_M with the coordinate expression: $\overline{x}^i = x^i$, $\overline{y}^i = h^i_j(x)y^j$. These equations with the added following ones $d\overline{x}^i = dx^i$, $d\overline{y}^i = h^i_{jk} y^j dx^k + h^i_j dy^j$, where we use $\frac{\partial f}{\partial x^i} := f_i$, determine the tangent map TH. Let Γ, $\Gamma^i_j(x, y) = \Gamma^i_{jk} y^j$, be a linear connection on Y. Then it is easy to deduce that the equations

$$\Gamma^i_{js} h^s_k - h^i_s \Gamma^s_{jk} = h^i_{kj}$$

are the coordinate conditions under which $TH(H\Gamma) \subset H\Gamma$. The solution Γ^i_{js} of these equations can but not has to exist. Let $\alpha(H\Gamma) \subset H\Gamma$. Then by (5'): $c^i_{kj} = \Gamma^i_{sj} a^s_k - {} - h^i_s \Gamma^s_{kj}$. If Γ is without torsion then the conditions $TH(H\Gamma) \subset H\Gamma$, $\alpha(H\Gamma) \subset H\Gamma$ lead to

$$\Gamma^i_{js}(h^s_k - a^s_k) = h^i_{kj} - c^i_{kj}. \tag{6}$$

If $H - A$ has sense (for instance in the case of $y = TM$) and if $H - A$ is regular then there is a unique solution Γ^i_{js} of (6). For example if $A = -H$, and H is regular then $\Gamma^i_{js} = \frac{1}{2}(h^i_{ks} - c^i_{ks})\tilde{h}^k_s$. In view of Proposition 10 we can say that if α is a symmetric VB-almost complex structure on TM such that $A = -H$, then there exists a unique symmetric linear connection such that $TH(H\Gamma) \subset H\Gamma$,
$\alpha(H\Gamma) \subset H\Gamma$. We will deal in detail with such an almost complex structure in our other paper.

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Department of mathematics, TU Zvolen, Masarykova 24,
960 53 Zvolen, Slovakia

E-mail address: dekret@vsld.tuzvo.sk

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METRICS ON SYSTEMS OF FINITE ALGEBRAS

Alfonz Haviar

Abstract. In this paper four different metrics on a system of n-element algebras of the same type are presented. For groupoids and lattices the maximal distance of algebras is also determined.

Introduction

In [1], [5] and [3], [4], metrics on systems of graphs and posets, respectively, are investigated. In this paper we show an analogous way of defining metrics on a system of pairwise non-isomorphic finite algebras of the same type.

In universal algebra, isomorphic algebras are not usually considered to be different. Assuming that two n-element algebras of the same type are not isomorphic one can seek for a bijection compatible with the operations as much as possible. Such an approach yields the first way of defining the metric. The concept of the homomorphism generalizes that of the isomorphism, and in our second approach, it motivates the definition of a measure of difference between algebras. Our third approach is based on the fact that two non-isomorphic algebras may have 'large' isomorphic subalgebras. The distance of algebras depends on the cardinality of these isomorphic subalgebras. As it turned out for systems of graphs, the approach based on subgraphs can be replaced by that based on supergraphs [2]. A similar idea can be applied also to the class of all finite algebras of the same type.

The second (homomorphic) metric and the third (substructure) metric can also be considered for finite algebras of the same type having different cardinalities, as in the proofs that these functions are metrics the cardinalities of algebras are not relevant. The first metric could also be modified (similarly as for graphs) for algebras of the same type but different cardinalities. However, it is hard to decide how much the metric depends on the difference of cardinalities of algebras and on the difference between algebraic properties of given algebras.

The concept of a metric reflects a 'distance' between classes containing isomorphic algebras. However, in order to simplify the terminology we will speak on a 'distance' between algebras.

We will particularly focus our attention to the metrics on systems of groupoids and lattices where we also determine the maximal distance of two algebras.

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Throughout this paper the set $N_n = \{0, 1, \ldots, n-1\}$ is taken as a universe of n-element algebra. By $\mathcal{S}_n$ we denote a system of pairwise non-isomorphic n-element algebras of the same type. By $\mathcal{G}_n$ ($\mathcal{L}_n$) we denote the system of all pairwise non-isomorphic n-element groupoids (lattices). The maximal distance between algebras on the system $\mathcal{G}_n$ ($\mathcal{L}_n$) in a metric d_j will be denoted by $D_j(\mathcal{G}_n)$ ($D_j(\mathcal{L}_n)$). The number D_j will be called the diameter of the system $\mathcal{G}_n$ ($\mathcal{L}_n$).

1. Isomorphism metric

Let $\mathbf{A} = (A, F_1, \ldots, F_m)$, $\mathbf{B} = (B, F'_1, \ldots, F'_m)$ be n-element algebras of the same type. We denote by $M(A, B)$ the set of all bijections of A onto B. Let $f \in M(A, B)$ and let F_j, $1 \leq j \leq m$ be a k-ary operation of $\mathbf{A}$. For $k \geq 1$ we put

$$D_j(f) = \{[a_1, \ldots, a_k] \in A^k; f(F_j(a_1, \ldots, a_k)) \neq F'_j(f(a_1), \ldots, f(a_k))\}$$

and for $k = 0$

$$D_j(f) = \begin{cases} \{F_j\} & \text{if} \quad f(F_j) \neq F'_j, \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let

$$D(f) = D_1(f) \cup \cdots \cup D_m(f),$$

and

$$d_{iso}(\mathbf{A}, \mathbf{B}) = min\{\mid D(f) \mid; f \in M(A, B)\}, \tag{1}$$

where $\mid D(f) \mid$ is the cardinality of the set $D(f)$.

Theorem 1.1. *The function d_{iso} given by (1) is a metric on the system $\mathcal{S}_n$ (of pairwise non-isomorphic n-element algebras of the same type).*

Proof. Clearly, $d_{iso}(\mathbf{A}, \mathbf{B}) = 0$ if and only if $\mathbf{A} = \mathbf{B}$.

If $f(F_j(a_1, \ldots, a_k)) \neq F'_j(f(a_1), \ldots, f(a_k))$ for a bijection $f : A \to B$, k-ary operation $F_j, k \geq 1$, and elements $a_1, \ldots a_k$ then $f^{-1}(F'_j(b_1, \ldots, b_k)) \neq F_j(f^{-1}(b_1), \ldots, f^{-1}(b_k))$ for the elements $b_1 = f(a_1), \ldots, b_k = f(a_k)$. It follows $d_{iso}(\mathbf{A}, \mathbf{B}) = d_{iso}(\mathbf{B}, \mathbf{A})$.

Let $d_{iso}(\mathbf{A}, \mathbf{B}) =\mid D(f) \mid$, $d_{iso}(\mathbf{B}, \mathbf{C}) =\mid D(g) \mid$ and $d_{iso}(\mathbf{A}, \mathbf{C}) =\mid D(h) \mid$. It is obvious that $\mid D(g \circ f) \mid \geq \mid D(h) \mid$. We shall have established $\mid D(f) \mid + \mid D(g) \mid \geq \mid D(h) \mid$ if we prove that

$$\mid D(f) \mid + \mid D(g) \mid \geq \mid D(g \circ f) \mid. \tag{1a}$$

The inequality (1a) follows easily from the fact that $[a_1, \ldots, a_k] \in D(f)$ or $[f(a_1), \ldots, f(a_k)] \in D(g)$, if $[a_1, \ldots, a_k] \in D(g \circ f)$ and $F_i \in D(f)$ or $F'_i \in D(g)$ if $F_i \in D(g \circ f)$, respectively.

Theorem 1.2. $D_{iso}(\mathcal{G}_n) = n^2 \quad if \quad n \geq 2$,
$D_{iso}(\mathcal{L}_n) = (n-2)^2 - (n-2) \quad if \quad n \geq 4.$

Proof. a) Of course, $d_{iso}(\mathbf{G}_1, \mathbf{G}_2) \leq n^2$ holds for any n-element groupoids $\mathbf{G}_1, \mathbf{G}_2$.
We define the operations $\circ$ and $*$ on the set $N_n = \{0, \ldots, n-1\}$, $n \geq 2$, by

$$x \circ y = x \quad \text{for every number } x$$

$$x * y = \begin{cases} y, & \text{if} \quad x \neq y \\ x+1, & \text{if} \quad x = y \end{cases}$$

(we compute modulo n). It follows immediately that $f(x \circ y) \neq f(x) * f(y)$ for any permutation f of N_n and any numbers $x, y \in N_n$. Therefore we have
$d_{iso}((M_n, o), (M_n, *)) = n^2$.
b) Let $\mathbf{L}_1 = (L_1, \vee, \wedge, 0, 1)$, $\mathbf{L}_2 = (L_2, \vee, \wedge, 0, 1)$ be n-element lattices. Let $f : L_1 \to L_2$ be a 0, 1-preserving bijection. It follows immediately that

(1b) $$| D(f) | \leq (n-2)^2 - (n-2).$$

The equality $d_{iso}(\mathbf{L}_1, \mathbf{L}_2) = (n-2)^2 - (n-2)$ holds if $\mathbf{L}_1$ is the n-element chain and $\mathbf{L}_2$ is the n-element lampion (a lattice of height 2 with n-2 atoms), $n \geq 4$. □

2. Homomorphism metric

Let $\mathbf{A}, \mathbf{B}$ be n-element algebras of the same type. If $f : A \to B$ is a homomorphism then $f(A)$ is a subuniverse of $\mathbf{B}$. We call $f(A)$ the homomorphic image of $\mathbf{A}$ in $\mathbf{B}$. If there is no homomorphism $f : A \to B$ we define the homomorphic image of $\mathbf{A}$ in $\mathbf{B}$ to be $\emptyset$.

Let $\mathcal{S}_n$ be a system of pairwise non-isomorphic n-element algebras of the same type. We define the distance of algebras $\mathbf{A}, \mathbf{B} \in \mathcal{S}_n$ by

(2) $$d_h(\mathbf{A}, \mathbf{B}) = | A | - | f(A) | + | B | - | g(B) |,$$

where f is a homomorphism $A \to B$ such that the cardinality of $f(A)$ is maximal possible. Analogously, for $g : B \to A$.

Theorem 2.1. *The function d_h given by (2) is a metric on the system $\mathcal{S}_n$.*

Proof. We see at once that $d_h(\mathbf{A}, \mathbf{B}) = 0$ iff $\mathbf{A} = \mathbf{B}$ and $d_h(\mathbf{A}, \mathbf{B}) = d_h(\mathbf{B}, \mathbf{A})$.
Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{S}_n$ and let
$f : A \to B, \quad g : B \to C, \quad h : A \to C,$
$F : B \to A, \quad G : C \to B, \quad H : C \to A,$
be such homomorphisms that $f(A), \ldots, H(C)$ are maximal homomorphic images. We want to prove the inequality

$$| A | - | h(A) | + | C | - | H(C) | \leq$$
$$\leq | A | - | f(A) | + | B | - | F(B) | + | B | - | g(B) | + | C | - | G(C) | .$$

It is sufficient to show that

$$(2a) \qquad \mid f(A) \mid + \mid g(B) \mid \leq \mid B \mid + \mid h(A) \mid$$

and

$$(2b) \qquad \mid F(B) \mid + \mid G(C) \mid \leq \mid B \mid + \mid H(C) \mid .$$

We are going to prove (2a) ((2b) can be proved in the same way). Since $\mid h(A) \mid \geq \mid g(f(A)) \mid$ it suffices to show that $\mid g(B) \mid - \mid g(f(A)) \mid \leq \mid B \mid - \mid f(A) \mid$, i.e.

$$(2c) \qquad \mid g(B) - g(f(A)) \mid \leq \mid B - f(A) \mid .$$

The inequality (2c) follows from

$$\mid B - f(A) \mid \geq \mid g(B - f(A)) \mid \geq \mid g(B) - g(f(A)) \mid .$$

If there are no homomorphisms from **A** to **B** or from **B** to **C** (i.e. if $f(A) = \emptyset$ or $g(B) = \emptyset$), the inequality (2a) also holds.

Remark. *We note that the proof runs if we drop the assumption that* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are algebras of the same cardinality.*

Theorem 2.2. $D_h(\mathcal{G}_n) = 2n$ *if* $n \geq 2$,
$D_h(\mathcal{L}_n) = 2n - 2$ *if* $n \geq 7$.

Proof. a)It is evident that $D_h(\mathcal{G}_n) \leq 2n$. We will find two non-isomorphic groupoids whose congruence lattices are trivial and the sets of idempotent elements are empty.

We define the operations $\circ$ and $*$ on the set $N_n = \{0, 1, \ldots, n-1\}$ $n \geq 2$ by

$$\begin{aligned} & i \circ (n-1) = i * (n-1) = i + 1, \\ & i \circ i = i * i = i + 1, \\ & i \circ k = i * k = k + 2 \quad \text{if} \quad i = k+2, \ldots n-1, \\ & i \circ k = 1, \quad i * k = k \quad \text{otherwise} \end{aligned}$$

(we compute modulo n). Now, we are going to show that the groupoids $(N_n, \circ)$ and $(N_n, *)$ have only trivial congruences. Let $\Theta \in Con(N_n, \circ)$, $i \Theta j$ and $i < j$. The equality $i+1 = j$ implies $i \circ (n-1) \Theta (i+1) \circ (n-1)$, i.e. $i+1 \Theta i+2$, analogously $i+2 \Theta i+3$, etc., hence $\Theta = N_n^2$. If $i+1 < j$ we have $i \circ i \Theta j \circ i$, i.e. $i+1 \Theta i+2$, and this again yields $\Theta = N_n^2$. In the same manner one can see that $(N_n, *)$ has trivial congruences. It is obvious that $(N_n, \circ)$ and $(N_n, *)$ are non-isomorphic and they do not contain any idempotent elements. Hence $d_h((N_n, \circ), (N_n, *)) = 2n$.

b) It is sufficient to find two non-isomorphic lattices such that they have no non-trivial congruences. It is immediate to check that the n-element lampion and the lattice depicted in Fig. 1 ($n \geq 7$) have only trivial congruences.

$\square$

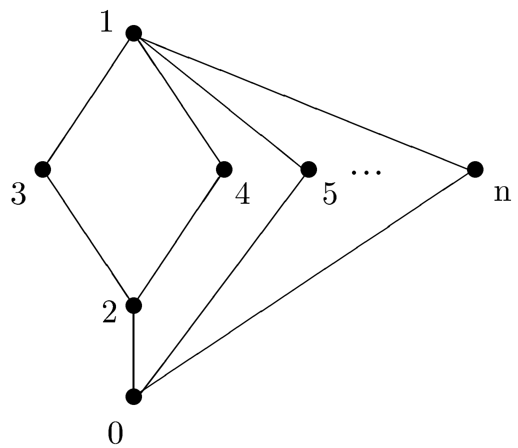

Fig. 1

3. Substructure metric.

Let $\mathbf{A}, \mathbf{B}$ be n-element algebras of the same type. We call an subalgebra $\mathbf{A}_1$ of $\mathbf{A}$ a *common subalgebra* of $\mathbf{A}$ and $\mathbf{B}$ if there exists a subalgebra $\mathbf{B}_1$ of $\mathbf{B}$ such that $\mathbf{A}_1$ and $\mathbf{B}_1$ are isomorphic. In this case we denote the universe A_1 of $\mathbf{A}_1$ by S_{AB}. Otherwise (i.e. if there are no isomorphic subalgebras of $\mathbf{A}$ and $\mathbf{B}$) we put $S_{AB} = \emptyset$. We define the distance of algebras $\mathbf{A}$ and $\mathbf{B}$ by

$$d_s(\mathbf{A}, \mathbf{B}) = \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid, \tag{3}$$

where $\mid S_{AB} \mid$ is the maximum of cardinalities of common subuniverses of $\mathbf{A}$ and $\mathbf{B}$.

Theorem 3.1. *The function d_s given by (3) is a metric on the system $\mathcal{S}_n$.*

Proof. We will prove only the triangle inequality. Let

$$\begin{aligned}
d_s(\mathbf{A}, \mathbf{B}) &= \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid, \\
d_s(\mathbf{B}, \mathbf{C}) &= \mid B \mid + \mid C \mid -2. \mid S_{BC} \mid, \\
d_s(\mathbf{A}, \mathbf{C}) &= \mid A \mid + \mid C \mid -2. \mid S_{AC} \mid,
\end{aligned}$$

and let f and h be embeddings of $\mathbf{S}_{AB}$ into $\mathbf{B}$ and $\mathbf{S}_{AC}$ into $\mathbf{C}$, respectively and let g be an embedding of $\mathbf{S}_{BC}$ into $\mathbf{C}$. It suffices to prove that

$$\mid B \mid + \mid S_{AC} \mid \geq \mid S_{AB} \mid + \mid S_{BC} \mid . \tag{3a}$$

It is easily seen that $B' = f(S_{AB}) \cap S_{BC}$ is a subuniverse of the algebra $\mathbf{B}$. Further, it is evident that

$$\mid B \mid + \mid B' \mid \geq \mid f(S_{AB}) \mid + \mid S_{BC} \mid = \mid S_{AB} \mid + \mid S_{BC} \mid . \tag{3b}$$

The subalgebras of **A** and **C** with subuniverses $f^{-1}(B')$ and $g(B')$ are isomorphic, therefore

(3c) $$|B'| \leq |S_{AC}|.$$

Combining (3c) with (3b) we have (3a).

If $S_{AB} = \emptyset$ or $S_{BC} = \emptyset$, the inequality (3a) is evident. $S_{AC} = \emptyset$ implies $B' = \emptyset$ and again (3a) holds. □

Theorem 3.2. $D_s(\mathcal{G}_n) = 2n$ *if* $n \geq 2$,
$D_s(\mathcal{L}_n) = 2n - 6$ *if* $n \geq 4$.

Proof. a) For example, the groupoids $(N_n, \circ)$ and $(N_n, *)$ with operations given by

$$x \circ x = x * x = x + 1,$$
$$x \circ y = x \quad \text{if} \quad x \neq y,$$
$$x * y = y \quad \text{if} \quad x \neq y,$$

(we compute modulo n) have the distance 2n. The details are left to the reader.

b) Every n-element lattice ($n \geq 4$) contains a 3-element chain. From this we have $D_s(\mathcal{L}_n) \leq 2n - 6$. The distance of the n-element chain and the n-element lampion is $2n - 6$.

Remark. We note that the proof runs if we drop the assumption that **A**, **B**, **C** are algebras of the same cardinality.

The next examples show that the metrics d_{iso}, d_h and d_s are independent.

Example 1. Let $(G, \circ)$, $(H, *)$ and $(K, .)$ be the groupoids given by Cayley's tables 1, 2 and 3, respectively.

o	a	b
a	b	a
b	b	a

Tab. 1

*	c	d
c	c	d
d	c	d

Tab. 2

·	0	1
0	0	0
1	0	1

Tab. 3

We can easily check that
$d_{iso}(\mathbf{G}, \mathbf{H}) = 4 = d_s(\mathbf{G}, \mathbf{H}) > d_h(\mathbf{G}, \mathbf{H}) = 3$,
$d_{iso}(\mathbf{H}, \mathbf{K}) = 1 < d_s(\mathbf{H}, \mathbf{K}) = 2 = d_h(\mathbf{H}, \mathbf{K})$,
$d_{iso}(\mathbf{G}, \mathbf{K}) = 3 = d_h(\mathbf{G}, \mathbf{K}) < d_s(\mathbf{G}, \mathbf{K}) = 4$.

Example 2. Let $(L_1, \vee, \wedge)$ be the 4-element lattice of height 2 (the lampion) and $(L_2, \vee, \wedge)$ the 4-element chain. It is obvious that
$d_{iso}(\mathbf{L}_1, \mathbf{L}_2) = 4 > d_h(\mathbf{L}_1, \mathbf{L}_2) = 3 > d_s(\mathbf{L}_1, \mathbf{L}_2) = 2$.

4. Superstructure metric

Let τ be a type of algebras. We define a metric on the system of all pairwise non-isomorphic n-elements algebras of the type τ as follows.

Let $\mathbf{A}, \mathbf{B}$ be n-element algebras of the type τ. We define the distance of $\mathbf{A}$ and $\mathbf{B}$ by

$$d_{Su}(\mathbf{A}, \mathbf{B}) = 2. \mid O_{AB} \mid - \mid A \mid - \mid B \mid, \tag{4}$$

where O_{AB} is a minimal algebra (with respect to the cardinality of its universe) of the type τ which contains subalgebras isomorphic to $\mathbf{A}$ and $\mathbf{B}$.

Theorem 4.1. *The function d_{Su} given by (4) is a metric on the system of all pairwise non-isomorphic n-element algebras of the type τ.*

Proof. We will show that (similarly as on a system of graphs)

$$d_{Su}(\mathbf{A}, \mathbf{B}) = d_s(\mathbf{A}, \mathbf{B}).$$

By (3)

$$d_s(\mathbf{A}, \mathbf{B}) = \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid,$$

where S_{AB} is a maximal algebra (with respect to the cardinality of its universe) such that there exist a subalgebra $\mathbf{A}_1$ of $\mathbf{A}$ which is isomorphic to $\mathbf{S}_{AB}$ and a subalgebra $\mathbf{B}_1$ of $\mathbf{B}$ which is isomorphic to $\mathbf{S}_{AB}$. Without loss of generality we can assume that $A_1 = B_1 = S_{AB}$. Let $C = A \cup B$ and let $\mathbf{F}$ be k-ary operation symbol of τ, $k \geq 1$. Fix an element $b \in A$. We define the operation F on the set C in the following way:

If $a_1, \ldots, a_k, a \in A$ and $F(a_1, \ldots, a_k) = a$ in $\mathbf{A}$ or
$a_1, \ldots, a_k, a \in B$ and $F(a_1, \ldots, a_k) = a$ in $\mathbf{B}$
then $F(a_1, \ldots, a_k) = a$. Otherwise $F(a_1, \ldots, a_k) = b$.

Now, we have

$$d_{Su}(\mathbf{A}, \mathbf{B}) \leq 2. \mid C \mid - \mid A \mid - \mid B \mid = 2.(\mid A \mid + \mid B \mid - \mid S_{AB} \mid) - \mid A \mid - \mid B \mid =$$
$$= \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid = d_s(\mathbf{A}, \mathbf{B}).$$

On the other hand, we can suppose that $A \subseteq O_{AB}$ and $B \subseteq O_{AB}$, whence

$$\mid O_{AB} \mid \geq \mid A \mid + \mid B \mid - \mid A \cap B \mid \geq \mid A \mid + \mid B \mid - \mid S_{AB} \mid .$$

Therefore,

$$d_{Su}(\mathbf{A}, \mathbf{B}) = 2. \mid O_{AB} \mid - \mid A \mid - \mid B \mid \geq \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid = d_s(\mathbf{A}, \mathbf{B}).$$

□

Unlike previous metrics d_{iso}, d_h, d_s, the function given by (4) may not be a metric on any system of non-isomorphic n-element algebras of the same type (like groups, rings, etc.). However, we can prove the next statement.

Theorem 4.2. *The function d_{Su} given by (4) is a metric on the system $\mathcal{L}_n$ of all pairwise non-isomorphic n-element lattices.*

Proof. Let $\mathbf{L}_1 = (L_1, \leq_1), \mathbf{L}_2 = (L_2, \leq_2)$ be lattices and $\mathbf{L}_{12}$ be a maximal lattice such that there exist a sublattice $\mathbf{L}_1'$ of $\mathbf{L}_1$ isomorphic to $\mathbf{L}_{12}$ and a sublattice $\mathbf{L}_2'$ of $\mathbf{L}_2$ isomorphic to $\mathbf{L}_{12}$. We can assume that $\mathbf{L}_1' = \mathbf{L}_2' = \mathbf{L}_{12}$ and $0, 1 \in L_{12}$. As the ordering on $L = L_1 \cup L_2$ we take the transitive closure of the union of the orderings $\leq_1$ and $\leq_2$. To finish the proof proceed similarly as in the proof of Theorem 4.1 $\square$

Corollary. $D_{Su}(\mathcal{G}_n) = 2n$ *if* $n \geq 2$,
$D_{Su}(\mathcal{L}_n) = 2n - 6$ *if* $n \geq 4$.

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Dept. of Mathematics, Faculty of Natural Sciences
Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

E-mail address: haviar@fhpv.umb.sk

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THE STUDY OF AFFINE COMPLETENESS FOR QUASI-MODULAR DOUBLE P-ALGEBRAS

Miroslav Haviar

Abstract. In this paper we study affine complete and locally affine complete algebras in the class of quasi-modular double p-algebras. We generalize Beazer's characterization of affine complete double Stone algebras with a non-empty bounded core [B 1983] to the class of quasi-modular double S-algebras with a non-empty bounded core. We prove that finite regular double p-algebras are the only finite affine complete quasi-modular double p-algebras with a non-empty core and that Post algebras of order 3 are the only affine complete quasi-modular double S-algebras with a non-empty finite core. In distributive case, we derive the Beazer result and we construct an example of an infinite regular double Stone algebra which is not affine complete. We finally show that the Post algebras of order 3 are the only locally affine complete (in a stronger sense of [P 1972]) quasi-modular double S-algebras with a non-empty bounded core.

1. Introduction.

One of the topics of universal algebra rapidly developed in the last decades has been the study of affine complete algebras. Let us recall that an n-ary function f on an algebra $\boldsymbol{A}$ is called *compatible* if for any congruence θ on $\boldsymbol{A}$, $a_i \equiv b_i\ (\theta)\ (a_i, b_i \in A)$, $i = 1, \dots, n$ yields $f(a_1, \dots, a_n) \equiv f(b_1, \dots, b_n)\ (\theta)$. Obviously, every polynomial function of $\boldsymbol{A}$, i.e. a function that can be obtained by composition of the basic operations of $\boldsymbol{A}$, the projections and the constant functions, is compatible. By H. Werner [W 1971], an algebra $\boldsymbol{A}$ is called *affine complete* if the only compatible functions on $\boldsymbol{A}$ are the polynomial ones. Hence one can imagine affine complete algebras as algebras having many congruences.

The first results in this topic are due to G. Grätzer. In [G 1962] he showed that every Boolean algebra is affine complete and in [G 1964] he characterized affine complete bounded distributive lattices as those which do not contain proper Boolean subintervals. In [G 1968] he formulated a problem of characterizing affine complete algebras which was later reformulated in [C-W 1981] as follows: characterize affine complete algebras in your favourite variety. In [C-W 1981] one can also find a list of particular varieties in which all affine complete members were characterized. Some new items in the list are mentioned in [Ha-Pl 1995].

Also a "local" version of affine completeness has been studied. Let us recall that an algebra $\boldsymbol{A}$ is said to be *locally affine complete* if any finite partial function in $A^n \to A$ (i.e.

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function whose domain is a finite subset of A^n) which is compatible (where defined) can be interpolated by a polynomial of A (see e.g. [P 1972] or [Kaa-P 1987]; the notion 'locally affine complete' has also another, weaker meaning in the literature - see e.g. [Sz 1986] or [Ha-Pl 1995].)

In [B 1982] R. Beazer characterized affine complete algebras in the class of Stone algebras with bounded dense filter and in [B 1983] he gave a similar characterization in the class of double Stone algebras with a non-empty bounded core. Locally affine complete Stone algebras (in the weaker sense of [Sz 1986]) were characterized in [Ha 1993] and affine complete algebras in the variety of all Stone algebras were recently described in [Ha-Pl 1995]. Another generalization of the first Beazer result, to the class of so-called principal p-algebras, was presented in [Ha 1995].

In this paper we generalize the second Beazer result and its consequences (3.1-3.3) into a larger class of all quasi-modular double S-algebras with a non-empty bounded core (Theorem 3.14). First we show that for a quasi-modular double p-algebra $\boldsymbol{L}$ with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, affine completeness of $\boldsymbol{L}$ yields affine completeness of $K(\boldsymbol{L})$ as a bounded lattice (Theorem 3.4). Consequently, we get that finite regular double p-algebras are the only finite affine complete quasi-modular double p-algebras with a non-empty core and that Post algebras of order 3 are the only affine complete quasi-modular double S-algebras with a non-empty finite core (3.7 and 3.8). In distributive case, we derive (3.15-3.17) the second Beazer result and its consequences. Then we construct an example of an infinite regular double Stone algebra which is not affine complete with regard to Beazer's question in [B 1983].

We finally show that Post algebras of order 3 are the only locally affine complete quasi-modular double S-algebras with a non-empty bounded core (3.19-3.20).

2. Preliminaries.

A *p-algebra (pseudocomplemented lattice* or *PCL)* is an algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and * is the unary operation of pseudocomplementation, i.e. $x \leq a^\star$ iff $x \wedge a = 0$. By a *distributive (modular)* p-algebra $(L; \vee, \wedge, ^\star, 0, 1)$ we mean that the lattice L is distributive (modular). Further, recall that a *Stone algebra* is a distributive p-algebra satisfying the Stone identity

$(S) \qquad x^\star \vee x^{\star\star} = 1.$

In general, p-algebras satisfying (S) are called *S-algebras.*

Besides distributive and modular p-algebras, a larger variety of *quasi-modular p-algebras* was introduced and studied [Ka-Me 1983]. This subvariety of p-algebras is defined by the identity

$$((x \wedge y) \vee z^{\star\star}) \wedge x = (x \wedge y) \vee (z^{\star\star} \wedge x).$$

It is known (see [Ka-Me 1983; 6.1]) that quasi-modular p-algebras satisfy the identity

$$x = x^{\star\star} \wedge (x \vee x^\star).$$

An algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ is called a (quasi-modular) *double p-algebra*, if $(L; \vee, \wedge, ^\star, 0, 1)$ is a (quasi-modular) p-algebra and $(L; \vee, \wedge, ^+, 0, 1)$ is a dual (quasi-modular) p-algebra, i.e. $x \geq a^+$ if and only if $a \vee x = 1$.

A *double S-algebra* is a double p-algebra satisfying the identities

$$x^\star \vee x^{\star\star} = 1 \text{ and } x^+ \wedge x^{++} = 0.$$

A *double Stone algebra* is a distributive double S-algebra. A double Stone algebra in which so-called determination principle,

$a^\star = b^\star$ and $a^+ = b^+$ implies $a = b$,

holds is called a *three-valued Lukasiewicz algebra*.

In a double p-algebra $\boldsymbol{L}$, the sets $B(\boldsymbol{L}) = \{x \in L;\ x = x^{\star\star}\}$ and $\overline{B}(\boldsymbol{L}) = \{x \in L;\ x = x^{++}\}$ give Boolean algebras $(B(\boldsymbol{L}); \triangledown, \wedge, ^\star, 0, 1)$ and $(\overline{B}(\boldsymbol{L}); \vee, \triangle, ^+, 0, 1)$ where $x \triangledown y = (x \vee y)^{\star\star}$ and $x \triangle y = (x \wedge y)^{\star\star}$. If $\boldsymbol{L}$ is a quasi-modular double S-algebra, then $B(\boldsymbol{L})$ $(= \overline{B}(\boldsymbol{L}))$ is a subalgebra of $\boldsymbol{L}$ (cf. [Ka-Me 1983; 6.8] and [Ka 1974]) and $x^{\star+} = x^{\star\star}$, $x^{+\star} = x^{++}$.

The sets $D(\boldsymbol{L}) = \{x \in L;\ x^\star = 0\}$ and $\overline{D}(\boldsymbol{L}) = \{x \in L;\ x^+ = 1\}$ form a filter and an ideal of $\boldsymbol{L}$, respectively. The set $K(\boldsymbol{L}) = D(\boldsymbol{L}) \cap \overline{D}(\boldsymbol{L})$ is the *core* of $\boldsymbol{L}$. The class of quasi-modular double S-algebras with non-empty core includes bounded lattices with a new zero and unit adjoined, Post algebras of order $n > 2$, injective double Stone algebras, etc.

Congruences on double p-algebras are lattice congruences preserving the operations * and $^+$. The congruence Φ of a double p-algebra defined by

$x \equiv y(\Phi)$ if and only if $x^\star = y^\star$ and $x^+ = y^+$

is called the *determination congruence*. A double p-algebra is *regular* (i.e. two congruence relations having a congruence class in common coincide) if and only if $\Phi = \omega$ (see [V 1972]). Regular double p-algebras form a variety defined by the identity $(x \wedge x^+) \vee (y \vee y^\star) = y \vee y^\star$. Further, a regular double p-algebra $\boldsymbol{L}$ is distributive (see [Ka 1973b]). In [B 1976] regular double p-algebras were shown to be congruence permutable, hence the variety of regular double p-algebras is arithmetical. A (quintuple) construction of regular double p-algebras was presented in [Ka 1974].

A special subclass (not a subvariety) of the variety of regular double Stone algebras (i.e. three-valued Lukasiewicz algebras) form *Post algebras of order* 3, which are defined by the condition $|K(\boldsymbol{L})| = 1$ (see e.g. [B 1983]).

Let $\boldsymbol{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ be a quasi-modular double p-algebra with a non-empty bounded core $K(L) = [k, l]$. Since $\boldsymbol{L}$ satisfies the identities $x = x^{\star\star} \wedge (x \vee x^\star)$ and $x = x^{++} \vee (x \wedge x^+)$, it obviously satisfies the equations $x = x^{\star\star} \wedge (x \vee k)$ and $x = x^{++} \vee (x \wedge l)$. Thus $\boldsymbol{L}$ satisfies the equation

(1) $\quad x = x^{++} \vee (x^{\star\star} \wedge (x \vee k) \wedge l)$.

In [Mu-En 1986; Theorem 5] it was shown that the filter $D(\boldsymbol{L})$ of a quasi-modular p-algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$ is a neutral element in the lattice $F(\boldsymbol{L})$ of all filters of $\boldsymbol{L}$. So if $D(L) = [k)$, then for all $x, y \in L$, $([x) \vee [y)) \wedge [k) = ([x) \wedge [k)) \vee ([y) \wedge [k))$ holds in $F(\boldsymbol{L})$. Consequently, $(x \wedge y) \vee k = (x \vee k) \wedge (y \vee k)$ for all $x, y \in L$. Thus in a quasi-modular double p-algebra $\boldsymbol{L}$ with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, the elements k, l are distributive.

For these and other properties of double p-algebras as well as for the standard rules of computation in double p-algebras we refer to [B 1976] or [Ka 1973b].

In the second part of this preliminary section we present a collection of results concerning (local) affine completeness of some classes of algebras which will frequently be used in our investigations.

We start with basic Grätzer's results.

2.1 Theorem ([G 1962]). *Any Boolean algebra is affine complete.*

Let us recall that a function $f : L^n \to L$ on a lattice $\boldsymbol{L}$ is *order-preserving* if $x_i \leq y_i$ $(x_i, y_i \in L,\ i = 1, \dots, n)$ implies $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$. It is well-known that every polynomial function on a lattice is order-preserving.

2.2 Theorem ([G 1964; Corollaries 1,3]). *Let $\boldsymbol{L}$ be a bounded distributive lattice. The following conditions are equivalent.*

(1) $\boldsymbol{L}$ is affine complete;
(2) every compatible function on $\boldsymbol{L}$ is order-preserving;
(3) $\boldsymbol{L}$ contains no proper Boolean interval.

When omiting the distributivity of $\boldsymbol{L}$, one can prove (at least) the following:

2.3 Proposition. *If a lattice $\boldsymbol{L}$ contains a Boolean interval $[a, b]$ $(a < b)$, then $\boldsymbol{L}$ is not affine complete.*

Proof. Define a function $f : L \to [a, b]$ by the rule $f(x) = ((x \vee a) \wedge b)'$, where $'$ denotes the complement in the Boolean interval [a,b]. For any non-trivial congruence $\Phi \in \mathrm{Con}(L)$ and $x \equiv y\ (\Phi)$ $(x,\ y \in L)$ we have $((x \vee a) \wedge b)' \equiv ((y \vee a) \wedge b)'\ (\Phi)$, i.e. f is a compatible function of $\boldsymbol{L}$. But f is not order-preserving because $f(a) = b,\ f(b) = a$, therefore f cannot be represented by a lattice polynomial. Hence $\boldsymbol{L}$ is not affine complete. □

2.4 Corollary. *A finite lattice $\boldsymbol{L}$ is affine complete if and only if $|L| = 1$.* □

If the property we study is the local affine completeness (in the sense of [P 1972]), then the trivial lattices are the only members of the variety of all lattices having this property:

2.5 Proposition. *A lattice $\boldsymbol{L}$ is locally affine complete if and only if $|L| = 1$.*

Proof. Let $\boldsymbol{L}$ be locally affine complete and let $a, b \in L,\ a < b$. The function $f = \{(a, b), (b, a)\}$ is a finite partial compatible function on $\boldsymbol{L}$, thus by hypothesis it can be interpolated on $\{a, b\}$ by a polynomial of $\boldsymbol{L}$, which is an order-preserving function. But we have $f(a) = b,\ f(b) = a$, a contradiction. □

On the other hand, there are varieties of which all members are locally affine complete. The following result (see [P 1979] or [P 1982] or [P 1991]) characterizes them as arithmetical, i.e. congruence-distributive and congruence-permutable (meaning that the congruence lattice of each algebra in such variety is distributive and every two congruences permute):

2.6 Theorem. *A variety V is arithmetical if and only if for each algebra $\boldsymbol{A} \in V$, a finite partial function f on $\boldsymbol{A}$ can be interpolated by a polynomial function of $\boldsymbol{A}$ just in the case f is Con($\boldsymbol{A}$)-compatible.*

This also yields that every finite algebra in an arithmetical variety is affine complete.

The following technical lemma will be used several times in the sequel (and we repeat its proof from [Ha 1992]):

2.7 Lemma. *Let $\boldsymbol{D} = (D, \vee, \wedge, f_1, \ldots, f_k, 0, 1)$ be any algebra such that its reduct $(D, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the algebra $\boldsymbol{D}$ is a subdirect product of 2-element algebras. Let $f', g' : D^n \to D$ be partial compatible functions with domains F and G ($F, G \subset D^n$), respectively, let $S := F \cap G$ and let $S \cap \{0,1\}^n \neq \varnothing$. For any $(0,1)$-homomorphism $h : D \to \{0,1\}$ between the algebra $\boldsymbol{D}$ and a 2-element algebra $\mathbf{2} = \{0,1\}$, denote $h(S) := \{(h(x_1), \ldots, h(x_n)) \in \{0,1\}^n;\ (x_1, \ldots, x_n) \in S\}$ and let $h(S) = h(S \cap \{0,1\}^n)$ hold. Then $f' \equiv g'$ identically on S if and only if $f' \equiv g'$ identically on $S \cap \{0,1\}^n$.*

Proof. Let $f' \equiv g'$ identically on $S \cap \{0,1\}^n$. Suppose on the contrary that there exists an n-tuple $(d_1, \ldots, d_n) \in S$ such that $f'(d_1, \ldots, d_n) = a \neq b = g'(d_1, \ldots, d_n)$. Since $a \neq b$ in $\boldsymbol{D}$ which is a subdirect product of 2-element algebras, there exists a 'projection map' $h : D \to \{0,1\}$, which is a $(0,1)$-homomorphism between the algebra $\boldsymbol{D}$ and some algebra $\mathbf{2} = \{0,1\}$, such that $h(a) \neq h(b)$. Define functions $f'_2,\ g'_2 : h(S) \to \{0,1\}$ by the following rules:

$$f'_2(h(x_1), \ldots, h(x_n)) = h(f'(x_1, \ldots, x_n)),$$
$$g'_2(h(x_1), \ldots, h(x_n)) = h(g'(x_1, \ldots, x_n)) \quad \text{where} \quad (x_1, \ldots, x_n) \in S.$$

Obviously, $f'_2,\ g'_2$ are well-defined, since f', g' preserve the kernel congruence of the homomorphism h. Obviously, $f'_2 \equiv g'_2$ identically on $h(S)$, because $h(S) = h(S \cap \{0,1\}^n)$, $h(0) = 0,\ h(1) = 1$ and $f' \equiv g'$ identically on $S \cap \{0,1\}^n$. Therefore $h(a) = h(f'(d_1, \ldots, d_n)) = f'_2(h(d_1), \ldots, h(d_n)) = g'_2(h(d_1), \ldots, h(d_n)) = h(g'(d_1, \ldots, d_n)) = h(b)$, a contradiction. Hence $f' \equiv g'$ identically on S and the proof is complete. $\square$

In order to abbreviate some expressions, we shall often use the notation $\tilde{\mathrm{x}}$ for an n-tuple $(x_1, \ldots, x_n)$, and $f(\tilde{\mathrm{x}})$ for $f(x_1, \ldots, x_n)$ in the next section. Further, $\tilde{\mathrm{x}}^\star$ and $\tilde{\mathrm{x}}^+$ will denote $(x_1^\star, \ldots, x_n^\star)$ and $(x_1^+, \ldots, x_n^+)$, respectively, $(\tilde{\mathrm{x}} \vee k) \wedge l$ will abbreviate $((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l)$, etc.

3. Afinne completeness.

We start this section with Beazer's characterization of affine complete double Stone algebras with a non-empty bounded core and its consequences.

3.1 Theorem ([B 1983; Theorem 5])**.** *Let $\boldsymbol{L}$ be a double Stone algebra having a non-empty bounded core $K(\boldsymbol{L})$. The following conditions are equivalent.*

(1) $\boldsymbol{L}$ is affine complete;
(2) $K(\boldsymbol{L})$ is an affine complete distributive lattice;
(3) No proper interval of $K(\boldsymbol{L})$ is Boolean.

3.2 Corollary ([B 1983, Corollary 6])**.** *Any Post algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ of order 3 is an affine complete double Stone algebra.*

3.3 Corollary ([B 1983, Corollary 7])**.** *A finite double Stone algebra having a non-empty core is affine complete if and only if it is a Post algebra of order 3.*

In the first part we generalize 3.3 to a larger class of double S-algebras.

3.4 Theorem. *Let $\boldsymbol{L}$ be a quasi-modular double p-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. If $\boldsymbol{L}$ is affine complete then $K(\boldsymbol{L})$ is an affine complete lattice.*

Proof. Let $\boldsymbol{L}$ be affine complete. Similarly as in [B 1983], for any compatible function $f_K : K(\boldsymbol{L})^n \to K(\boldsymbol{L})$ we define a function $f : L^n \to L$ by

$$f(x_1, \ldots, x_n) = f_K((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l).$$

Obviously, $f \restriction K(L)^n = f_K$ and f preserves the congruences of $\boldsymbol{L}$. Thus by hypothesis, f can be represented by a polynomial $p_0(x_1, \ldots, x_n)$ of $\boldsymbol{L}$. From now we proceed as follows: we apply the formulas $(x \wedge y)^\star = x^\star \bigtriangledown y^\star$, $(x \vee y)^\star = x^\star \wedge y^\star$, $(x \wedge y)^+ = x^+ \vee y^+$ and $(x \vee y)^+ = x^+ \bigtriangleup y^+$ in $p_0(\tilde{\mathrm{x}})$ everywhere it is possible (see Example 3.5 below) and we obtain a polynomial $p_1(x_1, \ldots, x_n)$ of the partial algebra $(L; \vee, \wedge, \bigtriangledown, \bigtriangleup, ^\star, ^+, 0, 1)$ with two partial operations $\bigtriangledown$ and $\bigtriangleup$ defined only for elements of $B(\boldsymbol{L})$ and $\overline{B}(\boldsymbol{L})$, respectively.

Let $\tilde{\mathrm{x}} \in K(\boldsymbol{L})^n$. Then $f_K(\tilde{\mathrm{x}}) = f(\tilde{\mathrm{x}}) = p_1(\tilde{\mathrm{x}})$, and moreover, in $p_1(\tilde{\mathrm{x}})$ we can put $x_i^+ = 1$, $x_i^\star = 0$ for all $i = 1, \ldots, n$. Hence each part of the form $(...)^\star$ or $(...)^+$ in $p_1(\tilde{\mathrm{x}})$ can be rewritten as a constant symbol equal to 0 or 1 if in the brackets were variables only, or as a constant of $B(\boldsymbol{L})$ or $\overline{B}(\boldsymbol{L})$ if there was at least one constant symbol of $\boldsymbol{L}$ in the brackets (see again 3.5). Rewriting the polynomial $p_1(\tilde{\mathrm{x}})$ in this way, we obtain a polynomial $p_2(\tilde{\mathrm{x}})$ of the lattice $(L; \vee, \wedge, 0, 1)$. If $a_1, \ldots, a_m$ are all constant symbols in $p_2(\tilde{\mathrm{x}})$, then $p_2(\tilde{\mathrm{x}})$ can be expressed as a term $t(\tilde{\mathrm{x}}, \tilde{\mathrm{a}})$ of the algebra $(L; \vee, \wedge, 0, 1, a_1, \ldots, a_m)$. Now using the lattice homomorphism $\varphi : L \to [k, l]$, $\varphi(\tilde{\mathrm{x}}) = (x \vee k) \wedge l$ (note that in Section 2 we showed that the elements k, l are distributive), we get

$$f_K(\tilde{\mathrm{x}}) = \varphi(t(\tilde{\mathrm{x}}, \tilde{\mathrm{a}})) = t(\varphi(x_1), \ldots, \varphi(x_n), \varphi(a_1), \ldots, \varphi(a_m))$$

hence $f_K(\tilde{\mathrm{x}})$ can be represented by a polynomial of the lattice $K(\boldsymbol{L})$. The proof is complete. $\square$

3.5 Example. We illustrate the method described in the proof of Theorem 3.4 on a simple example. Let $\boldsymbol{L}$ be a quasi-modular double p-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, $f_K(x_1, x_2, x_3)$ be a compatible function of the lattice $K(\boldsymbol{L})$ and let

$$p_0(x_1, x_2, x_3) = [(x_1 \wedge a^\star) \vee (x_2^+ \wedge b)]^+ \wedge x_3$$

be a polynomial of $\boldsymbol{L}$ representing the function $f(x_1, x_2, x_3) : L^3 \to L$ associated to the function f_K as in the proof of Theorem 3.4. In the first step, we get the polynomial

$$p_1(x_1, x_2, x_3) = [(x_1 \wedge a^\star)^+ \bigtriangleup (x_2^+ \wedge b)^+] \wedge x_3 = [(x_1^+ \vee a^{\star +}) \bigtriangleup (x_2^{++} \vee b^+)] \wedge x_3.$$

In the second step, by putting $x_1^+ = 1$, $x_2^{++} = 0$ we obtain a polynomial

$p_2(x_1, x_2, x_3) = b^+ \wedge x_3$, which is a term $t(x_1, x_2, x_3, b^+)$ of the algebra $(L; \wedge, \vee, 0, 1, b^+)$. Finally, we get

$$f_K(x_1, x_2, x_3) = t(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(b^+)) = [(b^+ \vee k) \wedge l] \wedge [(x_3 \vee k) \wedge l],$$

hence $f_K(x_1, x_2, x_3)$ is a polynomial function of the lattice $K(\boldsymbol{L})$. $\square$

3.6 Corollary. *Let $\boldsymbol{L}$ be a quasi-modular double p-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. If $K(\boldsymbol{L})$ contains a proper Boolean interval then $\boldsymbol{L}$ is not affine complete.*

Proof. The result follows from Theorem 3.4 and Proposition 2.3. $\square$

3.7 Corollary. *A finite quasi-modular double p-algebra with a non-empty core is affine complete if and only if it is a regular double p-algebra.*

Proof. The necessity follows from 3.4 and 2.4. Since the variety of all regular double p-algebras is arithmetical, all its finite members are affine complete by 2.6. $\square$

The next result generalizes 3.3 to a larger class of double S-algebras:

3.8 Corollary. *Let $\boldsymbol{L}$ be a quasi-modular double S-algebra with a non-empty finite core. Then $\boldsymbol{L}$ is affine complete if and only if $\boldsymbol{L}$ is a Post algebra of order 3.*

Proof. Affine completeness of $\boldsymbol{L}$ yields $|K(\boldsymbol{L})| = 1$ by 3.4 and 2.4. Hence $\boldsymbol{L}$ is a regular double p-algebra, so $\boldsymbol{L}$ is distributive. Thus $\boldsymbol{L}$ is a Post algebra of order 3. The converse follows from 3.2. □

3.9 Example. Take the lattice $\boldsymbol{M}_\infty$ of height 2 having an infinite number of atoms and any Boolean algebras $\boldsymbol{B}_1, \boldsymbol{B}_2$. The lattice $\boldsymbol{L}_1 = \boldsymbol{B}_1 \oplus \boldsymbol{M}_\infty \oplus \boldsymbol{B}_2$ ($\oplus$ means linear sum) obviously gives a quasi-modular double p-algebra with the core $K(\boldsymbol{L}_1) = \boldsymbol{M}_\infty$. (For $\boldsymbol{B}_1 = \boldsymbol{B}_2 = \mathbf{1}$, the quasi-modular double S-algebra $\boldsymbol{L}_1 = \mathbf{1} \oplus \boldsymbol{M}_\infty \oplus \mathbf{1}$ is depicted in Figure 1a.) By 3.6, $\boldsymbol{L}_1$ is not affine complete.

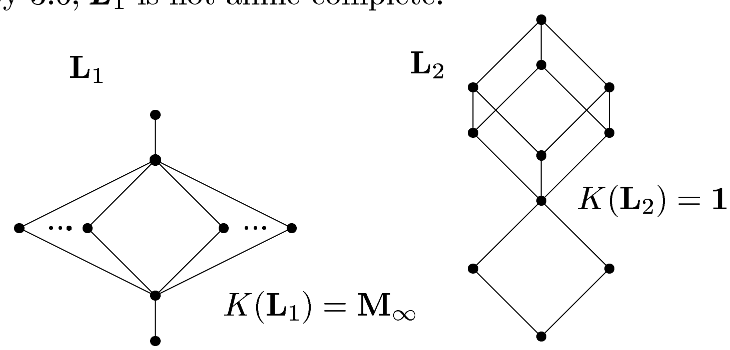

Figure 1a

Figure 1b

Now let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be finite Boolean algebras and $\boldsymbol{D}$ be a finite distributive lattice. Construct a quasi-modular double p-algebra $\boldsymbol{L}_2$ such that the zero of $\boldsymbol{B}_1$ will be identified with the unit of $\boldsymbol{D}$ and the zero of $\boldsymbol{D}$ will be identified with the unit of $\boldsymbol{B}_2$. By 3.7, $\boldsymbol{L}_2$ is affine complete if and only if $|D| = 1$. Hence the regular double p-algebra $\boldsymbol{L}_2$ in Figure 1b ($\boldsymbol{B}_1 = 2^3$, $\boldsymbol{B}_2 = 2^2$ and $\boldsymbol{D} = \mathbf{1}$) is affine complete. □

R. Beazer's technique employed in 3.1 was based on the fact that subdirectly irreducible double Stone algebras are very simple - the chains with at most four elements. This and also a 'good behaviour' of the operations * and $^+$ in double Stone algebras enabled him to find an exact form of the polynomials representing compatible functions. However, if we turn to a larger class of double S-algebras, which contains various subdirectly irreducible algebras, the situation becomes more complex and Beazer's method seems to be non-applicable.

Therefore we employ a technique based on the fact that in the class of quasi-modular double S-algebras with a non-empty bounded core, every element can be decomposed on two 'closed' elements and an element of the core - see the equation (1) in Section 2. Hence the elements from the range of any compatible function can be decomposed in this way, too. Since the set of all closed elements of a quasi-modular double S-algebra $\boldsymbol{L}$ forms a Boolean subalgebra $B(\boldsymbol{L})$ $(= \overline{B}(\boldsymbol{L}))$ and we assume that the core $K(\boldsymbol{L})$ is a bounded lattice, it would be natural to reduce the property of affine completeness of $\boldsymbol{L}$ into that of $B(\boldsymbol{L})$ $(= \overline{B}(\boldsymbol{L}))$ and $K(\boldsymbol{L})$. (This idea is, in fact, in accordance with a general idea of approaching quasi-modular p-algebras presented in [Ka 1980; p. 559].)

The main problem which arises when realizing the idea of the reduction is how to decompose a compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$ into (well-defined) functions of $B(\boldsymbol{L})$ ($= \overline{B}(\boldsymbol{L})$) and $K(\boldsymbol{L})$, respectively. As we shall see, the first part of this task concerning $B(\boldsymbol{L})$ ($= \overline{B}(\boldsymbol{L})$) can be quite easily managed, while the second is difficult so that we are forced to deal with partial functions of the lattice $K(\boldsymbol{L})$.

In the sequel, by $\boldsymbol{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ we always mean a quasi-modular double S-algebra having a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. In such case, the map $\varphi : L \to K(\boldsymbol{L})$, $\varphi(x) = (x \vee k) \wedge l$ is a lattice homomorphism. Further, we abbreviate $(\tilde{\mathrm{x}} \vee k) \wedge l$ as $\varphi(\tilde{\mathrm{x}})$.

To any compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$ we associate a partial function $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ as follows:

(2) $\quad f'_K(\varphi(\tilde{\mathrm{x}}), \varphi(\tilde{\mathrm{x}}^\star), \varphi(\tilde{\mathrm{x}}^{\star\star}), \varphi(\tilde{\mathrm{x}}^+), \varphi(\tilde{\mathrm{x}}^{++})) = \varphi(f(\tilde{\mathrm{x}})) \quad (\tilde{\mathrm{x}} \in L^n)$

and f'_K is undefined elsewhere.

3.10 Lemma. *The function f'_K defined above is a well-defined partial compatible function of the lattice $K(\boldsymbol{L})$.*

Proof. To show that f'_K preserves the congruences of $K(\boldsymbol{L})$ where defined, let θ_K be a congruence of $K(\boldsymbol{L})$ and $\varphi(x_i^j) \equiv \varphi(y_i^j)$ (θ_K) for $x_i, y_i \in L$, $i = 1, \dots, n$, $j \in \{-2, -1, 0, 1, 2\}$ where $x^0 = x$, $x^1 = x^+$, $x^2 = x^{++}$, $x^{-1} = x^\star$, $x^{-2} = x^{\star\star}$. We associate to the congruence θ_K an equivalence relation θ_L on $\boldsymbol{L}$ defined by

(3) $\quad x \equiv y$ (θ_L) if and only if $\varphi(x^j) \equiv \varphi(y^j)$ (θ_K) for all $j \in \{-2, -1, 0, 1, 2\}$.

Since $\boldsymbol{L}$ is a quasi-modular double S-algebra, i.e. $B(\boldsymbol{L})$ ($= \overline{B}(\boldsymbol{L})$) is a sublattice of $\boldsymbol{L}$, one can easily verify that θ_L is a congruence on $\boldsymbol{L}$. Hence we have $x_i \equiv y_i$ (θ_L), thus $f(\tilde{\mathrm{x}}) \equiv f(\tilde{\mathrm{y}})$ (θ_L) as f is compatible on $\boldsymbol{L}$. Now again by (3) $\varphi(f(\tilde{\mathrm{x}})) \equiv \varphi\ (f(\tilde{\mathrm{y}}))$ (θ_K), i.e. f'_K preserves the congruences of $K(\boldsymbol{L})$ where defined. To show that f'_K is well-defined, use the same method with $\theta_K = \triangle_{K(\boldsymbol{L})}$, the smallest congruence of K(**L**). $\square$

3.11 Definition. *We shall say that $\boldsymbol{L}$ satisfies an 'extension' property*

(E) *if for any compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$, the partial compatible function $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ defined by (2) can be extended to a total compatible function of the lattice $K(\boldsymbol{L})$.*

We will present two situations when the condition (E) is satisfied (and later on the third in 3.17).

3.12 Proposition. *If $\boldsymbol{L}$ is affine complete then $\boldsymbol{L}$ satisfies (E).*

Proof. Let f'_K be the function associated to a compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$. We define a function $f_1 : \boldsymbol{L}^n \to \boldsymbol{L}$ by $f_1(\tilde{\mathrm{x}}) = \varphi(f(\tilde{\mathrm{x}}))$. This is evidently compatible on $\boldsymbol{L}$, hence by hypothesis it can be represented by a polynomial $p(x_1, \dots, x_n)$ of $\boldsymbol{L}$. Using de Morgan laws for * and $^+$, $p(\tilde{\mathrm{x}})$ can be rewritten as $l(\tilde{\mathrm{x}}, \tilde{\mathrm{x}}^\star, \tilde{\mathrm{x}}^{\star\star}, \tilde{\mathrm{x}}^+, \tilde{\mathrm{x}}^{++})$ for some lattice polynomial $l(x_1, \dots, x_{5n})$ of $\boldsymbol{L}$. Further, using the homomorphism φ, one can show that for all $\tilde{\mathrm{x}} \in L^n$

$f'_K(\varphi(\tilde{\mathrm{x}}), \dots, \varphi(\tilde{\mathrm{x}}^{++})) = \varphi(f(\tilde{\mathrm{x}})) = f_1(\tilde{\mathrm{x}}) = p(\tilde{\mathrm{x}}) = l(\tilde{\mathrm{x}}, \tilde{\mathrm{x}}^\star, \tilde{\mathrm{x}}^{\star\star}, \tilde{\mathrm{x}}^+, \tilde{\mathrm{x}}^{++}) =$
$= \varphi(l(\tilde{\mathrm{x}}, \tilde{\mathrm{x}}^\star, \tilde{\mathrm{x}}^{\star\star}, \tilde{\mathrm{x}}^+, \tilde{\mathrm{x}}^{++})) = l'(\varphi(\tilde{\mathrm{x}}), \dots, \varphi(\tilde{\mathrm{x}}^{++}))$,

where $l'(x_1, \dots, x_{5n})$ is a polynomial of the lattice K(**L**). Then, of course, l' is the required total compatible extension of the partial function f'_K. $\square$

Let $\boldsymbol{L}$ be a quasi-modular double S-algebra with a non-empty core $K(\boldsymbol{L}) = [k, l]$ such that $K(\boldsymbol{L})$ is a Boolean lattice. Let $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ be the partial function from (2) and let S be its domain. Define a function $q(x_1, \ldots, x_{5n})$ on $K(\boldsymbol{L})$ by the rule

$$q(x_1, \ldots, x_{5n}) = \bigvee_{\tilde{a} \in S \cap \{k,l\}^{5n}} f'_K(a_1, \ldots, a_{5n}) \wedge y_1 \wedge \cdots \wedge y_{5n},$$

where $y_i = \begin{cases} x_i & \text{if } a_i = l \\ x'_i & \text{if } a_i = k. \end{cases}$

Obviously, $f'_K \equiv q$ identically on $S \cap \{k, l\}^{5n}$ and q is compatible on $\mathrm{K}(\mathbf{L})$. One can verify that $(K(\boldsymbol{L}); \vee, \wedge, k, l)$ with the partial compatible functions f'_K and q satisfy the assumptions of Lemma 2.7. Hence by the conclusion of Lemma 2.7 $f'_K \equiv q$ identically on S, thus the compatible function $q(x_1, \ldots, x_{5n})$ is a total extension of the function f'_K . So we have showed:

3.13 Proposition. *Let $\boldsymbol{L}$ be a quasi-modular double S-algebra with a non-empty core $K(\boldsymbol{L})$ which is a Boolean lattice. Then (E) is fulfilled in $\boldsymbol{L}$.*

Now we present a characterization theorem and its consequences.

3.14 Theorem. *Let $\boldsymbol{L}$ be a quasi-modular double S-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. Then $\boldsymbol{L}$ is affine complete if and only if $K(\boldsymbol{L})$ is an affine complete lattice and $\boldsymbol{L}$ satisfies (E).*

Proof. The necessity follows from Theorem 3.4 and Proposition 3.12. Now let $K(\boldsymbol{L})$ be an affine complete lattice and let $\boldsymbol{L}$ satisfy (E). Let $f : \boldsymbol{L}^n \to \boldsymbol{L}$ be a compatible function on $\boldsymbol{L}$. Since $\boldsymbol{L}$ satisfies the equation (1), we can write

(4) $\quad f(\tilde{x}) = f(\tilde{x})^{++} \vee (f(\tilde{x})^{\star\star} \wedge (f(\tilde{x}) \vee k) \wedge l) \quad$ for any $\tilde{x} = (x_1, \ldots, x_n) \in L^n$.

We shall show that the right side of (4) can be replaced by a polynomial of the algebra $\boldsymbol{L}$.

To replace $f(\tilde{x})^{\star\star}$ (and similarly $f(\tilde{x})^{++}$) in (4) by a polynomial of $\boldsymbol{L}$, we define a partial function $f'_B : B(\boldsymbol{L})^{2n} \to B(\boldsymbol{L})$ on the Boolean algebra $B(\boldsymbol{L})$ $(= \overline{B}(\boldsymbol{L})$) by

$$f'_B(\tilde{x}, \tilde{x}^+) = f(\tilde{x})^{\star\star} \quad (f(\tilde{x})^{++}) \quad (\tilde{x} \in L^n)$$

and f'_B is undefined elsewhere. Obviously, f'_B is well-defined since $x_i^\star = y_i^\star,\ x_i^+ = y_i^+,\ i = 1, \ldots, n$ yields $x_i \equiv y_i(\Phi)$ (the determination congruence), which follows $f(\tilde{x}) \equiv f(\tilde{y})(\Phi)$, thus $f(\tilde{x})^{\star\star} = f(\tilde{y})^{\star\star}$ $(f(\tilde{x})^{++} = f(\tilde{y})^{++})$. Further, for any congruence θ_B of $\mathrm{B}(\mathbf{L})$ we define an equivalence relation θ_L on L by $x \equiv y\ (\theta_L)$ if and only if $x^\star \equiv y^\star\ (\theta_B)$ and $x^+ \equiv y^+\ (\theta_B)$. Since $B(\boldsymbol{L})$ $(= \overline{B}(\boldsymbol{L}))$ is a subalgebra of $\boldsymbol{L}$, θ_L is obviously a congruence of $\boldsymbol{L}$ containing θ_B. Using θ_L, one can easily show that f'_B preserves the congruences of $B(\boldsymbol{L})$ where defined. Let S be the domain of f'_B, i.e.

$$S = \{(\tilde{x}^\star, \tilde{x}^+);\ \tilde{x} \in L^n\} \subset B(\boldsymbol{L})^{2n}.$$

Note that if $(\tilde{a}, \tilde{b}) = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in S \cap \{0,1\}^{2n}$ then $a_i = 1$ implies $b_i = 1$. One can easily verify that the function f'_B can be interpolated on the set $S \cap \{0, 1\}^{2n}$ by a Boolean polynomial function $b : B(\boldsymbol{L})^{2n} \to B(\boldsymbol{L})$ defined as follows:

$$b(x_1, \ldots, x_{2n}) = \bigvee_{(\tilde{a}, \tilde{b}) \in S \cap \{0,1\}^{2n}} (f'_B(\tilde{a}, \tilde{b}) \wedge x_1^{a_1} \wedge \cdots \wedge x_n^{a_n} \wedge x_{n+1}^{b_1} \wedge \cdots \wedge x_{2n}^{b_n}$$

where $x_i^1 = x_i,\ x_i^0 = x'_i = x_i^\star = x_i^+$. By Lemma 2.7, $f'_B \equiv b$ identically on the whole set S, hence for any $\tilde{x} \in L^n$ we have

$$f(\tilde{x})^{\star\star}\ (f(\tilde{x})^{++}) = f'_B(\tilde{x}^\star, \tilde{x}^+) = b(\tilde{x}^\star, \tilde{x}^+).$$

Therefore $f(\tilde{x})^{\star\star}$ (and similarly $f(\tilde{x})^{++}$) can be replaced in (4) by some polynomial $b_1(\tilde{x}^\star, \tilde{x}^+)$ ($b_2(\tilde{x}^\star, \tilde{x}^+)$) of the algebra $\boldsymbol{L}$.

Now we associate to $f(\tilde{x})$ the partial function $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ defined by (2). By (E) there exists a total compatible function $f_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ which extends f'_K. Affine completeness of $K(\boldsymbol{L})$ yields that f_K can be represented by a lattice polynomial $l(x_1, \ldots, x_{5n})$. Hence in (4) we have for any $\tilde{x} \in L^n$,

$$f(\tilde{x}) = b_2(\tilde{x}^\star, \tilde{x}^+) \vee (b_1(\tilde{x}^\star, \tilde{x}^+) \wedge l(\varphi(\tilde{x}), \varphi(\tilde{x}^\star), \varphi(\tilde{x}^{\star\star}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})))$$

where $\varphi(\tilde{x})$ means $(\tilde{x} \vee k) \wedge l := ((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l)$.

So f is a polynomial function of the algebra $\boldsymbol{L}$ and the proof is complete. $\square$

We shall finally derive the Beazer characterization of double Stone algebras with a non-empty bounded core.

3.15 Lemma. *Let $\boldsymbol{L}$ be a double Stone algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$ and $x, y \in L$. Then for the lattice homomorphism $\varphi : L \to K(\boldsymbol{L})$, $\varphi(x) = (x \vee k) \wedge l$ we have*

$$\varphi(x^\star) = \varphi(y^\star) \text{ if and only if } \varphi(x^{\star\star}) = \varphi(y^{\star\star}) \text{ and}$$
$$\varphi(x^+) = \varphi(y^+) \text{ if and only if } \varphi(x^{++}) = \varphi(y^{++}).$$

Proof. Let $\varphi(x^+) = \varphi(y^+)$. The identities $x^+ \wedge x^{++} = 0$ and $x^+ \vee x^{++} = 1$ imply $\varphi(x^+) \wedge \varphi(x^{++}) = k$, $\varphi(x^+) \vee \varphi(x^{++}) = l$ for any $x \in L$. Hence

$$\varphi(x^{++}) = (\varphi(y^+) \wedge \varphi(y^{++})) \vee \varphi(x^{++}) = (\varphi(x^+) \vee \varphi(x^{++})) \wedge (\varphi(y^{++}) \vee \varphi(x^{++})) = \varphi(y^{++}) \vee \varphi(x^{++}).$$

In the same way one can show $\varphi(y^{++}) = \varphi(y^{++}) \vee \varphi(x^{++})$. The converse statement as well as the proof of the first statement are analogous. $\square$

3.16 Lemma. *Let $\boldsymbol{L}$ be a double Stone algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, $k < l$ and let $x \in L$ such that $\varphi(x^\star), \varphi(x^{\star\star}), \varphi(x^+), \varphi(x^{++}) \in \{k, l\}$ for the lattice homomorphism $\varphi : L \to K(\boldsymbol{L})$, $\varphi(x) = (x \vee k) \wedge l$. Then $\varphi(x^\star) = l$ implies $\varphi(x^{\star\star}) = k$ and analogously, $\varphi(x^+) = l$ implies $\varphi(x^{++}) = k$.*

Proof. Let $\varphi(x^\star) = l$. It is obvious that $\varphi(x^{\star\star}) = l$ would yield $l = \varphi(x^\star) \wedge \varphi(x^{\star\star}) = \varphi(0) = k$, a contradiction. Analogously, if $\varphi(x^+) = l = \varphi(x^{++})$, then $l = \varphi(x^+) \wedge \varphi(x^{++}) = \varphi(0) = k$, using the dual Stone identity $x^+ \wedge x^{++} = 0$. $\square$

3.17 Proposition. *Let $\boldsymbol{L}$ be a double Stone algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$ such that $K(\boldsymbol{L})$ contains no proper Boolean interval. Then $\boldsymbol{L}$ satisfies (E).*

Proof. If $k = l$, then $\boldsymbol{L}$ is a Post algebra of order 3 and trivially, $\boldsymbol{L}$ satisfies (E). So let us further assume that $k < l$.

Let $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ be the partial compatible function associated to a compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$. Let $T = \{(\tilde{x} \vee k) \wedge l, \ldots, ((\tilde{x}^{++} \vee k) \wedge l);\ \tilde{x} \in L^n\}$ be the domain of f'_K. We shall show that f'_K can be interpolated on the set $T \cap \{k, l\}^{5n}$ by the following polynomial of the lattice K($\mathbf{L}$):

$$(5) \quad q(x_1, \ldots, x_{5n}) = \bigvee_{\tilde{b} \in T \cap \{k,l\}^{5n}} (f'_K(b_1, \ldots, b_{5n}) \wedge y_1 \wedge \cdots \wedge y_{5n}),$$

$$\text{where } \ y_i = \begin{cases} x_i, & \text{if } b_i = l \\ l, & \text{if } b_i = k. \end{cases}$$

Let $\tilde{x}$ be any (fixed) vector from $T \cap \{k,l\}^{5n}$. If $\tilde{b} \neq \tilde{x}$ and $b_j \neq x_j$ for some $n < j \leq 5n$, then either $b_j = l,\ x_j = k$ and then $f'_K(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n} = k$ or $b_j = k,\ x_j = l$ and then by Lemmas 3.15, 3.16 there exists s, $n < s \leq 5n$ such that $x_s = k, b_s = l$, thus again $f'_K(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n} = k$. Hence it suffices to take into account in (5) only conjunctions $f'_K(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n}$ such that $b_i = x_i$ for all $n < i \leq 5n$ and moreover, $b_i \leq x_i$ for $1 \leq i \leq n$. So

$$q(x_1, \ldots, x_{5n}) = \bigvee_{\tilde{b} \in T \cap \{k,l\}^{5n},\ \tilde{b} \leq \tilde{x}} (f'_K(b_1, \ldots, b_n, x_{n+1}, \ldots, x_{5n}).$$

In next we show that $f'_K(\tilde{b}) \leq f'_K(\tilde{x})$ for any $\tilde{b} \in T \cap \{k,l\}^{5n}$ such that $b_i = x_i$ for $i = n+1, \ldots, 5n$ and $b_i \leq x_i$ for $i = 1, \ldots, n$. For $s = 1, \ldots, n$ denote $u_s = b_s$ if $b_s = x_s$, otherwise $u_s = u$. We get a unary compatible function $g : K(\boldsymbol{L}) \to K(\boldsymbol{L}),\ g(u) = f'_K(u_1, \ldots, u_n, x_{n+1}, \ldots, x_{5n})$ and we want to show that $f'_K(\tilde{b}) = g(k) \leq g(l) = f'_K(\tilde{x})$. Since $g(k) \equiv g(u)\ (\theta_{\text{lat}}(k,u))$ and $g(u) \equiv g(l)\ (\theta_{\text{lat}}(u,l))$ for any $u \in K(\boldsymbol{L})$, we get

$g(u) \vee u = g(k) \vee u$ and

$g(u) \wedge u = g(l) \wedge u$.

This means that for any $u \in [g(l), g(k) \vee g(l)]$, g(u) is the relative complement of u in this interval, which is therefore Boolean. By hypothesis ($K(\boldsymbol{L})$ contains no Boolean interval) this implies $g(k) \leq g(l)$, what was to be proved. Hence

$q(x_1, \ldots, x_{5n}) = f'_K(x_1, \ldots, x_{5n})$ for any $\tilde{x} \in T \cap \{k,l\}^{5n}$, and by applying Lemma 2.7, $q(x_1, \ldots, x_{5n})$ is the required total compatible extension of the partial function f'_K. $\square$

From 3.14 and 3.17 we now get the Beazer result 3.1 and its consequences.

By Theorem 2.6 every finite algebra in an arithmetical variety, is affine complete. Hence, as Beazer concluded in [B 1983], any finite regular double p-algebra is affine complete. Afterwards he raised a question whether or not every infinite regular double p-algebra is affine complete.

Later on, K. Kaarli and A.F. Pixley [Kaa-P 1987] proved that an arithmetical variety of finite type is affine complete if and only if it has definable principal congruences and all its subdirectly irreducible members are finite and have no proper subalgebras. It is well-known that the variety of all regular double p-algebras has infinite subdirectly irreducible members, e.g. infinite Boolean algebras with a new unit adjoined. Moreover, every subdirectly irreducible regular double p-algebra having more than two elements has a proper subalgebra $\{0, 1\}$. Hence the variety of all regular double p-algebras is not affine complete, and consequently, it must exist an infinite regular double p-algebra which is not affine complete.

Next we construct an example of an infinite regular double Stone algebra which is not affine complete. This has been motivated by techniques in [P 1993].

3.18 Example. Let $\mathbf{3} = (\{0, a, 1\}; \vee, \wedge, ^\star, ^+, 0, 1)$ be the 3-element double Stone algebra with the core $\{a\}$. Let $\boldsymbol{L}$ be a subalgebra of $\mathbf{3}^\omega$ consisting of the sequences $\tilde{x} = (x_1, x_2, x_3, \ldots)$ which are 0 for all but finitely many n or are 1 for all but finitely many n. One can easily check that $\boldsymbol{L}$ is a regular double Stone algebra with an empty core.

Let $f : L \to L$ be defined componentwise as follows:

$$f(\tilde{x})_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ x_n & \text{if } n \text{ is even.} \end{cases}$$

We shall show that $f(\tilde{x}) \equiv f(\tilde{y})(\theta(\tilde{x}, \tilde{y}))$ for any $\tilde{x}, \tilde{y} \in L$. Let $\tilde{x}, \tilde{y} \in L$. By construction of $\boldsymbol{L}$, there is a natural number K such that

$\tilde{x} = (x_1, \ldots, x_K, x, x, x, \ldots), \quad x \in \{0, 1\}$
$\tilde{y} = (y_1, \ldots, y_K, y, y, y, \ldots), \quad y \in \{0, 1\}.$

For $x \in \{0, 1\}$, let $\underline{x}$ denote the constant sequence $(x, x, x, \ldots)$. Congruence-distributivity yields that congruences on finite subdirect products are 'skew-free', hence

$$\theta(\tilde{x}, \tilde{y}) = \theta(x_1, y_1) \times \cdots \times \theta(x_K, y_K) \times \theta(\underline{x}, \underline{y}).$$

Now it is clear that $f(\tilde{x}) \equiv f(\tilde{y})(\theta(\tilde{x}, \tilde{y}))$, thus f is a compatible function on $\boldsymbol{L}$. Suppose that $\boldsymbol{L}$ is affine complete. Then f is a polynomial function of $\boldsymbol{L}$, thus there is an $(m+1)$-ary term t of $\boldsymbol{L}$ and elements $\tilde{c}^1, \ldots, \tilde{c}^m \in L$ such that

$$f(\tilde{x}) = t(\tilde{x}, \tilde{c}^1, \ldots, \tilde{c}^m).$$

For the constants $\tilde{c}^1, \ldots, \tilde{c}^m$ there is a natural number N such that for $i = 1, \ldots, m$

$$\tilde{c}^i = (c^i_1, c^i_2, \ldots, c^i_N, c^i, c^i, c^i, \ldots), \quad c^i \in \{0, 1\}.$$

Take $\tilde{x} \in L$ such that $x_n = x_{n+1} = 0$ for some even $n > N$. Then we get

$$0 = x_n = f(\tilde{x})_n = t(0, c^1, \ldots, c^m) = f(\tilde{x})_{n+1} = 1,$$

a contradiction. Hence f cannot be represented by a polynomial of $\boldsymbol{L}$, so $\boldsymbol{L}$ is not affine complete. □

We finally turn to local affine completeness and we present 'local versions' of the previous results. As we shall see, much easier descriptions can be obtained.

3.19 Theorem. *Let $\boldsymbol{L}$ be a quasi-modular double p-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. $\boldsymbol{L}$ is locally affine complete if and only if $\boldsymbol{L}$ is a regular double p-algebra.*

Proof. If $\boldsymbol{L}$ is locally affine complete then analogously as in the proof of 3.4 (the only difference is that the functions f_K, f are finite partial compatible functions in this case) one can show that $K(\boldsymbol{L})$ is a locally affine complete lattice, thus by 2.5, $|K(\boldsymbol{L})| = 1$. Hence $\boldsymbol{L}$ is a regular double p-algebra. The converse follows from 2.6. □

Corollary 3.20. *Post algebras of order 3 are the only locally affine complete quasi-modular double S-algebras with a non-empty bounded core.*

Proof. If $\boldsymbol{L}$ is a locally affine complete quasi-modular double S-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, then by 3.19 $\boldsymbol{L}$ is a regular double Stone algebra (i.e. a three-valued Lukasiewicz algebra). Moreover, since $|K(\boldsymbol{L})| = 1$, $\boldsymbol{L}$ is a Post algebra of order 3. □

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Dept. of Mathematics, Matej Bel University,
Zvolenská 6, 974 01 Banská Bystrica, Slovakia

E-mail address: mhaviar@pdf.umb.sk

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NOTE ON ZEROS OF THE CHARACTERISTIC POLYNOMIAL OF BALANCED TREES

[1]Pavol Híc and [2]Roman Nedela

Abstract. A graph G is called integral if all the zeros of the characteristic polynomial $P(G;\lambda)$ are integers. A tree T is called balanced if the vertices at the same distance from the centre of T have the same degree. In the present paper we investigate the properties of the zeros of characteristic polynomials of balanced trees.

1.INTRODUCTION

A graph G is called *integral* if it has an integral spectrum, i.e. if all the zeros of the characteristic polynomial $P(G;\lambda)$ are integers. The identification of all integral graphs seems to be intractable. However, that of various families of integral graphs was investigated in [1, 3, 4, 5]. In [3] integral balanced trees were studied. A tree T is called *balanced* if the vertices at the same distance from the centre of T have the same degree. According to the parity of the diameter of a tree balanced trees split into two families. We shall code a balanced tree of diameter $2k$ by the sequence $(n_k, n_{k-1}, \ldots, n_1)$, where n_j $j = 1, \ldots, k$ denotes the number of succesors of a vertex at distance $k-j$ from the centre. In [3] it is proved that all zeros of the characteristic polynomial of the balanced tree with the sequence $(n_k, n_{k-1}, \ldots, n_1)$ are zeros of the following recursively defined polynomial $P_k(x)$:

Definition 1.

$$P_0(x) = x$$

$$P_1(x) = x^2 - n_1$$

$$P_j(x) = x.P_{j-1}(x) - n_j.P_{j-2}(x)$$

where $j = 2, \ldots, k$.

This fundamental observation allows us to reduce the study of spectra of balanced trees to the study of properties of polynomials $P_k(x)$. The aim of this note is to prove some basic results on the sequence $\{P_k(x)\}$ $k = 0, 1, \ldots$. Results proved here are used in [3].

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2.RESULTS

In what follows we always assume that a sequence $\{n_j\}$ $j = 1, 2 \ldots$ of positive integers is given. It is easy to verify by induction on k, that for the terms of the sequence $\{P_k(x)\}$ of polynomials defined by Definition 1 the following statements hold:

Proposition 1.
a. $P_k(0) > 0$, *for* $k \equiv 3 \pmod 4$;
b. $P_k(0) < 0$, *for* $k \equiv 1 \pmod 4$;
c. $P_k(0) = 0$, *for* $k \equiv 0$ *or* $2 \pmod 4$;
d. $P_k(x)$ *is decreasing in point 0 for* $k \equiv 2 \pmod 4$;
e. $P_k(x)$ *is increasing in point 0 for* $k \equiv 0 \pmod 4$.

Now, let x_i be the smallest positive zero of polynomial $P_i(x)$ (i=1,2, ...). Denote by $\{x_k\}$ the sequence of the smallest positive zeros corresponding to the sequence $\{P_k(x)\}$. The following theorem shows that the above notation is correct.

Theorem 1. *For every* $i \geq 1$ *there exists a positive zero of the polynomial* $P_i(x)$. *Moreover, using the above notation the following statements hold:*
a. $\{x_{2k+1}\}$ *is decreasing;*
b. $\{x_{2k}\}$ *is decreasing;*
c. $x_{2k+2} > x_{2k+1}$, *for k=0,1,* $\ldots$

Proof. a. We shall proceed by induction on k. If k=0, then from $P_1(x) = x^2 - n_1$ we have $x_1 = \sqrt{n_1}$. If k=1, then $P_3(x) = x^4 - (n_1 + n_2 + n_3)x^2 + n_1.n_3$. By Proposition 1.a we get for $x \in < 0, x_1 >$

$$P_3(0) > 0, \tag{1}$$

$$P_3(x_1) = x_1 P_2(x_1) - n_3 P_1(x_1) = x_1[x_1 P_1(x_1) - n_2 x_1] = -n_2 x_1^2 < 0. \tag{2}$$

Using (1) and (2) we deduce that there exists $y \in (0, x_1)$ for which $P_3(y) = 0$. It follows $x_3 < x_1$.

Now, let $x_1 > x_3 > \cdots > x_{2k-1} > 0$. We shall investigate the polynomial $P_{2k+1}(x)$. According to whether $2k + 1 \equiv 3$ or $1 \pmod 4$ we distinguish two cases (see Proposition 1):
Case 1. $P_{2k+1}(0) > 0$;
Case 2. $P_{2k+1}(0) < 0$.
We shall deal only with the Case 1 . The proof in the case 2 can be done similarly.

If $P_{2k+1}(0) > 0$, then by Proposition 1, $P_{2k-1}(0) < 0$ and it follows, that for every $x \in (0, x_{2k-1})$ we have $P_{2k-1}(x) < 0$ because of x_{2k-1} is the smallest positive zero of

$P_{2k-1}(x)$. Hence,

$$\begin{aligned} P_{2k+1}(x_{2k-1}) &= x_{2k-1}P_{2k}(x_{2k-1}) - n_{2k+1}P_{2k-1}(x_{2k-1}) = \\ &= x_{2k-1}P_{2k}(x_{2k-1}) = \\ &= x_{2k-1}[x_{2k-1}P_{2k-1}(x_{2k-1}) - n_{2k}P_{2k-2}(x_{2k-1})] = \\ &= -x_{2k-1}n_{2k}P_{2k-2}(x_{2k-1}). \end{aligned}$$

Further, substituting $x = x_{2k-1}$ into the equality

$$P_{2k-1}(x) = xP_{2k-2}(x) - n_{2k-1}P_{2k-3}(x)$$

we get

$$0 = x_{2k-1}P_{2k-2}(x_{2k-1}) - n_{2k-1}P_{2k-3}(x_{2k-1})$$

and

$$n_{2k-1}P_{2k-3}(x_{2k-1}) = x_{2k-1}P_{2k-2}(x_{2k-1}).$$

By Proposition 1 and the fact $x_{2k-1} \in (0, x_{2k-3})$ the left part of the last equation is positive and it follows

$$P_{2k-2}(x_{2k-1}) > 0.$$

Hence,

$$P_{2k+1}(x_{2k-1}) = -x_{2k-1}n_{2k}P_{2k-2}(x_{2k-1}) < 0.$$

Since $P_{2k+1}(0) > 0$, there exists $x_{2k+1} \in (0, x_{2k-1})$ which is a zero of $P_{2k+1}(x)$.

b. We shall proceed by induction on k. If k=1, then from $P_2(x) = x^3 - (n_1 + n_2).x$ it follows $x_2 = \sqrt{n_1 + n_2}$. If k=2, then $P_4(x) = x.P_3(x) - n_4.P_2(x)$. By Proposition 1.c and 1.e for $x \in < 0, x_2 >$ the polynomial $P_4(x)$ satisfies the following properties:

$$P_4(x^+) > 0, \text{ for some } x^+ \in O_{\epsilon^+}(0), \tag{3}$$

$$\begin{aligned} P_4(x_2) &= x_2P_3(x_2) - n_4P_2(x_2) = x_2[x_2P_2(x_2) - n_3P_1(x_2)] = \\ &= -x_2n_3P_1(x_2) = -x_2n_3(x_2^2 - n_1) < 0. \end{aligned} \tag{4}$$

Here $O_{\epsilon^+}(0)$ denotes a sufficiently small right open neighbourhood of 0. Using (3) and (4) we see that there exists $x_4 \in (0, x_2)$ such that x_4 is a zero of $P_4(x)$.

Now, let the statement hold for every $n < k$ i.e.

$$0 < x_{2k-2} < x_{2k-4} < \cdots < x_4 < x_2.$$

Consider the polynomial $P_{2k}(x)$. According to Proposition 1.d and 1.e we have to distinguish two cases:

$$k \text{ is odd and } P_{2k}(x^+) < 0, \text{ for } x^+ \in O_{\epsilon^+}(0); \tag{5}$$

$$(6) \qquad k \text{ is even and } P_{2k}(x^+) > 0, \text{ for } x^+ \in O_{\epsilon^+}(0).$$

We shall examine only Case (6). Case (5) can be handled in a similar way. Substituting $x = x_{2k-2}$ into the equation

$$P_{2k}(x) = xP_{2k-1}(x) - n_{2k}P_{2k-2}(x)$$

we have
(7)

$$\begin{aligned} P_{2k}(x_{2k-2}) &= x_{2k-2}P_{2k-1}(x_{2k-2}) = \\ &= x_{2k-2}[x_{2k-2}P_{2k-2}(x_{2k-2}) - n_{2k-1}P_{2k-3}(x_{2k-2})] = \\ &= -x_{2k-2}n_{2k-1}P_{2k-3}(x_{2k-2}) \end{aligned}$$

On the other hand, using the substitution $x = x_{2k-2}$ in the equation

$$P_{2k-2}(x) = xP_{2k-3}(x) - n_{2k-2}P_{2k-4}(x)$$

we have

$$(8) \qquad x_{2k-2}P_{2k-3}(x_{2k-2}) = n_{2k-2}P_{2k-4}(x_{2k-2})$$

Hence, using Proposition 1.d and 1.e $P_{2k-4}(x_{2k-2}) > 0$. Combining (7) and (8) we obtain

$$P_{2k}(x_{2k-2}) < 0.$$

According to $P_{2k}(x^+) > 0$ for $x^+ \in O_{\epsilon^+}(0)$ there exists $x_{2k} \in (0, x_{2k-2})$ such that x_{2k} is a zero of the polynomial $P_{2k}(x)$.

c. The statement is trivial for k=0, since $x_2 = \sqrt{n_1 + n_2} > \sqrt{n_1} = x_1$. Now, let the statement hold for every $n < k$; i.e.

$$(9) \qquad x_{2k} > x_{2k-1}$$

Suppose $n = k + 1$. We shall restrict ourselves to the case $P_{2k+2}(x^+) > 0$. The case $P_{2k+2}(x^+) < 0$ can be handled similarly. By Definition 1 we have

$$P_{2k+2}(x) = xP_{2k+1}(x) - n_{2k+2}P_{2k}(x)$$

According to Theorem 1.a and (9)

$$x_{2k+1} < x_{2k-1} < x_{2k}.$$

By Proposition 1.d and by the assumption $P_{2k+2}(x^+) > 0$. It follows $P_{2k}(x) < 0$ for every $x \in (0, x_{2k+1})$. On the other hand, $P_{2k+1}(x) > 0$ for every $x \in (0, x_{2k+1})$. Hence, $P_{2k+2}(x) > 0$, for $x \in (0, x_{2k+1})$ and it follows that the smallest zero x_{2k+2} of the polynomial $P_{2k+2}(x)$ is greater than x_{2k+1}. $\square$

Corollary 1. *If the polynomial $P_{2k+1}(x)$ has only integer zeros then 1 is not the zero of $P_{2k}(x)$.*

Proof. Let $x_{2k} = 1$ be the smallest zero of $P_{2k}(x)$. Using Theorem 1. a; and c;

$$1 = x_{2k} > x_{2k-1} > x_{2k+1} > 0,$$

for every $k = 1, 2, \ldots$. However, this contradicts the fact that x_{2k+1} is integer. □

A sequence $\{n_i\}_{i \in I}$, where I is an interval (finite or infinite) of integers ≥ 1 is called *integral* if the corresponding polynomials $P_j(x)$ $j = 1, \ldots$ have only integral zeros.

Corollary 2. *There is no infinite integral sequence.*

Corollary 3. *Every integral sequence $(n_k, n_{k-1}, \ldots, n_1)$ has a length $\leq min\{2\sqrt{n_1}, \sqrt{n_1 + n_2}\}$.*

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[1]Department of Mathematics, Informatics and Physics,

Faculty of Education,

Trnava University, Hornopotočná 23, 918 43 Trnava, Slovakia,

E-mail address: phic@ uvt.mtf.stuba.sk

[2]Department of Mathematics, Faculty of Natural Sciences,

Matej Bel University, Tajovského 40, 974 00 Banská Bystrica, Slovakia

E-mail address: nedela@ bb.sanet.sk

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THE EDGE DISTANCE IN SOME FAMILIES OF GRAPHS II

Pavel Hrnčiar and Gabriela Monoszová

Abstract. The edge distance between graphs is defined by the equality $d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + ||V_1| - |V_2||$ where $|A|$ is the cardinality of A and $E_{1,2}$ is the edge set of a maximal common subgraph of G_1 and G_2. Further, $\operatorname{diam} F_{p,q} = max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$ where $F_{p,q}$ denotes the set of all graphs with p vertices and q edges. In the paper we prove that for $p \in \{7, 8, 9\}$ $\operatorname{diam} F_{p,p+2} = 2p - 6$ and for $p \geqq 19$ and $p + 3 \leqq q \leqq \frac{3p}{2}$ $\operatorname{diam} F_{p,q} = 2q - 12$.

1. Preliminaries

A graph $G = (V, E)$ consists of a non-empty finite vertex set V and an edge set E. In this paper we consider undirected graphs without loops and multiple edges. A subgraph H of the graph G is a graph obtained from G by deleting some edges and vertices; notation: $H \subseteq G$. By $\triangle(G)$ we denote the maximal degree of vertices of the graph G. A graph G is a common subgraph of graphs G_1, G_2 if there exist graphs H_1, H_2 such that $H_1 \subseteq G_1$, $H_2 \subseteq G_2$ and $H_1 \cong G$, $H_2 \cong G$.

A maximal common subgraph is a common subgraph which contains the maximal number of edges.

The edge distance of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined (see [3]) by

$$d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + ||V_1| - |V_2|| \tag{1}$$

where $|E_1|$, $|E_2|$, $|V_1|$, $|V_2|$ are the cardinalities of the edge sets and the vertex sets, respectively and $|E_{1,2}|$ is the number of edges of a maximal common subgraph $G_{1,2}$ of the graphs G_1 and G_2.

Throughout this paper, by $F_{p,q}$ we denote the set of all graphs with p vertices and q edges. Further, $\operatorname{diam} F_{p,q} := \max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$. If $\operatorname{diam} F_{p,q} = d(G, H)$ and $c_{p,q}$ is the number of edges of a maximal common subgraph of the graphs G, H then

$$\operatorname{diam} F_{p,q} = 2q - 2c_{p,q}. \tag{2}$$

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Denote by v a firmly chosen vertex of a maximal degree in the considered graph G and by $v_1, v_2, \ldots, v_k$ the vertices adjacent to v (here $k = \triangle(G)$). Denote $U := \{v_1, v_2, \ldots, v_k\}$ and $U' := V - \{v, v_1, \ldots, v_k\}$. The subgraph of the graph G induced by the vertex set X ($X \subset V$) we denote by $G(X)$ and the set of its edges by $E(G(X))$ or briefly by $E(X)$. The subgraph of the graph G which contains all edges with one vertex in the set U and the other in the set U' is denoted by $G(U, U')$ and the set of its edges by $E(U, U')$.

This paper is a continuation of the articles [1] and [2] but it can be read independently on them.

2. Diameter of $F_{p,p+2}$

In [1] diam $F_{p,p+2}$ is determined for all p except of $p \in \{7, 8, 9\}$. In this section of the paper we will show that diam $F_{p,p+2} = 2p - 6$ for $p \in \{7, 8, 9\}$.

Lemma 2.1. *Let $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8\}$. If the graph G_1 without its isolated vertices is a subgraph of the graph K_5 then $|E_{1,2}| \geqq 5$.*

Proof. It si sufficient to show that G_2 has a subgraph with 5 vertices and with at least 5 edges. It is easy to check this fact by distinguishing the following cases:

a) $\triangle(G_2) = 3$

- (i) $|E(U)| \geqq 2$
- (ii) $|E(U)| = 1$
  If $|E(U, U')| = 0$ then $G_2(U')$ is the complete graph K_4.
- (iii) $|E(U)| = 0$
  If every vertex from U' has degree at most 1 in $G_2(U, U')$ then $G_2(U')$ has at least 3 edges.

b) $\triangle(G_2) \geqq 4$

- (i) $|E(U)| \geqq 1$
- (ii) There is a vertex in U' whose degree in $G_2(U, U')$ is at least 2
- (iii) If none of the previous two cases is valid then $G_2(U')$ is the complete graph K_3 and $|E(U, U')| = 3$. □

Lemma 2.2. *Let $G \in F_{p,p+2}$, $p \in \{7, 8, 9\}$ and $\triangle(G) \geqq 4$. Then G contains at least one of the graphs H_1, H_2 (Fig. 2.1).*

H_1 H_2

Fig. 2.1

Proof. Suppose that $|E(U)| = 0$ and simultaneously $|E(U, U')| = 0$. Then $|E(U')| = |U'| + 3$ which is impossible since $|U'| \leqq 4$. $\square$

Lemma 2.3. *Let $G \in F_{p,p+2}$, $p \in \{7, 8, 9\}$, $\triangle(G) = 4$ and $|E(U)| = |E(U')| = 0$. Then G contains the graphs H_3 and H_4 (Fig. 2.2).*

H_3 H_4

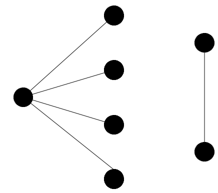
Fig. 2.2

Proof. Since $|E(U, U')| = |U'| + 3$, at least one of the following holds:

(i) there are at least 2 vertices of degree at least 2 in U'
(ii) there is a vertex of degree at least 3 and another vertex of non-zero degree in U' $\square$

Lemma 2.4. *If $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$ and $\triangle(G_1) = \triangle(G_2) = 4$ then $|E_{1,2}| \geqq 5$.*

Proof. In view of Lemma 2.2 it is sufficient to consider the case when exactly one of the graphs G_1, G_2 contains the graph H_1 and exactly one of them contains the graph H_2. Without loss of generality we can assume that the graph G_1 contains the graph H_1 and the graph G_2 contains the graph H_2. According to Lemma 2.1 we can also assume that the graph G_1 without its isolated vertices is not a subgraph of the graph K_5. According to the above facts, we have $|E(G_1(U'))| \geqq 1$. If $|E(G_2(U'))| \geqq 1$ then a common subgraph is the graph H_5 (Fig. 2.3).



H_5

Fig. 2.3

So let $|E(G_2(U'))| = 0$. According to Lemma 2.3 the graph G_2 contains the graphs H_3 and H_4. If $|E(G_1(U))| \geqq 3$ then the graph G_1 contains the graph H_3. In the opposite case we have $|E(G_1(U'))| \geqq |U'| + 1$ whence $|U'| = 4$. Then the graph G_1 contains the graph H_4 and the proof is finished. $\square$

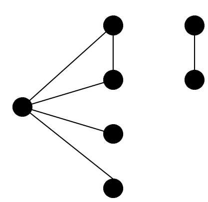
Lemma 2.5. *Let $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$, $\triangle(G_1) \geqq 4$ and $\triangle(G_2) \geqq 5$. Then $|E_{1,2}| \geqq 5$.*

Proof. The statement of the lemma is trivial if $\triangle(G_1) > 4$. Two cases are possible:

a) $|E(G_1(U, U'))| \geqq 1$,
b) $|E(G_1(U, U'))| = 0$.

In the case a) a common subgraph is the graph H_2. Obviously, if the graph G_2 did not contain the graph H_2 then it would be $|E(G_2(U))| = 0$ and $|E(G_2(U, U'))| = 0$. This yields $|E(G_2(U'))| > |U'|$ and it is impossible.

In the case b) we can assume according to Lemmas 2.1 and 2.2 that G_1 contains the graph H_6 (Fig. 2.4). Clearly, the statement holds if $|E(G_2(U))| \geqq 1$ or $|E(G_2(U, U'))| \geqq 1$. Now it is sufficient to realize that at least one of these inequalities must be valid for the graph G_2. $\square$



H_6

Fig. 2.4

Let $F = \{G \in F_{7,9} \cup F_{8,10} \cup F_{9,11}; \triangle(G) = 3\}$. Let us consider the next subsets of F:

F_1 contains all graphs which have the subgraph H_7 (Fig. 2.5),
F_2 contains all graphs from $F - F_1$ which have the subgraph H_8 (Fig. 2.5),

F_3 contains all graphs from $F - F_1$ which have the subgraph H_9 (Fig. 2.5),
F_4 contains all graphs from $F - F_1$ which have the subgraph H_{10} (Fig. 2.5),
F_5 contains all graphs from $F - F_1$ which have the subgraph H_{11} (Fig. 2.5).

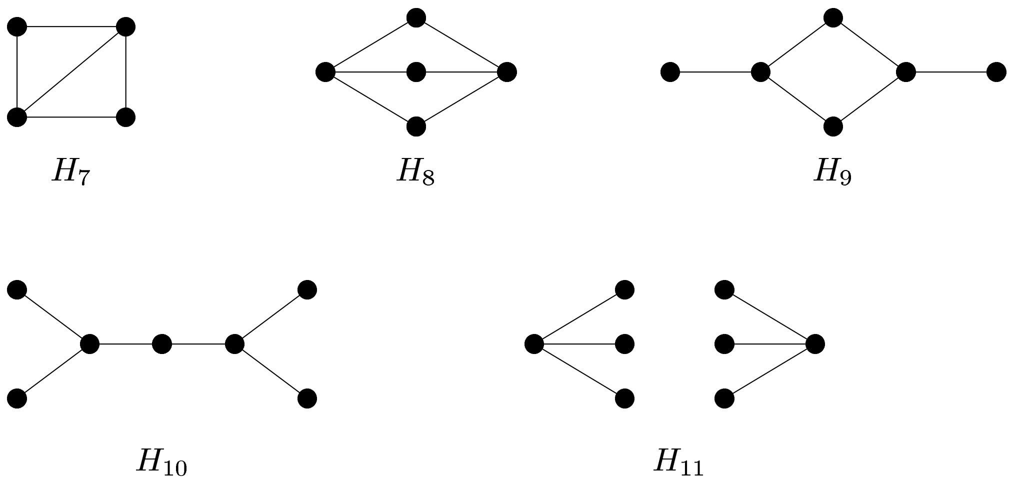
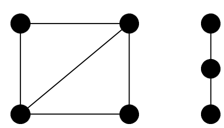


Fig. 2.5

Lemma 2.6. *$F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 = F$.*

Proof. Obviously, each graph $G \in F$ has at least 4 vertices of degree 3. If there are no two non-adjacent vertices of degree 3 then G contains H_7. If there are two non-adjacent vertices of degree 3 then G must contain at least one of the graphs H_8, H_9, H_{10} and H_{11}. $\square$

Lemma 2.7. *If $G \in F_1$ then G has the subgraph H_{12} in Fig. 2.6.*

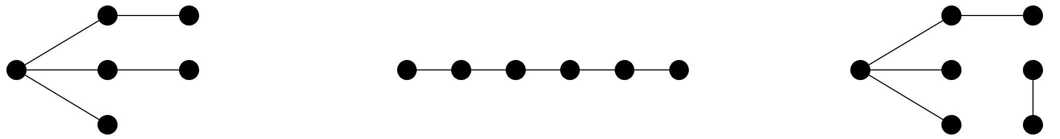


H_{12}

Fig. 2.6

Proof. We know that the graph G has the subgraph H_7 and apart from the vertices of this subgraph G it has other $p - 4$ vertices. The subgraph of the graph G induced by these $p - 4$ vertices has at least $p - 5$ edges and consequently it has a vertex of degree at least 2. $\square$

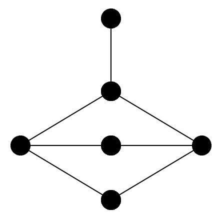
Lemma 2.8. *If $G \in F_2$ then it contains the graphs H_3, H_4, H_{13} and H_{14} (Figs. 2.2, 2.7). Moreover, if $p \neq 7$ then G contains also the graph H_{15} (Fig. 2.7).*



H_{13} H_{14} H_{15}

Fig. 2.7

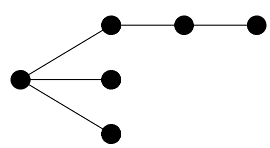
Proof. The component of the graph G which contains the graph H_8 must also contain a vertex which does not belong to H_8, i.e. G contains the graph H_{16} (Fig. 2.8). Obviously, if $p \neq 7$ then G has an edge which is not incident with any vertex of the subgraph H_8. $\square$



H_{16}

Fig. 2.8

Lemma 2.9. *If $G \in F_3$ then it contains the graphs H_3, H_4, H_{13} and H_{17} (Figs. 2.2, 2.7, 2.9).*

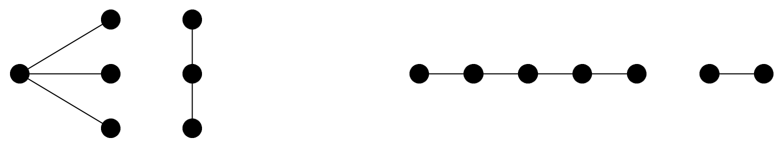


H_{17}

Fig. 2.9

Proof. Obviously, G contains the graphs H_3 and H_{17}. Since G contains the graph H_9 and has at least 9 edges, it has an edge which is not incident to any vertex of the circle in the considered subgraph H_9. Therefore G contains H_4. Apart from the edges of H_9 there must exist another edge in G which is incident with at least one vertex of H_9. It follows immediately that G contains H_{13}. $\square$

Lemma 2.10. *If $G \in F_4 \cup F_5$ then G contains the graphs H_{13}, H_{17}, H_{18} and H_{19}. (Figs. 2.7, 2.9, 2.10).*



H_{18} H_{19}

Fig. 2.10

Proof.

a) $G \in F_4$
Obviously, G has the subgraphs H_{17} and H_{18}. Further, the graph G has at least 3 edges except of the edges of the subgraph H_{10}. Each of these edges is incident with at least one vertex of the subgraph H_{10}. It follows that G has the subgraphs H_{13} and H_{19}.

b) $G \in F_5$
Obviously, G has the subgraph H_{18}. Since G can not have two non-trivial components it contains the subgraph H_{17}. If the graph G contains at least one of the subgraphs H_{20}, H_{21} (Fig. 2.11) then it also has the graphs H_{13} and H_{19}.

H_{20} H_{21}

Fig. 2.11

In the opposite case p=9 and G contains at least one of the subgraph H_{22}, H_{23} (Fig. 2.12). Hence G has the subgraph H_{19} and it is easy to verify that it also has the subgraph H_{13}. $\square$

H_{22} H_{23}

Fig. 2.12

Lemma 2.11. *If $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$ and $\triangle(G_1) = \triangle(G_2) = 3$ then $|E_{1,2}| \geqq 5$.*

Proof. The statement follows straightforwardly from Lemmas 2.6 - 2.10. $\square$

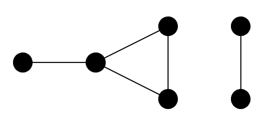
Lemma 2.12. *If $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$, $\triangle(G_1) = 3$, $\triangle(G_2) = 4$ then $|E_{1,2}| \geqq 5$.*

Proof. According to Lemma 2.1 we can assume that G_2 without its isolated vertices is not a subgraph of K_5. We distinguish several cases:

a) $G_1 \in F_1$
According to Lemma 2.7 G has the subgraph H_{12}. We distinguish 4 cases for the graph G_2:

(i) $|E(U)| \geqq 3$
A common subgraph is the graph H_7.

(ii) $|E(U)| = 2$
If the considered edges are adjacent then a common subgraph is H_7. If they are not adjacent then a common subgraph is H_{24} (Fig. 2.13).



H_{24}

Fig. 2.13

(iii) $|E(U)| = 1$
If $|E(U')| \geqq 1$ then a common subgraph is H_{24}. If $|E(U')| = 0$ then there exists a vertex of degree at least 2 in U' and at least one of the graphs H_7 and H_{24} is a common subgraph.

(iv) $|E(U)| = 0$
A common subgraph is H_{18}.

b) $G_1 \in F_2$
We show that G_2 contains at least one of the graphs from Lemma 2.8. We can assume that $|E(U)| \leqq 2$ and each vertex from U' has degree at most 1 in $G_2(U, U')$

(in opposite case a common subgraph is H_3). From this it follows that $|E(U')| \geqq 1$. If $p = 7$ then a common subgraph is H_{14}. So let $p \neq 7$. If $|E(U)| \geqq 1$ then a common subgraph is H_{15}. If $|E(U)| = 0$ then there exists a vertex of degree at least 2 in $G_2(U')$ and if moreover $|E(U,U')| \geqq 2$ then again H_{15} is a common subgraph. In the opposite case $p = 9$, $|E(U')| = 6$ and a common subgraph is H_4.

c) $G_1 \in F_3$

We show that G_2 contains at least one of the graphs from Lemma 2.9. If $|E(U,U') \cup E(U')| > |U'|$ then at least one vertex from U' has degree at least 2 in G_2. Hence it is easy to verify that at least one of the graphs H_3, H_4, H_{17} is a common subgraph. In the opposite case $|E(U)| \geqq 3$ and a common subgraph is H_3.

d) $G_1 \in F_4 \cup F_5$

We show that G_2 contains at least one of the graphs from Lemma 2.10 (i.e. H_{13}, H_{17}, H_{18}, H_{19}).

(i) $|E(U)| \geqq 1$

If $|E(U,U')| \geqq 1$ then a common subgraph is H_{13} or H_{17}. If $|E(U,U')| = 0$ then according to Lemma 2.1 we can assume that $|E(U')| \geqq 1$ and a common subgraph is H_{18} or H_{19}.

(ii) $|E(U)| = 0$

It is easy to check that G_2 contains the graph H_{18}. $\square$

Lemma 2.13. *If $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$, $\triangle(G_1) = 3$ and $\triangle(G_2) \geqq 5$ then $|E_{1,2}| \geqq 5$.*

Proof. We distinguish several cases:

a) $G_1 \in F_1$

We show that G_2 contains a subgraph of the graph H_{12} having 5 edges.

(i) $|E(U)| \leqq 2$

In this case $|E(U,U') \cup E(U')| \geqq |U'| + 1$ and therefore some vertex from U' has degree at least 2 in G_2. A common subgraph is H_{18}.

(ii) $|E(U)| \geqq 3$

If there exist two adjacent edges in $G_2(U)$ then a common subgraph is H_7. If there are no such edges then the common subgraph is H_{24}.

b) $G_1 \in F_2 \cup F_3$

According to Lemmas 2.8 and 2.9 it is sufficient to show that G_2 contains at least one of the graphs H_3 and H_{13}. Obviously, this is true if $|E(U)| \geqq 2$ or if some vertex from U' has degree at least 2 in $G_2(U,U')$. In the opposite case we have $|E(U')| \geqq 2$ and it follows that $p = 9$ and $|E(U,U')| \geqq 2$. Now it is easy to verify that G_2 contains H_3 or H_{13}.

c) $G_1 \in F_4 \cup F_5$

We show that G_2 contains at least one of the graphs H_{13}, H_{17} and H_{18} (see Lemma 2.10).

(i) $|E(U)| \geqq 2$

In this case G_2 contains at least one of the graphs H_{13}, H_{17}.

(ii) $|E(U)| \leqq 1$
In this case at least one vertex in U' has degree at least 2. Hence G_2 contains the graph H_{18}. $\square$

Theorem 2.14. $\operatorname{diam} F_{p,p+2} = 2p - 6$ *for* $p \in \{7, 8, 9\}$.

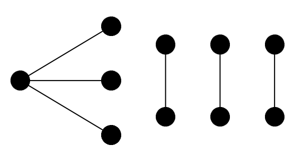
Proof. By Lemmas 2.4, 2.5, 2.11, 2.12 and 2.13 it suffices to find two graphs $G_1, G_2 \in F_{p,p+2}$ with $|E_{1,2}| = 5$. Such graphs are depicted in Fig. 2.14 (one component of G_1 is a circle). $\square$

Fig. 2.14

3. Diameter of $F_{p,p+3}$

In this section of the paper we will determine $\operatorname{diam} F_{p,p+3}$ for $p \geqq 19$.

Lemma 3.1. *If* $G \in F_{p,p+3}$, $p \geqq 17$ *and* $\triangle(G) = 3$ *then* G *contains the graph* H_{25} *(Fig. 3.1).*

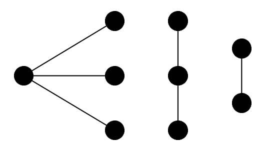


H_{25}

Fig. 3.1

Proof. Let v be a vertex of degree 3 in the graph G. If $p \geqq 7$ then there exists an edge w_1w_2 such that $w_i \notin \{v, v_1, v_2, v_3\}$, $i = 1, 2$. If $p \geqq 12$ then there exists an edge w_3w_4 such that $w_i \notin \{v, v_1, v_2, v_3, w_1, w_2\}$, $i = 3, 4$. If $p \geqq 17$ then there exists an edge w_5w_6 such that $w_i \notin \{v, v_1, v_2, v_3, w_1, w_2, w_3, w_4\}$, $i = 5, 6$. Thus G contains the graph H_{25}. □

Lemma 3.2. *If $G \in F_{p,p+3}$, $p \geqq 14$ and $\triangle(G) = 3$ then G contains the graph H_{26} (Fig. 3.2).*

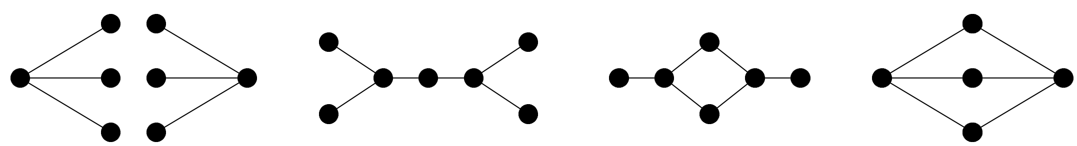


H_{26}

Fig. 3.2

Proof. If $p - 4 < 2(p - 6)$ i.e. $p > 8$ then $G(U')$ has a vertex w_1 of degree at least 2. Let w_2, w_3 are the vertices adjacent to w_1 in $G(U')$. If $p \geqq 14$ then there exists an edge such that neither of its vertices belongs to $\{v, v_1, v_2, v_3, w_1, w_2, w_3\}$ i.e. G contains the graph H_{26}. □

Lemma 3.3. *Let $G \in F_{p,p+3}$ and $\triangle(G) = 3$. Then G contains at least one of the graphs H_{27}, H_{28}, H_{29} and H_{30} (Fig. 3.3).*

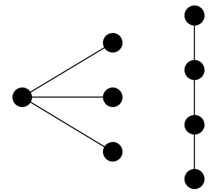


H_{27} H_{28} H_{29} H_{30}

Fig. 3.3

Proof. G has at least 6 vertices of degree 3. We thus get that there are two non-adjacent vertices of degree 3 in G. All four possible cases for these two vertices are depicted in the Fig. 3.3. □

Lemma 3.4. *Let $G \in F_{p,p+3}$ and $\triangle(G) = 3$. If G has at least two components with more edges than vertices then it contains the graphs H_{31} (Fig. 3.4) and H_{27}.*



H_{31}

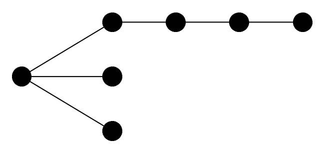
Fig. 3.4

Proof. Obviously, each of the considered components of the graph G contains a vertex of degree 3. Further it is sufficient to realize that such component has more than 3 edges. $\square$

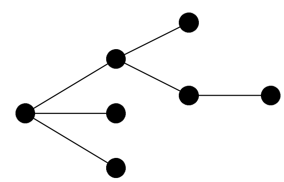
Lemma 3.5. *Let $G \in F_{p,p+3}$, $\triangle(G) = 3$ and G have only one component H having more edges than vertices.*

(a) *If H has the subgraph H_{27} then it contains at least one of the graphs H_{32} (Fig. 3.5) and H_{31}.*

(b) *If H has the subgraph H_{28} then it contains at least one of the graphs H_{32}, H_{33} (Fig. 3.5).*



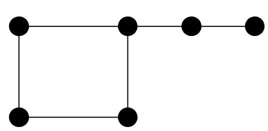
H_{32}



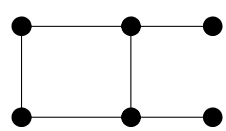
H_{33}

Fig. 3.5

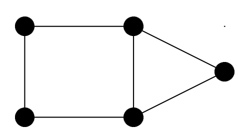
(c) *If H has the subgraph H_{29} then it contains at least one of the graphs H_{34}, H_{35}, H_{36} (Fig. 3.6).*



H_{34}



H_{35}

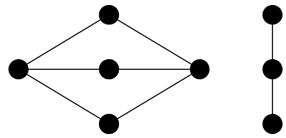


H_{36}

Fig. 3.6

Proof. The graph H has at least 3 edges more than vertices.
(a) and (b): It is sufficient to realize that H contains more than 6 edges.
(c): If H has exactly 6 vertices then it contains the graph H_{36}. If H has at least 7 vertices then it contains at least one of the graphs H_{34}, H_{35}. $\square$

Lemma 3.6. *Let $G \in F_{p,p+3}$, $p \geqq 8$ and $\triangle(G) = 3$. If G contains the graph H_{30} then it contains the graph H_{37} (Fig. 3.7).*

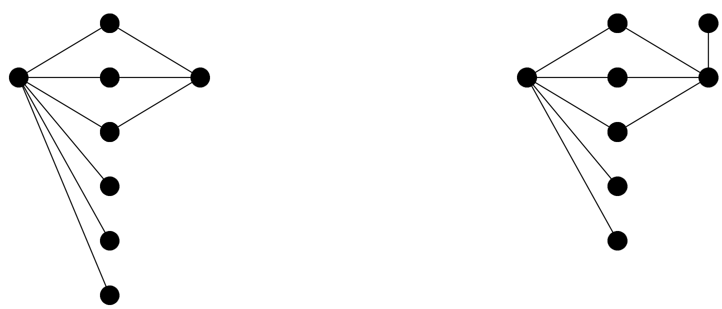


H_{37}

Fig. 3.7

Proof. There are another $p-5$ vertices in G besides the vertices of H_{30}. The subgraph H of the graph G induced by these $p-5$ vertices has at least $p-6$ edges. If H does not contain any vertex of degree 2 then $p-5 \geqq 2(p-6)$, i.e. $p \leqq 7$. □

Lemma 3.7. *If G contains at least one of the graphs H_{38}, H_{39} (Fig. 3.8) then it contains the graphs H_{27}, H_{28}, H_{29} and H_{30}.*

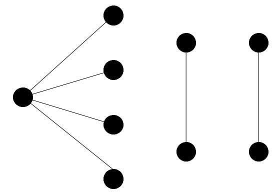


H_{38} H_{39}

Fig. 3.8

Proof. The statement is obvious. □

Lemma 3.8. *If $\triangle(G) = 4$ and $q \geqq 21$ then G contains the graph H_{40} (Fig. 3.9).*

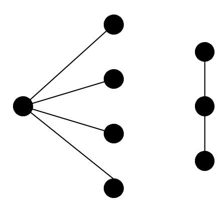


H_{40}

Fig. 3.9

Proof. $|E(U')| \geqq 5$ and hence $G(U')$ contains two independent edges. $\square$

Lemma 3.9. *If $G \in F_{p,p+3}$, $\triangle(G) = 4$ and $p \geqq 19$ then G contains the graph H_{41} (Fig. 3.10).*



H_{41}

Fig. 3.10

Proof. Obviously, if $|E(U)| + |E(U,U')| \leqq 10$ and G does not contain H_{41} then $p - 5 \geqq 2(p - 11)$ i.e. $p \leqq 17$. We can thus assume that $|E(U)| + |E(U,U')| \geqq 11$. Since $\triangle(G) = 4$ it holds $|E(U)| + |E(U,U')| \leqq 12$. We distinguish two cases:

(i) $|E(U)| + |E(U,U')| = 12$
In this case $|E(U)| = 0$ and every vertex from U has degree 3 in $G(U,U')$. We can assume that there are at most 2 vertices from U' of degree 0 in $G(U')$. In fact, in the opposite case it holds (if G does not contain H_{41}) $p - 8 \geqq 2(p - 13)$, i.e. $p \leqq 18$. If some vertex from U' has degree 2 or 3 in $G(U,U')$ then G contains the graph in Fig. 3.11.

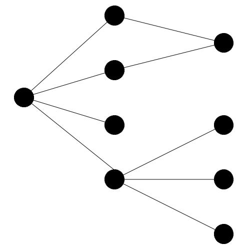


Fig. 3. 11

It follows that G contains the graph H_{41}. So, let no vertex from U' has degree 2 or 3 in $G(U,U')$. There are at most 3 vertices from U' which have degree 4 in $G(U,U')$. Let k be the number of them.

a) k=3
Since there are at most 2 vertices of degree 0 in $G(U')$ this case is impossible.

b) k=2
In this case there are exactly 4 vertices in U' of degree 1 in $G(U,U')$. At least 2 of these 4 vertices have degree at least 1 in $G(U')$ and it follows that G contains H_{41}.

c) $k \leqq 1$
The statement is obviuos.

(ii) $|E(U)| + |E(U,U')| = 11$
In this case $|E(U)| \leqq 1$. If there was an isolated vertex in $G(U')$ and G did not contain the graph H_{41} then it would hold $p - 6 \geqq 2(p-12)$, i.e. $p \leqq 18$. We can thus assume that no vertex in $G(U')$ is isolated. It follows that no vertex from U' has degree 4 in $G(U,U')$. The statement of the lemma holds if no vertex from U' has degree 2 or 3 in $G(U,U')$. If some vertex $u \in U'$ has degree 3 in $G(U,U')$ then it is sufficient to realize that the vertex from U not adjacent to the vertex u has degree at least 2 in $G(U,U')$. Since the vertex u is not isolated in $G(U')$ then G contains the graph H_{41}.
If some vertex from U' has degree 2 in $G(U,U')$ and is adjacent to vertices $v_1, v_2 \in U$ then it is sufficient to take into account that at least one of the vertices $v_3, v_4 \in U$ has degree 4 and is not adjacent to the vertex v_i for $i = 1, 2$. $\square$

Lemma 3.10. *Let $G_1, G_2 \in F_{p,p+3}$, $p \geqq 14$ and $\triangle(G_1) = \triangle(G_2) = 3$. Then $|E_{1,2}| \geqq 6$.*

Proof. The statement is a consequence of Lemma 3.2. $\square$

Lemma 3.11. *Let $G_1, G_2 \in F_{p,p+3}$, $p \geqq 18$ and $\triangle(G_1) = \triangle(G_2) = 4$. Then $|E_{1,2}| \geqq 6$.*

Proof. The statement is a consequence of Lemma 3.8. $\square$

Lemma 3.12. *Let $G_1, G_2 \in F_{p,p+3}$, $p \geqq 19$ and $\triangle(G_1) = 3$, $\triangle(G_2) \geqq 4$. Then $|E_{1,2}| \geqq 6$.*

Proof. First realize that if we prove that G_2 contains at least one of the graphs H_{25}, H_{26} then the statement of the lemma holds by Lemmas 3.1 and 3.2. We distinguish several cases for G_2:

a) There are 2 independent edges in $G_2(U')$

a_1) $\triangle(G_2) \geqq 5$

(i) If $|E(U)| \neq 0$ or $|E(U, U')| \neq 0$ then G_2 contains at least one of the graphs H_{25}, H_{26}.

(ii) If $|E(U)| = 0$ and $|E(U, U')| = 0$ then $|E(U')| \geqq 8$ and hence G_2 contains at least one of the graphs H_{25}, H_{26}.

a_2) $\triangle(G_2) = 4$
If $|E(U, U')| \neq 0$ or $|E(U')| \geqq 7$ then G_2 contains at least one of the graphs H_{25}, H_{26}; in the opposite case it holds $q \leqq 16$, a contradiction.

b) $|E(U')| \geqq 2$ and any two edges in $G_2(U')$ are adjacent edges in U'
In this case $G_2(U')$ contains the graph in Fig. 3.12

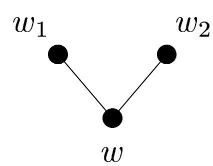


Fig. 3.12

b_1) there is an edge in $G_2(U, U')$ which is not incident with any vertex from $\{w_1, w_2, w\}$
In this case G_2 contains H_{26}.

b_2) $\triangle(G_2) \geqq 5$ and b_1) does not hold
We distinguish 3 subcases:

(i) $|E(U)| \neq 0$
In this case G_2 contains H_{26}.

(ii) $|E(U)| = 0$ and the vertex w has degree at least 3 in $G_2(U, U')$
The statement holds by Lemmas 3.7 and 3.3.

(iii) $|E(U)| = 0$ and the vertex w has degree at most 2 in $G_2(U, U')$
If $|E(U')| = 2$ then degree of w_1 or w_2 is at least 2 in $G_2(U, U')$ (since $|E(U, U')| \geqq 5$) and hence G_2 contains H_{26}. If $|E(U')| \geqq 3$ then again G_2 contains H_{26} (since $|E(U, U')| \geqq 4$).

b_3) $\triangle(G_2) = 4$ and b_1) does not hold
In this case $q \leqq 20$, a contradiction.

c) $|E(U')| = 1$

c_1) $\triangle(G_2) \geqq 6$

(i) $|E(U)| \geqq 2$
If there are two adjacent edges in U then G_2 contains H_{26}. Now let us consider the opposite case. If $\triangle(G_2) \geqq 7$ then G_2 contains H_{25}. If $\triangle(G_2) = 6$ then $|E(U,U')| \geqq 2$ and hence G_2 contains H_{25} or H_{26}.

(ii) $|E(U)| \leqq 1$
In this case $|E(U,U')| \geqq |U'|+2$ and if G_2 does not contain H_{26} then it contains at least one of the graphs in Fig. 3.13.

α β

Fig. 3.13

If G_2 contains the graph α then the lemma holds by Lemmas 3.7 and 3.3. If G_2 contains the graph β then G_2 contains each of the graphs H_{27}, H_{28}, H_{29}, H_{30} and the statement holds by Lemma 3.3.

c_2) $\triangle(G_2) = 5$
If there are at least two edges in $G_2(U,U')$ which are not adjacent to the edge of the graph $G_2(U')$ then G_2 contains at least one of the graphs H_{25}, H_{26}. In the opposite case at least 3 edges from $G_2(U,U')$ are incident with the same vertex of the edge of $G_2(U')$. Then the statement of the lemma follows from Lemmas 3.7 and 3.3.

c_3) $\triangle(G_2) = 4$
This case is not possible for $q \geqq 18$.

d) $|E(U')| = 0$

d_1) $\triangle(G_2) \geqq 6$

(i) there is a vertex in U' which has degree at least 2 and $|E(U,U')| \geqq 3$
Then G_2 contains at least one of the graphs in Fig. 3.14. In the case α) the statement of the lemma follows from Lemmas 3.7 and 3.3. In the case β) G_2 contains H_{26}. In case γ it holds $|E(U)|+|E(U,U')| \geqq 6$ and hence G_2 again contains H_{26}.

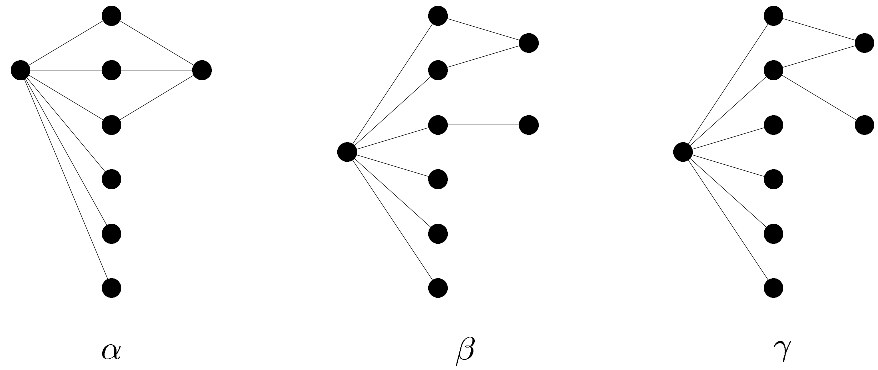


Fig. 3.14

(ii) there is a vertex in U' which has degree at least 2 and $|E(U,U')| = 2$ In this case $|E(U)| \geqq 3$ and if $\triangle(G_2) = 6$ then G_2 contains each of the graphs H_{27}, H_{28}, H_{29}, H_{30} (since $q \geqq 22$) and the statement of the lemma holds by Lemma 3.3. If $\triangle(G_2) \geqq 7$ then G_2 contains H_{26} or the graph H_{42} (Fig. 3.15).

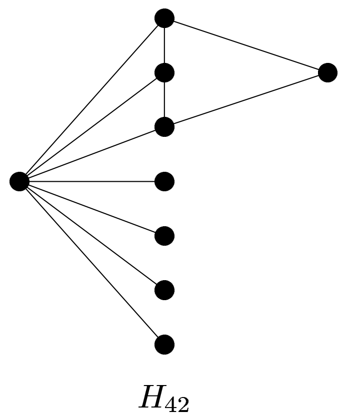


Fig. 3.15

The graph H_{42} contains the graphs H_{27}, H_{28} and H_{29}. If the graph G_1 does not contain any of the graphs H_{27}, H_{28} and H_{29} then it contains H_{30} by Lemma 3.3 and H_{37} by Lemma 3.6. The graphs H_{42} and H_{37} have a common subgraph which is depicted in Fig. 3.16.

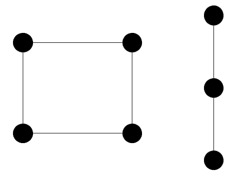


Fig. 3.16

(iii) every vertex from U' has degree at most 1 and there exists a vertex from U of degree at least 2 in $G_2(U, U')$
In this case $|E(U)| \geqq 4$. If G_2 does not contain H_{26} then the statement of the lemma follows from Lemmas 3.7 and 3.3.

(iv) there are no adjacent edges in $G_2(U, U')$ and $|E(U, U')| \geqq 2$
In this case $|E(U)| \geqq 4$. If $|E(U, U')| \geqq 3$ then G_2 contains H_{25}. If $|E(U, U')| = 2$ and $\triangle(G_2) \geqq 7$ then obviously a common subgraph is H_{25} or H_{26}. If $|E(U, U')| = 2$ and $\triangle(G_2) = 6$ then G_2 contains each of the graphs H_{27}, H_{28}, H_{29} and H_{30} (since $q \geqq 22$). Now use Lemma 3.3.

(v) $|E(U, U')| \leqq 1$
In this case we have $|E(U)| \geqq 4$. We distinguish three subcases:
1) $\triangle(G_2) \geqq 9$. If there is a vertex of degree at least 3 in $G_2(U)$ then the statement of the lemma holds by Lemmas 3.7 and 3.3. If there is no vertex of degree at least 3 in $G_2(U)$ and G_2 contains neither the graph H_{25} nor the graph H_{26} then G_2 contains the graph in Fig. 3.17.

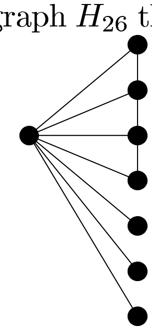


Fig. 3.17

The statement of the lemma follows from Lemmas 3.3, 3.4 and 3.5 (the component H from Lemma 3.5 contains at least one of the graphs H_{27}, H_{28}, H_{29} and H_{30}).
2) $\triangle(G_2) \in \{7, 8\}$
If $q \geqq 18$ then there exists a vertex of degree at least 3 in $G_2(U)$ and

hence G_2 contains the graph α in Fig. 3.14 and the statement of the lemma holds by Lemmas 3.7 and 3.3.
3) $\triangle(G_2) = 6$
This case is not possible for $q \geqq 22$.

d_2) $\triangle(G_2) = 5$
Since $|E(U')| = 0$ and $q \geqq 22$ there is a vertex from U which has degree at least 3 in $G_2(U, U')$. Now, if we realize that there exists an edge in $G_2(U, U')$ which is not incident with the considered vertex of degree at least 3 then we get that G_2 contains H_{26}.

d_3) $\triangle(G_2) = 4$
This case is not possible for $q \geqq 17$. $\square$

Lemma 3.13. *Let $G_1, G_2 \in F_{p,p+3}$, $\triangle(G_1) = 4$, $\triangle(G_2) \geqq 5$ and $p \geqq 19$. Then $|E_{1,2}| \geqq 6$.*

Proof. In wiew of Lemmas 3.8 and 3.9 it is sufficient to show that G_2 contains at least one of the graphs H_{40}, H_{41}.

a) $E(U') \neq 0$
If $\triangle(G_2) \geqq 6$ then the statement of the lemma is obvious. If $\triangle(G_2) = 5$ then it is sufficient to use the fact that $|E(U, U')| + |E(U')| > 1$.

b) $|E(U')| = 0$
If $\triangle(G_2) \geqq 6$ and $|E(U, U')| \geqq 2$ then the statement obviously holds. So, it is sufficient to consider two cases:

(i) $\triangle(G_2) \geqq 6$ and $|E(U, U')| \leqq 1$.
Obviously, the statement holds if $\triangle(G_2) \geqq 8$. If $\triangle(G_2) = 7$ then there exists a vertex of degree at least 2 in $G_2(U)$ and hence G_2 contains the graph H_{41}. The case $\triangle(G_2) = 6$ is impossible since $q \geqq 22$.

(ii) $\triangle(G_2) = 5$.
There exists a vertex from U of degree at least 2 in $G_2(U, U')$. $\square$

Lemma 3.14. *Let $G_1, G_2 \in F_{p,p+3}$, $\triangle(G_1) \geqq 5$, $\triangle(G_2) \geqq 5$, $p \geqq 19$. Then $|E_{1,2}| \geqq 6$.*

Proof. Obviously, the statement holds if $\triangle(G_1) \geqq 6$ and $\triangle(G_2) \geqq 6$. We distinguish two cases:

a) $\triangle(G_1) = \triangle(G_2) = 5$
It is sufficient to consider the case that none of the graphs H_{43}, H_{44} (Fig. 3.18) is a common subgraph of the graphs G_1 and G_2. So we can assume that $|E(G_1(U, U'))| = 0$ and $|E(G_2(U'))| = 0$.

H_{43} H_{44}

Fig. 3.18

We show that a common subgraph is the graph H_{41}. If the graph G_1 did not contain the graph H_{41} then it would hold $p-6 \geqq 2(p-12)$, i.e. $p \leqq 18$. The graph G_2 contains the graph H_{41} since $|E(U,U'| \geqq 14$.

b) $\triangle(G_1) = 5$ and $\triangle(G_2) \geqq 6$.
If the graph G_1 does not contain H_{44} then it contains each of the graphs H_{43}, H_{41}. The graph G_2 contains the graph H_{44} (the case $\triangle(G_2) > 6$ is trivial and if $\triangle(G_2) = 6$ then $|E(U,U')| + |E(U')| \geqq 1$). If G_2 does not contain H_{43} then $|E(U)| = 0$ and $|E(U,U')| = 0$. This implies $|E(U')| = |U'| + 4$ and hence G_2 contains the graph H_{41}. $\square$

Theorem 3.15. $\operatorname{diam} F_{p,p+3} = 2p-6$ *for* $p \geqq 19$.

Proof. In view of Lemmas 3.10 - 3.14 it suffices to find two graphs $G_1, G_2 \in F_{p,p+3}$ with $|E_{1,2}| = 6$. Such a graph G_1 is depicted in Fig. 3.19 and G_2 is an arbitrary graph for which $\triangle(G) = 3$. $\square$

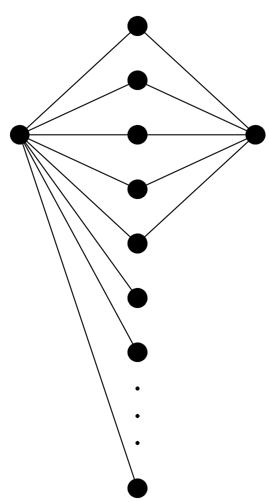


Fig. 3.19

4. Some other results about $\operatorname{diam} F_{p,q}$

Theorem 4.1.

a) *(i)* $\operatorname{diam} F_{5,8} = 2$
   (ii) $\operatorname{diam} F_{6,9} = 6$
   (iii) $\operatorname{diam} F_{8,11} = 12$

b) *If* $p+3 \leqq q \leqq \frac{3p}{2}$ *and* $7 \leqq p \leqq 18$ *then* $\operatorname{diam} F_{p,q} \in \{2q-12, 2q-10\}$.

c) *If* $p+3 \leqq q \leqq \frac{3p}{2}$ *and* $p \geqq 19$ *then* $\operatorname{diam} F_{p,q} = 2q-12$.

Proof.

a) (i) According to Theorems 5 and 2 from [3] we get
$$\operatorname{diam} F_{5,8} = \operatorname{diam} F_{5,2} = 2.$$

(ii) According to Theorem 5 from [2] we have $c_{6,6} = 3$. Now by using Theorem 5 from [3] we get
$$\operatorname{diam} F_{6,9} = \operatorname{diam} F_{6,6} = 2.6 - 2.3 = 6.$$

(iii) According to Lemma 2.14 we have $c_{8,10} = 5$. Since $c_{8,11} \geqq c_{8,10}$, it is sufficient to find two graphs $G_1, G_2 \in F_{8,11}$ with $|E_{1,2}| = 5$. Such graphs are depicted in Fig. 4.1.

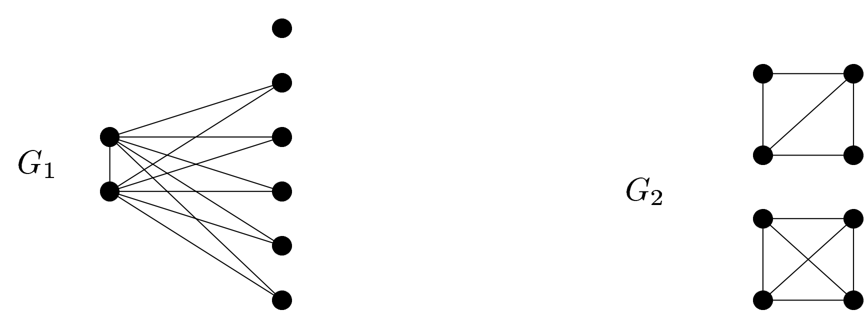


Fig. 4.1

b) According to Lemma 2.14 and Theorem 14 from [1] we have $c_{p,p+2} = 5$ for $7 \leqq p \leqq 18$. It implies $c_{p,q} \geqq 5$ for $q \geqq p+3$. To show that $c_{p,q} \leqq 6$ it is sufficient to find two graphs $G_1, G_2 \in F_{p,q}$ with $|E_{1,2}| = 6$. The graph G_1 is depicted in Fig. 4.2 and G_2 is an arbitrary graph for which $\triangle(G) = 3$.

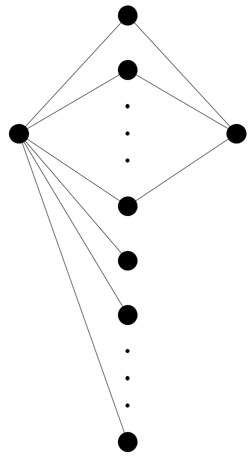


Fig. 4.2

c) By Theorem 3.15 we get $c_{p,q} \geqq 6$. To show that $c_{p,q} = 6$ it is sufficient to find two graphs $G_1, G_2 \in F_{p,q}$ with $|E_{1,2}| = 6$. The graph G_1 is depicted in Fig. 4.2 and G_2 is an arbitrary graph for which $\triangle(G) = 3$. $\square$

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Dept. of Mathematics, Faculty of Natural Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

E-mail address: monosz@fhpv.umb.sk

E-mail address: hrnciar@fhpv.umb.sk

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ON THE S-DISTANCE BETWEEN POSETS

Pavel Klenovčan

Abstract. V. Baláž, V. Kvasnička and J. Pospíchal [1] proved that the distances based on maximal common subgraph and minimal common supergraph are identical. Here we shall study an analogy for posets.

Throughout this paper all partially ordered sets are assumed to be finite. In [2] a metric on a system of isomorphism classes of posets, which have the same cardinality, is defined. Without loss of generality we can suppose that all posets are defined on the same set P. We will often write a poset R instead of a poset (P,R).

Let $B(P)$ be the set of all bijective maps of P onto itself. For any $f \in B(P)$ and posets $(P,R),(P,S)$ we denote by $d_f(R,S)$ the number defined by

$$d_f(R,S) = |f(R) \setminus S| + |S \setminus f(R)|, \tag{1}$$

where $f(R) = \{[f(a), f(b)];\ [a,b] \in R\}$ (cf. [2]). Since the posets (P,R) and $(P,f(R))$ are isomorphic, then

$$d_f(R,S) = |R| + |S| - 2|f(R) \cap S|. \tag{2}$$

The *distance* of the posets $(P,R),(P,S)$ is defined by

$$d(R,S) = \min\{d_f(R,S);\ f \in B(P)\}. \tag{3}$$

If we identify isomorphic posets, then (3) defines a *metric* on the set of all (finite, non-isomorphic) posets defined on the same set P.

If a map $f \in B(P)$ is an isotone map of a poset (P,R) onto a poset (P,S), then $f(R) \subseteq S$ and $d(R,S) = d_f(R,S) = |S| - |R|$ (cf. Remark 2 in [2]).

The following lemma is easy to verify (cf. Lemma 1 in [2]).

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Lemma 1. *For any posets $(P,R),(P,S)$ and any maps $f,g \in B(P)$ the following properties are satisfied:*

$$\begin{array}{ll} \text{(i)} & d_f(R,S) = d_g(R,S) \quad iff \quad |f(R)\cap S| = |g(R)\cap S|, \\ \text{(ii)} & d_f(R,S) < d_g(R,S) \quad iff \quad |f(R)\cap S| > |g(R)\cap S|, \\ \text{(iii)} & |f(R)\cap S| = |R \cap f^{-1}(S)|. \end{array}$$

Let (P,R), (P,S) be posets and let $f \in B(P)$. If $d_f(R,S) = d(R,S)$, f is said to be an *optimal map* of (P,R) onto (P,S) (cf. Definition in [2]). From Lemma 1 it follows that f is an optimal map if and only if $|f(R)\cap S|$ is maximal. Any isotone map $f \in B(P))$ is optimal (Remark 2 in [2]).

Let (P,R), (P,R') be posets. The poset (P,R') is called a *w-subposet* of the poset (P,R) if there is a map $f \in B(P)$ with $f(R') \subseteq R$.

If a poset (P,R') is a w-subposet of posets (P,R), (P,S) then we will say that (P,R') is a *common w-subposet* of (P,R) and (P,S).

Let $\{(P,R_i);\ i \in I\}$ be a set all common w-subposet of posets (P,R), (P,S). If there is $m \in I$ with $|R_i| \le |R_m|$ for each $i \in I$, then we will say that (P,R_m) is a *maximal common w-subposet* (MCWS) of posets (P,R), (P,S).

Let (P,Q) be a MCWS of posets (P,R), (P,S). The *s-distance* between posets (P,R), (P,S) is the number defined by

$$d^s(R,S) = |R| + |S| - 2|Q|. \tag{4}$$

Lemma 2. *Let (P,R), (P,S) be posets. Then $d(R,S) = d^s(R,S)$.*

Proof. If $f \in B(P)$ is an optimal map of (P,R) onto (P,S) then $d(R,S) = d_f(R,S) = |R| + |S| - 2|f(R)\cap S|$. It suffices to show that the poset $(P, f(R)\cap S)$ is an MCWS of posets (P,R), (P,S).

If $[a,b] \in f(R)\cap S$, then $[f^{-1}(a), f^{-1}(b)] \in R$, $f^{-1} \in B(P)$ and $[\mathrm{id}_P(a), \mathrm{id}_P(b)] \in S$, $id_P \in B(P)$. Thus $(P, f(R)\cap S)$ is a common w-subposet of (P,R), (P,S). Suppose on the contrary that the poset $(P, f(R)\cap S)$ is not an MCWS. Let (P,Q) be an MCWS. Then $|f(R)\cap S| < |Q|$ and there are optimal maps $h,g \in B(P)$ with $h(Q) \subseteq R$ and $g(Q) \subseteq S$. From this we have

$$g(Q) = gh^{-1}h(Q) \subseteq gh^{-1}(R)$$

and so

$$g(Q) \subseteq gh^{-1}(R) \cap S$$

which gives

$$|Q| = |g(Q)| \le |gh^{-1}(R)\cap S|.$$

We thus get

$$|f(R)\cap S| < |gh^{-1}(R)\cap S|.$$

Therefore by (ii)

$$d_f(R,S) > d_{gh^{-1}}(R,S), \quad gh^{-1} \in B(P),$$

a contradiction. $\square$

The next theorem follows from Lemma 2 immediately.

Theorem 3. *Let $\mathcal{F}_n$, $n \in N$, be a system of all (non-isomorphic) posets on a set P of the cardinality n. Then the function d^s on the system $\mathcal{F}_n$ given by (4) is a metric.*

Let (P, R), (P, R') be posets. If a poset (P, R') is a w-subposet of a poset (P, R) then we will say that (P, R) is a *w-owerposet* of a poset (P, R'). If a poset (P, R) is a w-overposet of posets (P, S), (P, Q) then (P, R) will be called a *common w-owerposet* of (P, S) and (P, Q). Let $\{(P, R_i);\ i \in I\}$ be a set all common w-overposet of posets (P, S), (P, Q). If there is $m \in I$ with $|R_m| \leq |R_i|$ for each $i \in I$, then we will say that (P, R_m) is a *minimal common w-owerposet* (mCWO) of posets (P, S), (P, Q).

Let (P, M) be an mCWS of posets (P, R), (P, S). We denote by $d^o(R, S)$ the number defined by

(5) $$d^o(R, S) = 2|M| - |R| - |S|.$$

Proposition 4. *Let (P, R), (P, S) be posets. Then $d(R, S) \leq d^o(R, S)$.*

Proof. Let (P, M) be a mCWO of posets (P, R), (P, S). If $f \in B(P)$ is an optimal map of (P, R) onto (P, S), then

$$\begin{aligned} d(R, S) = d_f(R, S) &= |R| + |S| - 2|f(R) \cap S| = \\ &= |f(R)| + |S| - 2|f(R) \cap S| = |f(R)| + |S| - 2(|f(R)| + |S| - |f(R) \cup S|) = \\ &= 2|f(R) \cup S| - |f(R)| - |S| = 2|f(R) \cup S| - |R| - |S|. \end{aligned}$$

Since $|f(R) \cup S|$ is minimal if and only if $|f(R) \cap S|$ is maximal,

$$d(R, S) = 2|f(R) \cup S| - |R| - |S| \leq 2|M| - |R| - |S| = d^o(R, S).$$

Proposition 5. *Let (P, Q) be an mCWO of posets (P, R), (P, S). If $d(R, S) = d_f(R, S)$ and $|Q| = |f(R) \cup S|$, then there exists a map $f' \in B(P)$ with $d_{f'}(R, S) = d(R, S)$ and $(P, f'(R) \cup S)$ is a poset isomorphic to a poset (P, Q).*

Proof. Since (P, Q) is an mCWO of posets (P, R), (P, S) there are isotone maps $g, h \in B(P)$ with $g(R) \subseteq Q$, $h(S) \subseteq Q$ and so

$$d_g(R, Q) = d(R, Q) = |Q| - |R|, \qquad d_h(S, Q) = d(S, Q) = |Q| - |S|.$$

From (iii) we have $d(Q, S) = d_{h^{-1}}(Q, S) = |Q| - |S|$. By assumption $|Q| = |f(R) \cup S|$ we obtain

$$\begin{aligned} d(R, Q) + d(S, Q) =& d_g(R, Q) + d_{h^{-1}}(Q, S) = 2|Q| - |R| - |S| = \\ =& 2|f(R) \cup S| - |R| - |S| = 2(|f(R)| + |S| - |f(R) \cap S| - |R| - |S|) = \\ =& 2(|R| + |S| - |f(R) \cap S| - |R| - |S|) = |R| + |S| - 2|f(R) \cap S| = \\ =& d_f(R, S) = d(R, S). \end{aligned}$$

As in the proof of Theorem 1 in [2] we obtain

$$d_{h^{-1}g}(R,S) \leq d_g(R,Q) + d_{h^{-1}}(Q,S) = d(R,S).$$

Thus

$$d_{h^{-1}g}(R,S) = d(R,S)$$

and so $h^{-1}g \in B(P)$ is an optimal map of the poset (P,R) onto the poset (P,S).

It remains to prove that the relation structure $(P, h^{-1}g(R) \cup S)$ is isomorphic to the poset (P,Q). For all $[x,y] \in h^{-1}g(R) \cup S$ we put $\psi([x,y]) = [h(x), h(y)]$.

a) If $[x,y] \in S$, then $\psi([x,y]) = [h(x), h(y)] \in Q$, since h is an isotone map of (P,S) onto (P,Q).

b) If $[x,y] \in h^{-1}g(R) \setminus S$, then there is $[a,b] \in R$ with $[x,y] = [h^{-1}g(a), h^{-1}g(b)]$ and thus $\psi([x,y]) = [h(x), h(y)] = [hh^{-1}g(a), hh^{-1}g(b)] = [g(a), g(b)] \in Q$, since g is an isotone map of (P,R) onto (P,Q).

By the above, ψ is a map of $h^{-1}g(R) \cup S$ to Q. It is obvious that the map ψ is injective. Since $|h^{-1}g(R) \cup S| = |Q|$, the map ψ is bijective. Thus ψ^{-1} is a bijective map of Q onto $h^{-1}g(R) \cup S$ and for $[u,v] \in Q$,

$$\psi^{-1}([u,v]) = [h^{-1}(u), h^{-1}(v)] \in h^{-1}g(R) \cup S.$$

From this it follows that the relation structure $(P, h^{-1}g(R) \cup S)$ is a poset isomorphic to the poset (P,Q). $\square$

Example 6. Let (P,R), (P,S), (P,T) be posets with $|R| = |S| = 16$, $|T| = 15$ given in Figure.

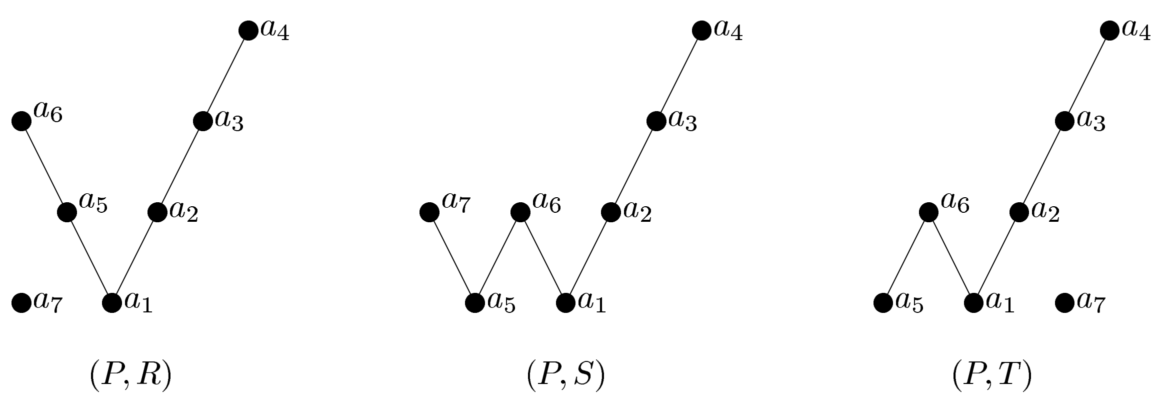

Fig.

The map $\mathrm{id}_P \in B(P)$ is the only optimal map of the poset (P,R) onto the poset (P,S).

The union $\mathrm{id}_P(R) \cup S$ is not a partial ordering, since $[a_1, a_5], [a_5, a_7] \in \mathrm{id}_P(R) \cup S$ but $[a_1, a_7] \notin \mathrm{id}_P(R) \cup S$. Let (P, Q) be an mCWO of the posets $(P, R), (P, S)$. By Proposition 5, from $|\,\mathrm{id}_P(R) \cup S| = 17$ it follows $|Q| \geq 18$. Thus $d^o(R, S) = 2|Q| - |R| - |S| \geq 4$. The poset $(P, \mathrm{id}_P(R) \cup S \cup \{[a_1, a_7]\})$ with $|\,\mathrm{id}_P(R) \cup S \cup \{[a_1, a_7]\}| = 18$ is a mCWO of the posets $(P, R), (P, S)$ and so $d^o(R, S) = 4$.

Since the poset (P, T) is a w-subposet of the posets $(P, R), (P, S)$, we have $d^o(R, T) = 1$, $d^o(T, S) = 1$. From this it follows that

$$2 = d^o(R, T) + d^o(T, S) < d^o(R, S) = 4.$$

Therefore d^o is not a metric.

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Dept. of Mathematics, Matej Bel University,

Zvolenská 6, 974 01 Banská Bystrica, SLOVAKIA

E-mail adress: klenovca@pdf.umb.sk

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A NOTE ON THE DISTANCE POSET OF POSETS

Judita Lihová

Abstract. Let F_ω^* be the system of all non–isomorphic finite orders of a countable set P, ordered in such a way that $R \leq S$ if $f(R) \subseteq S$ for a bijective map $f : P \to P$. There are investigated some properties of $(F_\omega^*, \leq)$.

In [3] a metric d on the system F_n of isomorphism classes of ordered sets of the same finite cardinality n has been introduced. In [4] there is shown that this metric coincides with the distance–metric on the covering graph of F_n. The system F_n can be partially ordered. By the help of the above mentioned metric the author proves in [4] that the ordered system F_n is graded, i.e. all maximal chains with the same endpoints have the same length. The ordered system F_n, as each finite partially ordered set, is a multilattice. A natural question arises. Is F_n a metric multilattice with respect to d, in the sense of [5]?

In this note there is proved that F_n is not a metric multilattice with respect to any metric by showing that F_n is not a modular multilattice. In the second part some properties of the ordered system of all finite orders of the same infinite set P are mentioned.

0. Basic notions

A partially ordered set $(M, \leq)$ is said to be a multilattice if, whenever $a, b \in M, u \in M, u \geq a, u \geq b$, there exists a minimal upper bound u' of $\{a, b\}$ with $u' \leq u$, and dually. If, moreover, $(M, \leq)$ is a directed set, then $(M, \leq)$ is called a directed multilattice.

Let $a \vee b$ $(a \wedge b)$ denote the set of all minimal upper bounds of $\{a, b\}$ (maximal lower bounds of $\{a, b\}$). A multilattice $(M, \leq)$ is distributive if

$$a, b, c \in M, (a \wedge b) \cap (a \wedge c) \neq \emptyset, (a \vee b) \cap (a \vee c) \neq \emptyset \Rightarrow b = c,$$

and modular if

$$a, b, c \in M, b \leq c, (a \wedge b) \cap (a \wedge c) \neq \emptyset, (a \vee b) \cap (a \vee c) \neq \emptyset \Rightarrow b = c.$$

(For the above definitions see [1].)

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By a metric multilattice is meant a multilattice with a metric d fulfilling the following conditions (cf. [5]):

M1. $a \le b \le c$ implies $d(a,b) + d(b,c) = d(a,c)$,

M2. if $u \in a \wedge b, v \in a \vee b$, then $d(a,b) = d(u,v)$.

In [5] there is proved:

0.1. Theorem. *A metric multilattice is modular.*

0.2. Theorem. *A directed modular multilattice of locally finite length is a metric multilattice.*

1. Properties of F_n

Let $F_n (n \in N)$ be the set of all (non–isomorphic) orders of a set P of cardinality n. Set $R \le S$ $(R, S \in F_n)$ if there exists a permutation f of P satisfying $f(R) \subseteq S$ (the symbol $f(R)$ denotes the set $\{[f(a), f(b)] : [a,b] \in R\}$). In other words, $R \le S$ means that there exists an isotone bijection of (P, R) onto (P, S). The poset $(F_n, \le)$ is called the distance poset (of orders of an n–element set) (cf. [4]).

The following theorem is proved in [4].

1.1. Theorem. *The distance poset $(F_n, \le)$ is a graded poset with the least element and the greatest element.*

The diagrams of $(F_3, \le)$ and $(F_4, \le)$ are depicted in Fig. 1 and Fig. 2, respectively. Evidently $(F_1, \le)$ is a one element set and $(F_2, \le)$ is a two element chain. The least element of $(F_n, \le)$ is the discrete order, i.e. the order in which only comparable elements are the couples of equal elements and the greatest element is the linear order. Let us remark that $(F_n, \le)$, as a finite bounded partially ordered set, is a directed multilattice.

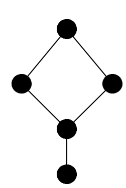

Fig. 1

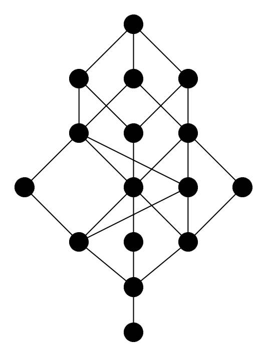

Fig. 2

If $R, S \in F_n, R \le S$ and S covers R, we will write $R \prec S$. The following lemma proved in [4] will be useful.

1.2. Lemma. *Let $R, S \in F_n, R \leq S$, f be a permutation of P satisfying $f(R) \subseteq S$. Then $R \prec S$ if and only if $S - f(R) = \{[a, b]\}$, where $a \prec_S b$.*

It is easy to see that $(F_1, \leq), (F_2, \leq)$ and $(F_3, \leq)$ are distributive lattices. In contrast with this, there holds:

1.3. Lemma. *If $n \geq 4$, then $(F_n, \leq)$ is not a lattice.*

Proof. Let $R, S \in F_n$ be as in Fig. 3 and Fig. 4, respectively. Using 1.2 it is easy to see that U shown in Fig. 5 and its dual U^δ are covered by R, S and V in Fig. 6 and its dual V^δ cover both R and S. Hence U and U^δ are maximal lower bounds of $\{R, S\}$ and V, V^δ are minimal upper bounds of $\{R, S\}$.

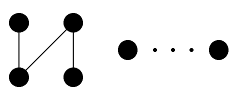

Fig. 3

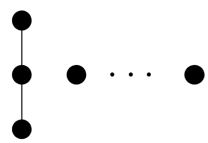

Fig. 4

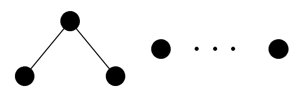

Fig. 5

As we have remarked, $(F_n, \leq)$ is a multilattice, hence for any $R, S \in F_n$ and any $U, V \in F_n$ satisfying $U \leq R$, $S \leq V$ there exists a maximal lower bound U' of $\{R, S\}$ and a minimal upper bound V' of $\{R, S\}$ with $U \leq U'$ and $V' \leq V$. In 1.4 and 1.6 there is described the set of all maximal lower bounds of $\{R, S\}$ and the set of all minimal upper bounds of $\{R, S\}$, respectively.

1.4. Lemma. *Let $R, S, U \in F_n, U \leq R, U \leq S$. Then $U \in R \wedge S$ if and only if for each couple of permutations f, g of P with $f(U) \subseteq R, g(U) \subseteq S$ there is $f^{-1}(R) \cap g^{-1}(S) = U$.*

Proof. Clearly for any permutation h of P and any order T of P, $h^{-1}(T)$ is an order of P and further the intersection of two orders of P is an order of P less than or equal to each of them. So if $U \leq R, U \leq S$, then for each couple of permutations f, g of P with $f(U) \subseteq R, g(U) \subseteq S$ there is $U \subseteq f^{-1}(R) \cap g^{-1}(S)$, $f^{-1}(R) \cap g^{-1}(S) \leq R$, $f^{-1}(R) \cap g^{-1}(S) \leq S$. Now if U is a maximal lower bound of $\{R, S\}$, then $U = f^{-1}(R) \cap g^{-1}(S)$. Conversely, if $U < U' \leq R, S$ and h, f_1, g_1 are permutations of P such that $h(U) \subset U', f_1(U') \subseteq R, g_1(U') \subseteq S$, then $f_1(h(U)) \subseteq R, g_1(h(U)) \subseteq S, h^{-1}(f_1^{-1}(R)) \cap h^{-1}(g_1^{-1}(S)) = h^{-1}(f_1^{-1}(R) \cap g_1^{-1}(S)) \supseteq h^{-1}(U') \supset U$.

Let us remark that it can happen that $f_1^{-1}(R) \cap g_1^{-1}(S) = U$ for some permutations f_1, g_1 of P and at the same time $f_2^{-1}(R) \cap g_2^{-1}(S) \supset U$ for some other permutations f_2, g_2 of P, as the following example shows.

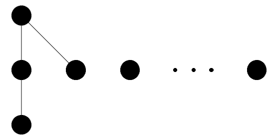

Fig. 6

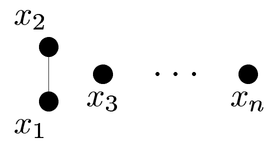

Fig. 7

1.5. Example. Let U, R, S be as in Fig. 7, Fig. 8 and Fig. 9, respectively. Define i to be the identity map on $P = \{x_1, \ldots, x_n\}, g = (x_3x_4)$. Then $i^{-1}(R) \cap i^{-1}(S) = U$ while $i^{-1}(R) \cap g^{-1}(S) \supset U$.

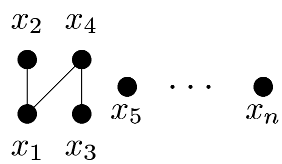

Fig. 8

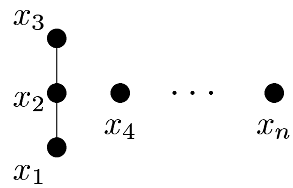

Fig. 9

Analogously can be proved:

1.6. Lemma. *Let $R, S, V \in F_n, R \leq V, S \leq V$. Then $V \in R \vee S$ if and only if for each couple of permutations f, g of P with $f(R) \subseteq V, g(S) \subseteq V$, V is the transitive cover of $f(R) \cup g(S)$.*

Considering the same R, S as in 1.5 and V as in Fig. 10, $V = i(R) \cup g_1(S)$, but V properly contains the transitive cover of $i(R) \cup g_2(S)$ for $g_1 = (x_1x_3x_4), g_2 = (x_3x_4)$.

Now we are going to investigate $(F_n, \leq)$ for $n \geq 4$ from the view of its distributivity and modularity. Obviously $(F_4, \leq)$ it is not distributive and a straightforward testing yields that $(F_4, \leq)$ is modular.

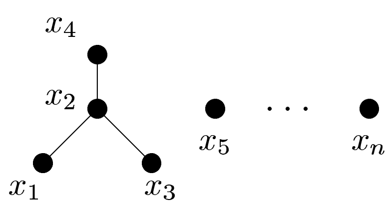

Fig. 10

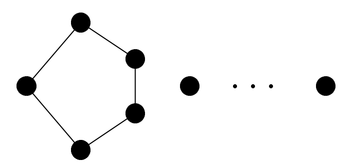

Fig. 11

1.7. Theorem. *If $n \geq 5$, then the multilattice $(F_n, \leq)$ is not modular.*

Proof. Let R, S, T, U, V be as in Fig. 11, 12, 13, 14 and 15, respectively. Then $U < S \prec V, U \prec R \prec T \prec V$, by 1.2, and $V \in S \vee R$, because $S \ngeq R$. Let us suppose that there exists $U' \in F_n$ satisfying $U' > U, U' \leq S, T$. Since S, T are incomparable orders, using 1.1 we obtain that U' must be covered by S and T. If we find all orders covered by S, using

1.2, we see that the order in Fig. 16 is the only one covered also by T, but it is not greater than U. We have a contradiction.

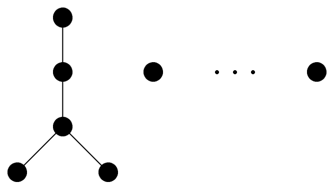

Fig. 12

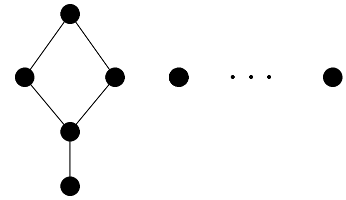

Fig. 13

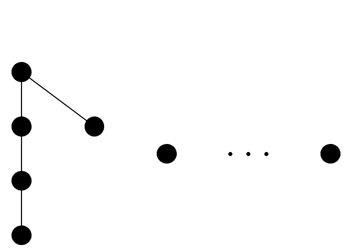

Fig. 14

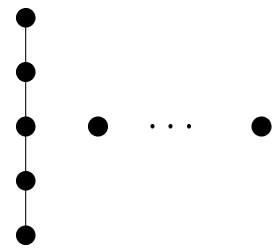

Fig. 15

Using 0.1 and 0.2 we obtain:

1.8. Corollary. *If $n \geq 5$, then the multilattice $(F_n, \leq)$ is not a metric multilattice. $(F_4, \leq)$ is a metric multilattice, $(F_1, \leq), (F_2, \leq)$ and $(F_3, \leq)$ are metric lattices.*

Note that if $n \leq 4$, the metric d introduced in [3] (for the definition see below) fulfils the conditions M1 and M2.

2. Distance poset F^*_ω

In this section P will be any countable set (in fact, it could be of any infinite cardinality). An order R of P will be said to be finite if R contains only finitely many couples of distinct elements. Let F^*_ω denote the system of all (non–isomorphic) finite orders of P. It can be partially ordered by

$$R \leq S \text{ if there exists a bijective map } f : P \to P \text{ with } f(R) \subseteq S.$$

Evidently $(F^*_\omega, \leq)$ has the least element (the discrete order of P), but it contains no maximal elements, so it is of infinite length.

Denote by F'_n the set of all orders $R \in F^*_\omega$ with the property that there exists an n–element subset P' of P satisfying

$$[a, b] \in R, a \neq b \Rightarrow \{a, b\} \subseteq P'.$$

2.1. Theorem. *For each $n \in N(F_n, \leq)$ is isomorphic to $(F'_n, \leq)$. F'_n is an interval of F^*_ω with the discrete order of P as the least element. Further $F'_1 \subset F'_2 \subset F'_3 \subset \dots$ and $F^*_\omega = \cup_{n \in N} F'_n$.*

This statement is evident.

The preceding theorem yields immediately that $(F^*_\omega, \leq)$ is of locally finite length and graded. So $(F^*_\omega, \leq)$ is a directed multilattice. If $R, S \in F^*_\omega$, let us denote by $R \vee_\omega S (R \wedge_\omega S)$, $R \vee_n S (R \wedge_n S)$ the set of all minimal upper bounds (maximal lower bounds) of $\{R, S\}$ in $(F^*_\omega, \leq)$ and $(F'_n, \leq)$, respectively. It is easy to verify:

2.2. Theorem. *Let $R, S \in F^*_\omega$ and let n_0 be the least positive integer such that both R and S belong to F'_{n_0}. Then $R \wedge_\omega S = R \wedge_{n_0} S, R \vee_\omega S = \cup_{n \geq n_0} R \vee_n S$.*

One can see that for any $R, S \in F^*_\omega$ the set $R \wedge_\omega S$ is finite. Since some $R, S \in F^*_\omega$ can have minimal upper bounds in various F'_n (cf. the following example), it is not quite evident that the same holds for the set $R \vee_\omega S$.

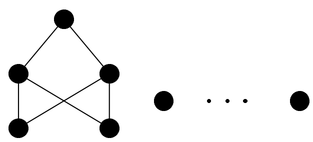

Fig. 16

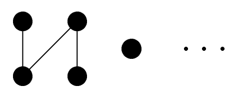

Fig. 17

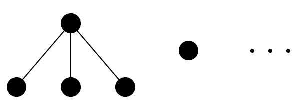

Fig. 18

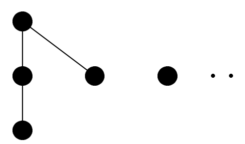

Fig. 19

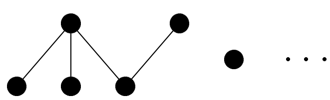

Fig. 20

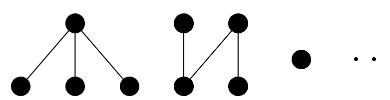

Fig. 21

2.3. Example. Let R, S be as in Fig. 17 and Fig. 18, respectively. Then each of Figures 19, 20, 21 represents a minimal upper bound of $\{R, S\}$.

2.4. Theorem. *For any $R, S \in F^*_\omega$ the set $R \vee_\omega S$ is finite.*

Proof. Let $R, S \in F^*_\omega$, $V \in R \vee_\omega S$. We are going to show that V contains at most card $R+$ card S couples of elements a, b with $a \prec_V b$. Suppose that this is not the case. Let f, g be bijective maps $P \to P$ satisfying $f(R) \subseteq V, g(S) \subseteq V$. Then there exist a, b with

$a \prec_V b$ such that $[a,b] \notin f(R) \cup g(S)$. Then in view of 1.2 $V - \{[a,b]\}$ is an order covered by V. Evidently $f(R) \subseteq V - \{[a,b]\}$, $g(S) \subseteq V - \{[a,b]\}$, so $V - \{[a,b]\}$ is an upper bound of $\{R, S\}$ less than V, a contradiction.

Using 1.7 and 2.2 we obtain:

2.5. Theorem. *The multilattice $(F_\omega^*, \leq)$ is not modular.*

In view of 0.1 we have:

2.6. Corollary. *The multilattice $(F_\omega^*, \leq)$ is not a metric multilattice.*

Nevertheless, there can be introduced a metric into F_ω^*, but not satisfying both M1 and M2. Namely, the metric d on the system F_n of all non–isomorphic orders of an n–element set P_n, defined in [3] by

$$d(R,S) = \min \{d_f(R,S) : f \text{ is a permutation of } P_n\},$$

where $d_f(R,S) = \operatorname{card}(f(R) - S) + \operatorname{card}(S - f(R))$, evidently yields a metric on F_ω^*, too.

In [4] there is proved that if $R, S \in F_n$, $d(R,S) = \delta(R,S)$, where $\delta(R,S)$ is the distance of vertices R, S of the covering graph of F_n (Th. 2.2). Further by 2.1 of [2] $\delta(R,S) = h(R) - h(S)$ (h denotes the height) provided that $S \leq R$, thanks to the fact that $(F_n, \leq)$ is a graded poset.

So we have:

2.7. Theorem. *An order $R \in F_\omega^*$ has the height k in the partially ordered set $(F_\omega^*, \leq)$ if and only if card $\{[a,b] \in R : a \neq b\} = k$.*

Proof. Evidently the height of R in $(F_\omega^*, \leq)$ is the same as in F_n', if $R \in F_n'$. Therefore $h(R) = k$ if and only if $d(R,D) = k$ with D being the discrete order. But obviously $d(R,D) = \operatorname{card}\{[a,b] \in R : a \neq b\}$.

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Department of Geometry and Algebra, University of P. J. Šafárik, Jesenná 5, 041 54 Košice, Slovakia

E-mail address: lihova@duro.upjs.sk

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CONSISTENT ORTHOGONAL ATOMIC PARTITIONS

Peter Maličký

Abstract. The present paper defines consistent orthogonal atomic partitions of two elements of an orthollatice. Then this notion is studied in the realm of ortholattices associated with a vector space with a scalar product over an ordered field.

This paper deals with ortholattices in which every element is a union of finitely many pairwise orthogonal atoms. Let us recall the notion of an ortholattice - [1], p. 75.

Definition 1. Let L be a lattice with element 0 and 1 and a mapping $\perp: L \to L, a \mapsto$ $\mapsto a^\perp$ such that

$$\begin{array}{ll} 0 \le a \le 1 & \text{for any } a \in L \\ a^{\perp\perp} = a & \text{for any } a \in L \\ (a \wedge b)^\perp = a^\perp \vee b^\perp,\ (a \vee b)^\perp = a^\perp \wedge b^\perp & \text{for any } a, b \in L \\ a \wedge a^\perp = 0,\ a \vee a^\perp = 1 & \text{for any } a \in L\ . \end{array}$$

Then L is said to be an ortholattice. We define $a \perp b$ if and only if $a \le b^\perp$, we say that a and b are orthogonal in this case.

Definition 2. Let L be an ortholattice. A sequence $a_1, \dots, a_k$ of pairwise orthogonal atoms of L is said to be an orthogonal atomic partition of an element $u \in L$, if $u = a_1 \vee \dots \vee a_k$.
If v is another element of L with an orthogonal atomic partition $b_1, \dots, b_m$, then these two orthogonal atomic partitions are said to be consistent if $a_i \perp b_j$ for $1 \le i \le k$, $1 \le j \le m$ and $i \ne j$. (Relation $a_i \perp b_i$ is not required.)

The definition of consistent orthogonal atomic partitions is motivated by Theorem 1.

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Example 1. Let F be an ordered field and X be a finite dimensional vector space over F with a scalar product $\cdot$, i.e. symmetric bilinear positively defined form with values in F. The case $F = \mathbb{R}$ is well known, see [2], pp. 432 – 465. Many results and basic notions may be easily extended to the general case. We need mainly Gram-Schmidt ortogonalization, i.e. the following procedure. Let $u_i, \ldots, u_m$ be linearly independent vectors of X. Define vectors $a_1, \ldots, a_m$ by induction

$$a_1 = u_1 \quad \text{and} \quad a_k = u_k - \sum_{i=1}^{k-1} \frac{(u_k \cdot a_i)}{(a_i \cdot a_i)} a_i \quad \text{for} \quad 1 < k \leq m \ .$$

Then we obtain an orthogonal system of vectors which generate the same subspace as $u_1, \ldots, u_m$. So, any subspace of X has an orthogonal basis. When any nonnegative element of the field F has a square root, then any orthogonal system may be orthonormalized.

Let $L(X)$ be the lattice of all subspaces of X and put $U^\perp = \{v \in X : \ u \cdot v = 0$ for all $u \in U\}$ for a subspace U of X.
Then we obtain an ortholattice in which any element has an orthogonal atomic partition, because any subspace of X has an orthogonal basis. Example 2 shows that consistent orthogonal atomic partitions of two subspaces need not exist. On the other hand Theorem 2 gives a sufficient condition under which consistent orthogonal partitions exist.

Definition 3. Let F be an ordered field and X be a finite dimensional vector space over F with a scalar product $\cdot$. A linear operator $A : X \to X$ is said to be selfadjoint if $(Ax) \cdot y = x \cdot Ay$ for all $x, y \in X$.
Let U be a subspace and $P : X \to X$ be a selfadjoint linear operator such that $P^2 = P$ and $P(X) = U$, then P is said to be the orthogonal projection onto U.

Proposition 1. *Let F be an ordered field, X be a finite dimensional vector space over F with a scalar product and $a_1, \ldots, a_k$ be an orthogonal basis of a subspace U of X. Then the formula $Px = \sum_{i=1}^{k_1} \frac{(x \cdot a_i)}{(a_i \cdot a_i)} a_i$ defines uniquely the orthogonal projection onto U.*

Theorem 1. *Let X be a finite dimensional real vector space with a scalar product and U and V be subspaces of X with $\dim(U) = k_1$ and $\dim(V) = k_2$. Then there are bases $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ of U and V respectively such that*

(1) $\quad ||a_i|| = 1 = ||b_j|| \quad$ *for* $1 \leq i \leq k_1$ *and* $j \leq k_2$

(2) $\quad a_i \cdot a_j = 0 \quad$ *for* $1 \leq i < j \leq k_1$

(3) $\quad b_i \cdot b_j = 0 \quad$ *for* $1 \leq i < j \leq k_2$

(4) $\quad a_i \cdot b_j = 0 \quad$ *for* $i \neq j$, $1 \leq i \leq k_1$ *and* $1 \leq j \leq k_2$

(5) $\quad a_i \cdot b_i \geq 0 \quad$ *for* $1 \leq i \leq \min(k_1, k_2)$

The following Lemma shows the properties which the bases $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ must have.

Lemma 1. *Let F be an ordered field, X be a finite dimensional vector space over F with a scalar product, U and V be subspaces of X with bases $a_1, \ldots, a_{k_1}$ and $b_1, \ldots b_{k_2}$, which satisfy relations (2) - (4). Then all a_i are eigenvectors for PQ and all b_j are eigenvectors for QP, where P and Q denote orthogonal projections onto U and V respectively.*

Proof. Since $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ form orthogonal bases of U and V then $Px = = \sum_{i=1}^{k_1} \frac{(x \cdot a_i)}{(a_i \cdot a_i)} a_i$ and $Qx = \sum_{i=1}^{k_2} \frac{(x \cdot b_i)}{(b_i \cdot b_i)} b_i$ for all $x \in X$ by Proposition 1.
We may assume $k_1 \leq k_2$. For $1 \leq i \leq k_1$ relation (4) implies

$$Pb_i = \frac{(b_i \cdot a_i)}{(a_i \cdot a_i)} a_i \quad \text{and} \quad Qa_i = \frac{(a_i \cdot b_i)}{(b_i \cdot b_i)} b_i \ .$$

Therefore

$$PQa_i = \frac{(a_i \cdot b_i)^2}{(a_i \cdot a_i)(b_i \cdot b_i)} a_i \quad \text{and} \quad QPb_i = \frac{(a_i \cdot b_i)^2}{(a_i \cdot a_i)(b_i \cdot b_i)} b_i \ ,$$

which means that a_i and b_j are eigenvectors for PQ and QP. If $k_1 < i \leq k_2$ then $Pb_i = 0$ and $QPb_i = 0$ and b_i is an eigenvector for QP.

Proof of Theorem 1. Let P and Q be the orthogonal projections onto U and V respectively. Let $u_1 \in U$ and $u_2 \in U$. Then $(PQu_1) \cdot u_2 = (Qu_1) \cdot (Pu_2) = Qu_1 \cdot u_2 = = u_1 \cdot Qu_2 = Pu_1 \cdot Qu_2 = u_1 \cdot (PQu_2)$, which means that the restriction of PQ onto U is a selfadjoint linear operator. There is an orthonormal basis $a_1, \ldots, a_k$ of U such that $PQa_i = \lambda_i a_i$ for some $\lambda_i \in \mathbb{R}$, see [2], p. 461. Now, consider the elements Qa_i. Let $i \neq j$ and $i, j \in \{1, \ldots, k_1\}$. Then $a_i \cdot Qa_j = Pa_i \cdot Qa_j = a_i \cdot PQa_j = a_i \cdot (\lambda_j a_j) = = \lambda_j (a_i \cdot a_j) = 0$ and

$$(Qa_i \cdot Qa_j) = (a_i \cdot Q^2 a_j) = (a_i \cdot Qa_j) = 0 \ .$$

Put $k_3 = \dim(Q(U))$. Obviously $k_3 \leq \min(k_1, k_2)$. We may assume that $Qa_i \neq 0$ for $1 \leq i \leq k_3$ and $Qa_i = 0$ for $k_3 < i \leq k_1$. Put $b_j = \frac{Qa_j}{||Qa_j||}$ for $1 \leq j \leq k_3$. If $k_3 = k_2$, the proof is complete. If $k_3 < k_2$, take an orthonormal basis $c_1, \ldots, c_{k_2 - k_3}$ of $V \cap Q(U)^\perp$ and put $b_{k_3 + j} = c_j$ for $1 \leq j \leq k_3 - k_2$. It is sufficient to verify $a_i \cdot c_j = 0$ for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2 - k_3$. We have $a_i \cdot c_j = a_i \cdot (Qc_j) = (Qa_i) \cdot c_j = 0$.

Example 2. Let X be $\mathbb{Q}^4$ with the standard scalar product, $u_1 = (1,0,0,0)$, $u_2 = = (0,1,0,0)$, $v_1 = (1,1,1,1)$, $v_2 = (1,-2,-2,3)$, U and V be linear spans of u_1, u_2 and v_1, v_2 respectively. Note that $u_1 \perp u_2$ and $v_1 \perp u_2$. Using Proposition 1, it is easy to see that $PQu_1 = \frac{1}{36}(11u_1 + 5u_2)$ and $PQu_2 = \frac{1}{36}(5u_1 + 17u_2)$. It means that the restriction of PQ onto U has the matrix $\frac{1}{36}\left(\begin{smallmatrix} 11 & 5 \\ 5 & 17 \end{smallmatrix}\right)$ with respect to the basis $\{u_1, u_2\}$. The eigenvalues of this matrix are $\frac{1}{36}(14 \pm \sqrt{34})$, which are irrational and PQ has no eigenvectors in U. By Lemma 1. consistent orthogonal atomic partitions of U and V do not exist.

Example 2 and Lemma 1 indicate that the existence of consistent atomic partitions for subspaces U and V is connected with the solvability of algebraic equations over the field F. In fact, in Theorem 1 is essential only the fact, that the field or real numbers is a maximal ordered field ([3], pp. 276 – 282), which means that every polynomial is a product of linear polynomials and quadratic polynomials with the negative discriminant. The following theorems shows that the condition of maximality of the ordered field F may be weakened according to $\dim(X)$.

Theorem 2. *Let X be a n-dimensional vector space with a scalar product over an ordered field F with the property, that any polynomial of the degree $\leq [\frac{n}{2}]$ is a product of linear polynomials and quadratic polynomials with the negative discriminant. Then for any vector subspaces U and V of X there are bases $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ satisfying relations (2) - (5). Relation (1) may be satisfied whenever every nonnegative element of F has a square root, which is automatically satisfied for $n \geq 4$.*

Proof. The case $n \leq 3$ may be easily studied. So, assume $n \geq 4$. In this case any nonnegative $\alpha \in F$ must have a square root, because $g(\lambda) = \lambda^2 - \alpha$ is a polynomial of degree $\leq [\frac{n}{2}]$ with the nonnegative discriminant and it must be reducible. We may assume $\dim(U) \leq \dim(V)$. We have

$$n \geq \dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V) \geq [\dim(U) - \dim(U \cap V)] + \\ + [\dim(V) - \dim(U \cap V)] \geq 2[\dim(U) - \dim(U \cap V)] \ ,$$

which implies $[\dim(U) - \dim(U \cap V)] \leq [\frac{n}{2}]$.

Analogically to Theorem 1. the restriction of PQ onto U is a selfadjoint linear operator. Denote this restriction by A. Let $f(\lambda)$ be the characteristic polynomial of A. For $x \in U \cap V$ we have $Ax = PQx = x$. It means that $f(\lambda)$ is divisible by $(\lambda - 1)^m$, where $m = \dim(U \cap V)$. Therefore $f(\lambda) = (\lambda - 1)^m g(\lambda)$, where $\deg(g) = \deg(f) - m = = \dim(U) - \dim(U \cap V) \leq [\frac{n}{2}]$. Since $\deg(g) \leq [\frac{n}{2}]$, the polynomial g is a product of linear and quadratic polynomials with the negative discriminant. Let K be the complexification of F. Then $g(\lambda) = \prod_{i=1}^{\deg g} (\lambda - \lambda_i)$, where $\lambda_i \in K$. Every λ_i is an eigenvalue of the selfadjoint operator A and in fact $\lambda_i \in F$. (The proof of this fact is fully analogical to the case $F = \mathbb{R}$. For this case see [2], p. 460.) Now, analogically to the real case ([2], p. 461) using induction it may be constructed an orthonormal basis $a_1, \ldots, a_{k_1}$ of U consisting of eigenvectors of A. The proof may be finished in the same way as the proof of Theorem 1.

Let K be either the complexification of an ordered field F or the quaternion algebra over F. All results of this paper may be reformulated for the case, when F is replaced by K. The axioms for the scalar product are the following

$$\begin{aligned}
&x \cdot y \in K \text{ for all } \ x, y \in X \\
&y \cdot x = (x \cdot y)^* \\
&(\alpha x) \cdot y = \alpha (x \cdot y) \\
&(x_1 + x_2) \cdot y = x_1 \cdot y + x_2 \cdot y \\
&x \cdot x > 0 \text{ for } x \neq 0 \ .
\end{aligned}$$

(If $x \cdot y = \alpha + \beta i$, then $(x \cdot y)^* = \alpha - \beta i$. If $x \cdot y = \alpha + \beta i + \gamma j + \delta k$, then $(x \cdot y)^* = \alpha - \beta i - \gamma j - \delta k$.)

In the reformulation of Theorem 2 F has the same property and X is a vector space over K.

It would be interesting to characterize the class of ortholattices in which any two elements have consistent orthogonal atomic partitions. By the results of the present paper this problem is connected with eigenvalues when it is considered in the realm of ortholattices associated with a vector space with a scalar product over an ordered field. So, in a general case this problem may be difficult. Therefore any partial result will be interesting.

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Dept. of Mathematics, Faculty of Sciences, Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

E-mail address: malicky@fhpv.umb.sk

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STRONGLY IRREDUCIBLE STRINGS

Bohuslav Sivák

Abstract. The "strong irreducibility" of strings is defined and it is proved that certain special strings of 0's and 1's are strongly irreducible. This fact has found an application (see [1]) in the study of discrete dynamic systems by the methods of the symbolic dynamics.

1. Introduction

Let T be a finite set of symbols. The string over T is a finite sequence of symbols from the set T. The length of the string w is the number of symbols in the string w, this number will be denoted by $|w|$. For example, $|abac| = 4$. The empty string will be denoted by ε. Trivially, $|\varepsilon| = 0$. The concatenation of the strings u and v will be denoted by $u \cdot v$ or by uv. For example, the concatenation of the strings $u = 001$, $v = 10$ is the string $uv = 00110$. The concatenation of several identical strings will be written in the form of the formal power: $u^0 = \varepsilon$, $u^1 = u$, $u^2 = uu$, etc.

Definition. Let B be a string over T. The string B is called reducible iff it can be written in the form

$$B = W^k, \qquad k \geq 2.$$

The string B is called irreducible iff it is not reducible. The string B is called strongly irreducible iff the following two conditions are satisfied:

(1) - the string B is irreducible,
(2) - the string $A^m B$ is irreducible for every $m \geq 2$ and every irreducible string $A \neq B$.

Examples. Put $T = \{0, 1\}$. Following strings over T are reducible:

$$\varepsilon,\ 00,\ 11,\ 000,\ 111,\ 0000,\ 0101,\ 1010,\ 1111.$$

Following strings are irreducible:

$$0,\ 1,\ 01,\ 10,\ 001,\ 010,\ 011,\ 100,\ 101,\ 0010.$$

It is easy to see that the string 01 is strongly irreducible. The string 1001 is irreducible but not strongly irreducible. In fact,

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$$(010)^2 \cdot 1001 = (01001)^2.$$

Similarly, the string 10101011 is irreducible but not strongly irreducible. In fact,

$$(01)^3 \cdot 10101011 = (0101011)^2.$$

This example can be generalized: for every integer $m \geq 2$, the string $(10)^m \cdot 11$ is irreducible but not strongly irreducible.

Definition. Let U be a string of the length $\geq k$. The string $Pref_k(U)$ is the prefix of the length k. Similarly, the string $Postf_k(U)$ is the postfix of the length k.

Examples. The string 011010 has the following prefixes:

$$\varepsilon,\ 0,\ 01,\ 011,\ 0110,\ 01101,\ 011010.$$

The same string has the following postfixes:

$$\varepsilon,\ 0,\ 10,\ 010,\ 1010,\ 11010,\ 011010.$$

We leave to the reader the verification that the prefixes and the postfixes of the string 1111 are idenical.

Lemma 1.1. *Let D, E be arbitrary strings and let x, y be positive integers such that $D^x = E^y$. Then there exist positive integers i, j and a string Z such that*

$$D = Z^i,\ E = Z^j.$$

Remark. A common generalization of our Lemma 1.1, Lemma 1.2 and Lemma 1.3 is proved in [2].

Proof of Lemma 1.1. We can assume that the strings D, E are nonempty. Let $c = (x, y)$ be the greatest common divisor of the integers x, y. Then

$$x = c \cdot j,\ y = c \cdot i,\ (i, j) = 1.$$

By the assumption of the lemma,

$$D^{c \cdot j} = E^{c \cdot i},$$
$$c \cdot j \cdot |D| = c \cdot i \cdot |E|,$$
$$j \cdot |D| = i \cdot |E|.$$

By the last equality, there exists a positive integer h such that

$$|D| = h \cdot i, \quad |E| = h \cdot j.$$

The string $D^j = E^i$ can be uniquely written in the form

$$D^j = E^i = Z_1 \cdot Z_2 \ldots Z_{i \cdot j},$$
$$|Z_1| = |Z_2| = \cdots = |Z_{i \cdot j}| = h.$$

By these equalities,

$$D = Z_1 \ldots Z_i,$$
$$D = Z_{i+1} \ldots Z_{2 \cdot i},$$
$$\ldots\ldots\ldots$$
$$D = Z_{i \cdot (j-1)+1} \ldots Z_{i \cdot j},$$

Consequently, $Z_p = Z_q$ whenever the difference of the indexes p, q is a multiple of i. Similarly,

$$E = Z_1 \dots Z_j,$$
$$E = Z_{j+1} \dots Z_{2\cdot j},$$
$$\dots\dots\dots$$
$$E = Z_{(i-1)\cdot j+1} \dots Z_{i\cdot j}$$

and $Z_p = Z_q$ whenever the difference of the indexes p, q is a multiple of j. We know that $(i, j) = 1$. It follows that all of the strings $Z_1,\ Z_2, \dots, Z_{i\cdot j}$ are identical.

Lemma 1.2. *Let D, E be irreducible strings and let x, y be positive integers such that $D^x = E^y$. Then*

$$x = y, \quad D = E.$$

Proof. It suffices to apply Lemma 1.1.

Lemma 1.3. *Let D, E be irreducible strings and let x, y be positive integers such that $D^x \cdot E^y = E^y \cdot D^x$. Then $D = E$.*

Proof. Suppose the assertion of this lemma is false. Then we can choose a counterexample (D, E, x, y) such that the length of the string $D^x \cdot E^y$ is minimal. We can assume the inequality $|D| < |E|$ in this counterexample. Several powers of the string D can be prefixes of the string E (trivially, D^0 is a prefix of the string E). Let q be the maximal integer such that the string D^q is a prefix of the string E. Then we can write E in the form

$$E = D^q \cdot F, \qquad D \text{ is not a prefix of } F.$$

Substituting the last equation into the assumption of lemma, we obtain

$$D^x \cdot (D^q \cdot F)^y = (D^q \cdot F)^y \cdot D^x.$$

Put $W = (D^q \cdot F)^{y-1}$. Then we can write

$$D^{x+q} \cdot F \cdot W = D^q \cdot F \cdot W \cdot D^x,$$
$$D^x \cdot (F \cdot W) = (F \cdot W) \cdot D^x.$$

The string D is not a prefix of F, and so the string F is a prefix of D. Now it is easy to check that $|F \cdot W| < |E^y|$. By our assumption of the minimality, the string $F \cdot W$ is a power of the string D:

$$F \cdot (D^q \cdot F)^{y-1} = D^z, \quad z \geq 1,$$
$$E^y = (D^q \cdot F)^y = D^q \cdot F \cdot W = D^{q+z},$$

contrary to Lemma 1.2.

Lemma 1.4. *Let D, E be arbitrary strings such that the string DE is irreducible and $DE = ED$. Then exactly one of the strings D, E is empty.*

Proof. It suffices to apply Lemma 1.3.

2. The fundamental theorem

Lemma 2.1. *Let A, C be irreducible strings and let B be arbitrary string such that*

$$A^m \cdot B = C^k, \quad m \geq 2, \quad k \geq 2, \quad |A| < |C| < m \cdot |A|.$$

Then there exist non-empty strings F, G and a non-negative integer s such that

$$A = (FG)^{s+1} \cdot F, \quad C = A^{m-1} \cdot FG,$$
$$B = GF \cdot A^{m-2} \cdot FG \cdot C^{k-2}.$$

Proof. The string A is a prefix of the string C and the string C is a prefix of the string A^m. Therefore we can write the string C in the following form:

$$C = A^r \cdot D, \quad 0 < |D| < |A|, \quad 1 \leq r < m.$$

(The equality $D = \varepsilon$ would contradict the irreducibility of C.) Substituting the equality $C = A^r \cdot D$ into the assumption of lemma, we obtain

$$A^m \cdot B = (A^r \cdot D)^k,$$
$$A^m \cdot B = A^r \cdot D \cdot (A^r \cdot D)^{k-1},$$
$$A^{m-r} \cdot B = D \cdot (A^r \cdot D)^{k-1}.$$

It follows immediately that the string D is a prefix of A:

$$A = D \cdot E, \quad |E| > 0,$$
$$DE \cdot (DE)^{m-r-1} \cdot B = D \cdot (DE)^r \cdot D^{k-1},$$
$$E \cdot (DE)^{m-r-1} \cdot B = (DE)^r \cdot D^{k-1},$$

The inequality $m-r-1 > 0$ would contradict Lemma 1.4. It follows that $r = m-1$ and

$$E \cdot B = (DE)^{m-1} \cdot D^{k-1}.$$

Let s be the maximal integer such that the string D^s is a prefix of the string E. Then there exists a string F such that

$$E = D^s \cdot F, \quad D \text{ is not a prefix of } F.$$

The strings A, C, $E \cdot B$ can be written as follows:

$$A = D \cdot E = D^{s+1} \cdot F,$$
$$C = A^{m-1} \cdot D,$$
$$D^s \cdot F \cdot B = ((D^{s+1} \cdot F)^{m-1} \cdot D)^{k-1}.$$

From this we conclude that

$$F \cdot B = DF \cdot (D^{s+1} \cdot F)^{m-2} \cdot D \cdot ((D^{s+1} \cdot F)^{m-1} \cdot D)^{k-2}.$$

We know that the string D is not a prefix of F, and so the string F is a (proper) prefix of D:

$$D = F \cdot G, \quad 0 < |G| < |D|.$$

Substituting this equality into the preceding ones, we complete the proof.

Lemma 2.2. *Let A, C be irreducible strings and let B be arbitrary string such that*
$$A^m \cdot B = C^k, \quad m \geq 2, \quad k \geq 2, \quad |C| > m \cdot |A|.$$

Then there exists a non-empty string D such that
$$C = A^m \cdot D, \quad B = D \cdot C^{k-1}.$$

Proof. The string A^m is a prefix of the string C. It follows that there exists a string D such that $C = A^m \cdot D$. (The string D is non-empty because C is irreducible.) Substituting this equation into the assumption of lemma, we obtain
$$A^m \cdot B = (A^m \cdot D)^k,$$
$$A^m \cdot B = A^m \cdot D \cdot (A^m \cdot D)^{k-1},$$
$$B = D \cdot (A^m \cdot D)^{k-1} = D \cdot C^{k-1}.$$

Theorem 2.1. *Let A, C be irreducible strings and let B be arbitrary string such that*
$$A^m \cdot B = C^k, \quad m \geq 2, \quad k \geq 2.$$

Then there is satisfied exactly one of the following three conditions:

(1) $C = A$, $B = A^{k-m}$,

(2) *there exist non-empty strings F, G and a non-negative integer s such that*
$$A = (FG)^{s+1} \cdot F, \quad C = A^{m-1} \cdot FG,$$
$$B = GF \cdot A^{m-2} \cdot FG \cdot C^{k-2},$$

(3) *there exists a non-empty string D such that*
$$C = A^m \cdot D, \quad B = D \cdot C^{k-1}.$$

Proof. According to Lemma 2.1 and Lemma 2.2, we can suppose that
$$0 < |C| < |A|.$$

The string A is a prefix of the string C^k and so it can be written in the form
$$A = C^r \cdot D, \quad r \geq 1, \quad 0 < |D| < |C|.$$

Substituting this equality into the assumption of the lemma, we obtain
$$(C^r \cdot D)^m \cdot B = C^k,$$
$$C^r \cdot D \cdot (C^r \cdot D)^{m-1} \cdot B = C^k,$$
$$D \cdot (C^r \cdot D)^{m-1} \cdot B = C^{k-r}.$$

Therefore the string D is a proper prefix of the string C:
$$C = D \cdot E, \quad |E| > 0,$$
$$D \cdot ((DE)^r \cdot D)^{m-1} \cdot B = (DE)^{k-r},$$
$$D \cdot ((DE)^r \cdot D)^{m-1} \cdot B = DE \cdot (DE)^{k-r-1},$$
$$((DE)^r \cdot D)^{m-1} \cdot B = E \cdot (DE)^{k-r-1}.$$

Applying the prefix of the length $|DE|$, we obtain $DE = ED$, contrary to Lemma 1.4.

Theorem 2.2. *Let the string B be irreducible but not strongly irreducible. Then B can be written in the form*
$$B = G \cdot F \cdot H \cdot F \cdot G,$$

where H is an arbitrary string and F, G are non-empty strings.

Proof. Apply Theorem 2.1.

Corollary. *Every string of the length 3 containing at least two different symbols is strongly irreducible.*

Remark. The condition in Theorem 2.2 is necessary but not sufficient. For example, the string 10101 is strongly irreducible and the string 11111 is not irreducible.

3. Applications to concrete strings

Put $T = \{0, 1\}$. Every non-negative integer $j < 2^n$ can be uniquely written in the form of a string over T of the length n. This string will be denoted by $Cod(n, j)$. For $z \in T$, put

$$B(n, z) = Cod(n, 0) \cdot Cod(n, 1) \dots Cod(n, 2^n - 1) \cdot z.$$

Example. Put $n = 3$. Then

$$\begin{aligned} Cod(3,0) &= 000, \quad Cod(3,1) = 001, \\ Cod(3,2) &= 010, \quad Cod(3,3) = 011, \\ Cod(3,4) &= 100, \quad Cod(3,5) = 101, \\ Cod(3,6) &= 110, \quad Cod(3,7) = 111, \\ B(3,0) &= 0000010100111001011101110, \\ B(3,1) &= 0000010100111001011101111. \end{aligned}$$

Lemma 3.1. *The string $B(n, z)$ is irreducible.*

Proof. For any string w over T, the number of the occurences of the symbols x in w is usually denoted by $\#_x(w)$. It is evident that

$$|\#_0(B(n, z)) - \#_1(B(n, z))| = 1.$$

If the string $B(n, z)$ would not be irreducible, we could write

$$B(n, z) = K^m, \quad m \geq 2$$

and both numbers $\#_0(B(n, z))$, $\#_1(B(n, z))$ would be multiples of m, a contradiction.

Lemma 3.2. *The string $B(n, z)$ can not be written in the form*

$$B(n, z) = G \cdot F \cdot H \cdot F \cdot G,$$

where H is an arbitrary string and F, G are non-empty strings.

Proof. Suppose, contrary to our claim, that the string $B(n,z)$ can be written in the form

$$B(n,z) = G \cdot F \cdot H \cdot F \cdot G,$$

where H is an arbitrary string and F, G are non-empty strings. Let us denote

$$d = |GF| = |FG|.$$

The symbol 1 occurs in the strings GF and FG. Consequently, $d \geq 2n$. Moreover, it is obvious that

$$Pref_{2n}(B(n,z)) = 0^{2n-1} \cdot 1,$$
$$Postf_{2n+1}(B(n,z)) \in \{1^{n-1}01^n0, 1^{n-1}01^{n+1}\}$$

and the string 0^{2n-1} has only one occurence in $B(n,z)$ - in the role of the prefix. The rest of this proof is left to the reader.

Theorem 3.1. *For every positive integer n and every $z \in \{0,1\}$, the string $B(n,z)$ is strongly irreducible.*

Proof. It suffices to apply Theorem 2.2, Lemma 3.1 amd Lemma 3.2.

References

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Department of Computer Science, Faculty of Natural Sciences

Matej Bel University, Tajovského 40, 974 01 Banská Bystrica, Slovakia

E-mail address: sivak@fhpv.umb.sk

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