Acta Univ. M. Belii
Math. no.3(1995), pp.9-16

METRICS ON SYSTEMS OF FINITE ALGEBRAS

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Abstract. In this paper four different metrics on a system of n-element algebras of the same type are presented. For groupoids and lattices the maximal distance of algebras is also determined.

Introduction

In [1], [5] and [3], [4], metrics on systems of graphs and posets, respectively, are investigated. In this paper we show an analogous way of defining metrics on a system of pairwise non-isomorphic finite algebras of the same type.

In universal algebra, isomorphic algebras are not usually considered to be different. Assuming that two n-element algebras of the same type are not isomorphic one can seek for a bijection compatible with the operations as much as possible. Such an approach yields the first way of defining the metric. The concept of the homomorphism generalizes that of the isomorphism, and in our second approach, it motivates the definition of a measure of difference between algebras. Our third approach is based on the fact that two non-isomorphic algebras may have 'large' isomorphic subalgebras. The distance of algebras depends on the cardinality of these isomorphic subalgebras. As it turned out for systems of graphs, the approach based on subgraphs can be replaced by that based on supergraphs [2]. A similar idea can be applied also to the class of all finite algebras of the same type.

The second (homomorphic) metric and the third (substructure) metric can also be considered for finite algebras of the same type having different cardinalities, as in the proofs that these functions are metrics the cardinalities of algebras are not relevant. The first metric could also be modified (similarly as for graphs) for algebras of the same type but different cardinalities. However, it is hard to decide how much the metric depends on the difference of cardinalities of algebras and on the difference between algebraic properties of given algebras.

The concept of a metric reflects a 'distance' between classes containing isomorphic algebras. However, in order to simplify the terminology we will speak on a 'distance' between algebras.

We will particularly focus our attention to the metrics on systems of groupoids and lattices where we also determine the maximal distance of two algebras.

1991 *Mathematics Subject Classification.* 08A62, 28A99.
Key words and phrases. Metric, finite algebra.

Throughout this paper the set $N_n = \{0, 1, \ldots, n-1\}$ is taken as a universe of n-element algebra. By $\mathcal{S}_n$ we denote a system of pairwise non-isomorphic n-element algebras of the same type. By $\mathcal{G}_n$ ($\mathcal{L}_n$) we denote the system of all pairwise non-isomorphic n-element groupoids (lattices). The maximal distance between algebras on the system $\mathcal{G}_n$ ($\mathcal{L}_n$) in a metric d_j will be denoted by $D_j(\mathcal{G}_n)$ $(D_j(\mathcal{L}_n))$. The number D_j will be called the diameter of the system $\mathcal{G}_n$ ($\mathcal{L}_n$).

1. Isomorphism metric

Let $\mathbf{A} = (A, F_1, \ldots, F_m)$, $\mathbf{B} = (B, F_1', \ldots, F_m')$ be n-element algebras of the same type. We denote by $M(A, B)$ the set of all bijections of A onto B. Let $f \in M(A, B)$ and let F_j, $1 \le j \le m$ be a k-ary operation of $\mathbf{A}$. For $k \ge 1$ we put

$$D_j(f) = \{[a_1, \ldots, a_k] \in A^k; f(F_j(a_1, \ldots, a_k)) \neq F_j'(f(a_1), \ldots, f(a_k))\}$$

and for $k = 0$

$$D_j(f) = \begin{cases} \{F_j\} & \text{if} \quad f(F_j) \neq F_j', \\ \emptyset, & \text{otherwise.} \end{cases}$$

Let

$$D(f) = D_1(f) \cup \cdots \cup D_m(f),$$

and

$$d_{iso}(\mathbf{A}, \mathbf{B}) = min\{\mid D(f) \mid; f \in M(A, B)\}, \tag{1}$$

where $\mid D(f) \mid$ is the cardinality of the set $D(f)$.

Theorem 1.1. *The function d_{iso} given by (1) is a metric on the system $\mathcal{S}_n$ (of pairwise non-isomorphic n-element algebras of the same type).*

Proof. Clearly, $d_{iso}(\mathbf{A}, \mathbf{B}) = 0$ if and only if $\mathbf{A} = \mathbf{B}$.

If $f(F_j(a_1, \ldots, a_k)) \neq F_j'(f(a_1), \ldots, f(a_k))$ for a bijection $f : A \to B$, k-ary operation $F_j, k \ge 1$, and elements $a_1, \ldots a_k$ then $f^{-1}(F_j'(b_1, \ldots, b_k)) \neq F_j(f^{-1}(b_1), \ldots, f^{-1}(b_k))$ for the elements $b_1 = f(a_1), \ldots, b_k = f(a_k)$. It follows $d_{iso}(\mathbf{A}, \mathbf{B}) = d_{iso}(\mathbf{B}, \mathbf{A})$.

Let $d_{iso}(\mathbf{A}, \mathbf{B}) =\mid D(f) \mid$, $d_{iso}(\mathbf{B}, \mathbf{C}) =\mid D(g) \mid$ and $d_{iso}(\mathbf{A}, \mathbf{C}) =\mid D(h) \mid$. It is obvious that $\mid D(g \circ f) \mid \ge \mid D(h) \mid$. We shall have established $\mid D(f) \mid + \mid D(g) \mid \ge \mid D(h) \mid$ if we prove that

$$\mid D(f) \mid + \mid D(g) \mid \ge \mid D(g \circ f) \mid . \tag{1a}$$

The inequality (1a) follows easily from the fact that $[a_1, \ldots, a_k] \in D(f)$ or $[f(a_1), \ldots, f(a_k)] \in D(g)$, if $[a_1, \ldots, a_k] \in D(g \circ f)$ and $F_i \in D(f)$ or $F_i' \in D(g)$ if $F_i \in D(g \circ f)$, respectively.

Theorem 1.2. $D_{iso}(\mathcal{G}_n) = n^2 \quad if \quad n \geq 2$,
$D_{iso}(\mathcal{L}_n) = (n-2)^2 - (n-2) \quad if \quad n \geq 4.$

Proof. a) Of course, $d_{iso}(\mathbf{G}_1, \mathbf{G}_2) \leq n^2$ holds for any n-element groupoids $\mathbf{G}_1, \mathbf{G}_2$.
We define the operations $\circ$ and $*$ on the set $N_n = \{0, \dots, n-1\}$, $n \geq 2$, by

$$x \circ y = x \quad \text{for every number } x$$

$$x * y = \begin{cases} y, & \text{if } x \neq y \\ x+1, & \text{if } x = y \end{cases}$$

(we compute modulo n). It follows immediately that $f(x \circ y) \neq f(x) * f(y)$ for any permutation f of N_n and any numbers $x, y \in N_n$. Therefore we have
$d_{iso}((M_n, o), (M_n, *)) = n^2$.

b) Let $\mathbf{L}_1 = (L_1, \vee, \wedge, 0, 1)$, $\mathbf{L}_2 = (L_2, \vee, \wedge, 0, 1)$ be n-element lattices. Let $f : L_1 \to L_2$ be a 0, 1-preserving bijection. It follows immediately that

(1b) $$| D(f) | \leq (n-2)^2 - (n-2).$$

The equality $d_{iso}(\mathbf{L}_1, \mathbf{L}_2) = (n-2)^2 - (n-2)$ holds if $\mathbf{L}_1$ is the n-element chain and $\mathbf{L}_2$ is the n-element lampion (a lattice of height 2 with n-2 atoms), $n \geq 4$. □

2. Homomorphism metric

Let $\mathbf{A}, \mathbf{B}$ be n-element algebras of the same type. If $f : A \to B$ is a homomorphism then $f(A)$ is a subuniverse of $\mathbf{B}$. We call $f(A)$ the homomorphic image of $\mathbf{A}$ in $\mathbf{B}$. If there is no homomorphism $f : A \to B$ we define the homomorphic image of $\mathbf{A}$ in $\mathbf{B}$ to be $\emptyset$.

Let $\mathcal{S}_n$ be a system of pairwise non-isomorphic n-element algebras of the same type. We define the distance of algebras $\mathbf{A}, \mathbf{B} \in \mathcal{S}_n$ by

(2) $$d_h(\mathbf{A}, \mathbf{B}) = | A | - | f(A) | + | B | - | g(B) |,$$

where f is a homomorphism $A \to B$ such that the cardinality of $f(A)$ is maximal possible. Analogously, for $g : B \to A$.

Theorem 2.1. *The function d_h given by (2) is a metric on the system $\mathcal{S}_n$.*

Proof. We see at once that $d_h(\mathbf{A}, \mathbf{B}) = 0$ iff $\mathbf{A} = \mathbf{B}$ and $d_h(\mathbf{A}, \mathbf{B}) = d_h(\mathbf{B}, \mathbf{A})$.
Let $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathcal{S}_n$ and let
$f : A \to B, \quad g : B \to C, \quad h : A \to C,$
$F : B \to A, \quad G : C \to B, \quad H : C \to A,$
be such homomorphisms that $f(A), \dots, H(C)$ are maximal homomorphic images. We want to prove the inequality

$$| A | - | h(A) | + | C | - | H(C) | \leq$$
$$\leq | A | - | f(A) | + | B | - | F(B) | + | B | - | g(B) | + | C | - | G(C) | .$$

It is sufficient to show that

(2a) $$| f(A) | + | g(B) | \leq | B | + | h(A) |$$

and

(2b) $$| F(B) | + | G(C) | \leq | B | + | H(C) | .$$

We are going to prove (2a) ((2b) can be proved in the same way). Since $| h(A) | \geq | g(f(A)) |$ it suffices to show that $| g(B) | - | g(f(A)) | \leq | B | - | f(A) |$, i.e.

(2c) $$| g(B) - g(f(A)) | \leq | B - f(A) | .$$

The inequality (2c) follows from

$$| B - f(A) | \geq | g(B - f(A)) | \geq | g(B) - g(f(A)) | .$$

If there are no homomorphisms from **A** to **B** or from **B** to **C** (i.e. if $f(A) = \emptyset$ or $g(B) = \emptyset$), the inequality (2a) also holds.

Remark. *We note that the proof runs if we drop the assumption that* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are algebras of the same cardinality.*

Theorem 2.2. $D_h(\mathcal{G}_n) = 2n$ *if* $n \geq 2$,
$D_h(\mathcal{L}_n) = 2n - 2$ *if* $n \geq 7$.

Proof. a)It is evident that $D_h(\mathcal{G}_n) \leq 2n$. We will find two non-isomorphic groupoids whose congruence lattices are trivial and the sets of idempotent elements are empty.

We define the operations $\circ$ and $*$ on the set $N_n = \{0, 1, \dots, n-1\}$ $n \geq 2$ by

$$\begin{aligned} & i \circ (n-1) = i * (n-1) = i + 1, \\ & i \circ i = i * i = i + 1, \\ & i \circ k = i * k = k + 2 \quad \text{if} \quad i = k + 2, \dots n - 1, \\ & i \circ k = 1, \quad i * k = k \quad \text{otherwise} \end{aligned}$$

(we compute modulo n). Now, we are going to show that the groupoids $(N_n, \circ)$ and $(N_n, *)$ have only trivial congruences. Let $\Theta \in Con(N_n, \circ)$, $i \Theta j$ and $i < j$. The equality $i + 1 = j$ implies $i \circ (n-1) \Theta (i+1) \circ (n-1)$, i.e. $i + 1 \Theta i + 2$, analogously $i + 2 \Theta i + 3$, etc., hence $\Theta = N_n^2$. If $i + 1 < j$ we have $i \circ i \Theta j \circ i$, i.e. $i + 1 \Theta i + 2$, and this again yields $\Theta = N_n^2$. In the same manner one can see that $(N_n, *)$ has trivial congruences. It is obvious that $(N_n, \circ)$ and $(N_n, *)$ are non-isomorphic and they do not contain any idempotent elements. Hence $d_h((N_n, \circ), (N_n, *)) = 2n$.

b) It is sufficient to find two non-isomorphic lattices such that they have no non-trivial congruences. It is immediate to check that the n-element lampion and the lattice depicted in Fig. 1 ($n \geq 7$) have only trivial congruences.

$\square$

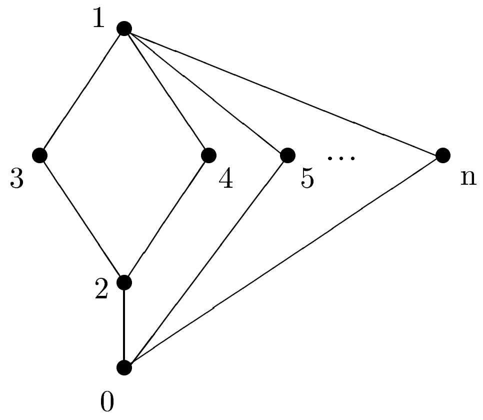

Fig. 1

3. Substructure metric.

Let $\mathbf{A}, \mathbf{B}$ be n-element algebras of the same type. We call an subalgebra $\mathbf{A}_1$ of $\mathbf{A}$ a *common subalgebra* of $\mathbf{A}$ and $\mathbf{B}$ if there exists a subalgebra $\mathbf{B}_1$ of $\mathbf{B}$ such that $\mathbf{A}_1$ and $\mathbf{B}_1$ are isomorphic. In this case we denote the universe A_1 of $\mathbf{A}_1$ by S_{AB}. Otherwise (i.e. if there are no isomorphic subalgebras of $\mathbf{A}$ and $\mathbf{B}$) we put $S_{AB} = \emptyset$. We define the distance of algebras $\mathbf{A}$ and $\mathbf{B}$ by

$$d_s(\mathbf{A}, \mathbf{B}) = \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid, \tag{3}$$

where $\mid S_{AB} \mid$ is the maximum of cardinalities of common subuniverses of $\mathbf{A}$ and $\mathbf{B}$.

Theorem 3.1. *The function d_s given by (3) is a metric on the system $\mathcal{S}_n$.*

Proof. We will prove only the triangle inequality. Let

$$\begin{aligned} d_s(\mathbf{A}, \mathbf{B}) &= \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid, \\ d_s(\mathbf{B}, \mathbf{C}) &= \mid B \mid + \mid C \mid -2. \mid S_{BC} \mid, \\ d_s(\mathbf{A}, \mathbf{C}) &= \mid A \mid + \mid C \mid -2. \mid S_{AC} \mid, \end{aligned}$$

and let f and h be embeddings of $\mathbf{S}_{AB}$ into $\mathbf{B}$ and $\mathbf{S}_{AC}$ into $\mathbf{C}$, respectively and let g be an embedding of $\mathbf{S}_{BC}$ into $\mathbf{C}$. It suffices to prove that

$$\mid B \mid + \mid S_{AC} \mid \geq \mid S_{AB} \mid + \mid S_{BC} \mid . \tag{3a}$$

It is easily seen that $B' = f(S_{AB}) \cap S_{BC}$ is a subuniverse of the algebra $\mathbf{B}$. Further, it is evident that

$$\mid B \mid + \mid B' \mid \geq \mid f(S_{AB}) \mid + \mid S_{BC} \mid = \mid S_{AB} \mid + \mid S_{BC} \mid . \tag{3b}$$

The subalgebras of **A** and **C** with subuniverses $f^{-1}(B')$ and $g(B')$ are isomorphic, therefore

(3c) $$| B' | \leq | S_{AC} | .$$

Combining (3c) with (3b) we have (3a).

If $S_{AB} = \emptyset$ or $S_{BC} = \emptyset$, the inequality (3a) is evident. $S_{AC} = \emptyset$ implies $B' = \emptyset$ and again (3a) holds. □

Theorem 3.2. $D_s(\mathcal{G}_n) = 2n$ *if* $n \geq 2$,
$D_s(\mathcal{L}_n) = 2n - 6$ *if* $n \geq 4$.

Proof. a) For example, the groupoids $(N_n, \circ)$ and $(N_n, *)$ with operations given by

$$x \circ x = x * x = x + 1,$$
$$x \circ y = x \quad \text{if} \quad x \neq y,$$
$$x * y = y \quad \text{if} \quad x \neq y,$$

(we compute modulo n) have the distance 2n. The details are left to the reader.

b) Every n-element lattice ($n \geq 4$) contains a 3-element chain. From this we have $D_s(\mathcal{L}_n) \leq 2n - 6$. The distance of the n-element chain and the n-element lampion is $2n - 6$.

Remark. We note that the proof runs if we drop the assumption that **A**, **B**, **C** are algebras of the same cardinality.

The next examples show that the metrics d_{iso}, d_h and d_s are independent.

Example 1. Let $(G, \circ)$, $(H, *)$ and $(K, .)$ be the groupoids given by Cayley's tables 1, 2 and 3, respectively.

o	a	b
a	b	a
b	b	a

Tab. 1

*	c	d
c	c	d
d	c	d

Tab. 2

·	0	1
0	0	0
1	0	1

Tab. 3

We can easily check that
$d_{iso}(\mathbf{G}, \mathbf{H}) = 4 = d_s(\mathbf{G}, \mathbf{H}) > d_h(\mathbf{G}, \mathbf{H}) = 3$,
$d_{iso}(\mathbf{H}, \mathbf{K}) = 1 < d_s(\mathbf{H}, \mathbf{K}) = 2 = d_h(\mathbf{H}, \mathbf{K})$,
$d_{iso}(\mathbf{G}, \mathbf{K}) = 3 = d_h(\mathbf{G}, \mathbf{K}) < d_s(\mathbf{G}, \mathbf{K}) = 4$.

Example 2. Let $(L_1, \vee, \wedge)$ be the 4-element lattice of height 2 (the lampion) and $(L_2, \vee, \wedge)$ the 4-element chain. It is obvious that
$d_{iso}(\mathbf{L}_1, \mathbf{L}_2) = 4 > d_h(\mathbf{L}_1, \mathbf{L}_2) = 3 > d_s(\mathbf{L}_1, \mathbf{L}_2) = 2$.

4. Superstructure metric

Let τ be a type of algebras. We define a metric on the system of all pairwise non-isomorphic n-elements algebras of the type τ as follows.

Let $\mathbf{A}, \mathbf{B}$ be n-element algebras of the type τ. We define the distance of $\mathbf{A}$ and $\mathbf{B}$ by

$$d_{Su}(\mathbf{A}, \mathbf{B}) = 2. \mid O_{AB} \mid - \mid A \mid - \mid B \mid, \tag{4}$$

where O_{AB} is a minimal algebra (with respect to the cardinality of its universe) of the type τ which contains subalgebras isomorphic to $\mathbf{A}$ and $\mathbf{B}$.

Theorem 4.1. *The function d_{Su} given by (4) is a metric on the system of all pairwise non-isomorphic n-element algebras of the type τ.*

Proof. We will show that (similarly as on a system of graphs)

$$d_{Su}(\mathbf{A}, \mathbf{B}) = d_s(\mathbf{A}, \mathbf{B}).$$

By (3)

$$d_s(\mathbf{A}, \mathbf{B}) = \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid,$$

where S_{AB} is a maximal algebra (with respect to the cardinality of its universe) such that there exist a subalgebra $\mathbf{A}_1$ of $\mathbf{A}$ which is isomorphic to $\mathbf{S}_{AB}$ and a subalgebra $\mathbf{B}_1$ of $\mathbf{B}$ which is isomorphic to $\mathbf{S}_{AB}$. Without loss of generality we can assume that $A_1 = B_1 = S_{AB}$. Let $C = A \cup B$ and let $\mathbf{F}$ be k-ary operation symbol of τ, $k \geq 1$. Fix an element $b \in A$. We define the operation F on the set C in the following way:

If $a_1, \ldots, a_k, a \in A$ and $F(a_1, \ldots, a_k) = a$ in $\mathbf{A}$ or
$a_1, \ldots, a_k, a \in B$ and $F(a_1, \ldots, a_k) = a$ in $\mathbf{B}$
then $F(a_1, \ldots, a_k) = a$. Otherwise $F(a_1, \ldots, a_k) = b$.

Now, we have

$$d_{Su}(\mathbf{A}, \mathbf{B}) \leq 2. \mid C \mid - \mid A \mid - \mid B \mid = 2.(\mid A \mid + \mid B \mid - \mid S_{AB} \mid) - \mid A \mid - \mid B \mid =$$
$$= \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid = d_s(\mathbf{A}, \mathbf{B}).$$

On the other hand, we can suppose that $A \subseteq O_{AB}$ and $B \subseteq O_{AB}$, whence

$$\mid O_{AB} \mid \geq \mid A \mid + \mid B \mid - \mid A \cap B \mid \geq \mid A \mid + \mid B \mid - \mid S_{AB} \mid .$$

Therefore,

$$d_{Su}(\mathbf{A}, \mathbf{B}) = 2. \mid O_{AB} \mid - \mid A \mid - \mid B \mid \geq \mid A \mid + \mid B \mid -2. \mid S_{AB} \mid = d_s(\mathbf{A}, \mathbf{B}).$$

□

Unlike previous metrics d_{iso}, d_h, d_s, the function given by (4) may not be a metric on any system of non-isomorphic n-element algebras of the same type (like groups, rings, etc.). However, we can prove the next statement.

Theorem 4.2. *The function d_{Su} given by (4) is a metric on the system $\mathcal{L}_n$ of all pairwise non-isomorphic n-element lattices.*

Proof. Let $\mathbf{L}_1 = (L_1, \leq_1), \mathbf{L}_2 = (L_2, \leq_2)$ be lattices and $\mathbf{L}_{12}$ be a maximal lattice such that there exist a sublattice $\mathbf{L}_1'$ of $\mathbf{L}_1$ isomorphic to $\mathbf{L}_{12}$ and a sublattice $\mathbf{L}_2'$ of $\mathbf{L}_2$ isomorphic to $\mathbf{L}_{12}$. We can assume that $\mathbf{L}_1' = \mathbf{L}_2' = \mathbf{L}_{12}$ and $0, 1 \in L_{12}$. As the ordering on $L = L_1 \cup L_2$ we take the transitive closure of the union of the orderings $\leq_1$ and $\leq_2$. To finish the proof proceed similarly as in the proof of Theorem 4.1 $\square$

Corollary. $D_{Su}(\mathcal{G}_n) = 2n$ *if* $n \geq 2$,
$D_{Su}(\mathcal{L}_n) = 2n - 6$ *if* $n \geq 4$.

References

1. V. Baláž, J. Koča, V. Kvasnička, and M. Sekanina, *A metric for graphs*, Čas. pěs. mat. . **111** (1986), 431–433.
2. V. Baláž, V. Kvasnička and J. Pospíchal,, *Dual approach for edge distance between graphs*, Čas. pěs. mat. **114** (1989), 155–159.
3. P. Klenovčan, *The distance poset of posets*, Acta Univ. M. Belii **2** (1994), 43–48.
4. B. Zelinka, *Distances between partially ordered sets*, Math. Bohemica **118** (1993), 167–170.
5. B. Zelinka, *Distances between directed graphs*, Čas. pěs. mat. **112** (1987), 359–367.

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(Received November 6, 1995)