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THE STUDY OF AFFINE COMPLETENESS FOR QUASI-MODULAR DOUBLE P-ALGEBRAS

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Abstract. In this paper we study affine complete and locally affine complete algebras in the class of quasi-modular double p-algebras. We generalize Beazer's characterization of affine complete double Stone algebras with a non-empty bounded core [B 1983] to the class of quasi-modular double S-algebras with a non-empty bounded core. We prove that finite regular double p-algebras are the only finite affine complete quasi-modular double p-algebras with a non-empty core and that Post algebras of order 3 are the only affine complete quasi-modular double S-algebras with a non-empty finite core. In distributive case, we derive the Beazer result and we construct an example of an infinite regular double Stone algebra which is not affine complete. We finally show that the Post algebras of order 3 are the only locally affine complete (in a stronger sense of [P 1972]) quasi-modular double S-algebras with a non-empty bounded core.

1. Introduction.

One of the topics of universal algebra rapidly developed in the last decades has been the study of affine complete algebras. Let us recall that an n-ary function f on an algebra $\boldsymbol{A}$ is called *compatible* if for any congruence θ on $\boldsymbol{A}$, $a_i \equiv b_i$ (θ) $(a_i, b_i \in A)$, $i = 1, \ldots, n$ yields $f(a_1, \ldots, a_n) \equiv f(b_1, \ldots, b_n)$ (θ). Obviously, every polynomial function of $\boldsymbol{A}$, i.e. a function that can be obtained by composition of the basic operations of $\boldsymbol{A}$, the projections and the constant functions, is compatible. By H. Werner [W 1971], an algebra $\boldsymbol{A}$ is called *affine complete* if the only compatible functions on $\boldsymbol{A}$ are the polynomial ones. Hence one can imagine affine complete algebras as algebras having many congruences.

The first results in this topic are due to G. Grätzer. In [G 1962] he showed that every Boolean algebra is affine complete and in [G 1964] he characterized affine complete bounded distributive lattices as those which do not contain proper Boolean subintervals. In [G 1968] he formulated a problem of characterizing affine complete algebras which was later reformulated in [C-W 1981] as follows: characterize affine complete algebras in your favourite variety. In [C-W 1981] one can also find a list of particular varieties in which all affine complete members were characterized. Some new items in the list are mentioned in [Ha-Pl 1995].

Also a "local" version of affine completeness has been studied. Let us recall that an algebra $\boldsymbol{A}$ is said to be *locally affine complete* if any finite partial function in $A^n \to A$ (i.e.

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function whose domain is a finite subset of A^n) which is compatible (where defined) can be interpolated by a polynomial of A (see e.g. [P 1972] or [Kaa-P 1987]; the notion 'locally affine complete' has also another, weaker meaning in the literature - see e.g. [Sz 1986] or [Ha-Pl 1995].)

In [B 1982] R. Beazer characterized affine complete algebras in the class of Stone algebras with bounded dense filter and in [B 1983] he gave a similar characterization in the class of double Stone algebras with a non-empty bounded core. Locally affine complete Stone algebras (in the weaker sense of [Sz 1986]) were characterized in [Ha 1993] and affine complete algebras in the variety of all Stone algebras were recently described in [Ha-Pl 1995]. Another generalization of the first Beazer result, to the class of so-called principal p-algebras, was presented in [Ha 1995].

In this paper we generalize the second Beazer result and its consequences (3.1-3.3) into a larger class of all quasi-modular double S-algebras with a non-empty bounded core (Theorem 3.14). First we show that for a quasi-modular double p-algebra $\boldsymbol{L}$ with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, affine completeness of $\boldsymbol{L}$ yields affine completeness of $K(\boldsymbol{L})$ as a bounded lattice (Theorem 3.4). Consequently, we get that finite regular double p-algebras are the only finite affine complete quasi-modular double p-algebras with a non-empty core and that Post algebras of order 3 are the only affine complete quasi-modular double S-algebras with a non-empty finite core (3.7 and 3.8). In distributive case, we derive (3.15-3.17) the second Beazer result and its consequences. Then we construct an example of an infinite regular double Stone algebra which is not affine complete with regard to Beazer's question in [B 1983].

We finally show that Post algebras of order 3 are the only locally affine complete quasi-modular double S-algebras with a non-empty bounded core (3.19-3.20).

2. Preliminaries.

A *p-algebra (pseudocomplemented lattice* or *PCL)* is an algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$ where $(L; \vee, \wedge, 0, 1)$ is a bounded lattice and * is the unary operation of pseudocomplementation, i.e. $x \leq a^\star$ iff $x \wedge a = 0$. By a *distributive (modular)* p-algebra $(L; \vee, \wedge, ^\star, 0, 1)$ we mean that the lattice L is distributive (modular). Further, recall that a *Stone algebra* is a distributive p-algebra satisfying the Stone identity

$(S) \qquad x^\star \vee x^{\star\star} = 1.$

In general, p-algebras satisfying (S) are called *S-algebras*.

Besides distributive and modular p-algebras, a larger variety of *quasi-modular p-algebras* was introduced and studied [Ka-Me 1983]. This subvariety of p-algebras is defined by the identity

$$((x \wedge y) \vee z^{\star\star}) \wedge x = (x \wedge y) \vee (z^{\star\star} \wedge x).$$

It is known (see [Ka-Me 1983; 6.1]) that quasi-modular p-algebras satisfy the identity

$$x = x^{\star\star} \wedge (x \vee x^\star).$$

An algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ is called a (quasi-modular) *double p-algebra*, if $(L; \vee, \wedge, ^\star, 0, 1)$ is a (quasi-modular) p-algebra and $(L; \vee, \wedge, ^+, 0, 1)$ is a dual (quasi-modular) p-algebra, i.e. $x \geq a^+$ if and only if $a \vee x = 1$.

A *double S-algebra* is a double p-algebra satisfying the identities

$x^\star \vee x^{\star\star} = 1$ and $x^+ \wedge x^{++} = 0.$

A *double Stone algebra* is a distributive double S-algebra. A double Stone algebra in which so-called determination principle,

$a^\star = b^\star$ and $a^+ = b^+$ implies $a = b$,

holds is called a *three-valued Lukasiewicz algebra.*

In a double p-algebra $\boldsymbol{L}$, the sets $B(\boldsymbol{L}) = \{x \in L;\ x = x^{\star\star}\}$ and $\overline{B}(\boldsymbol{L}) = \{x \in L;\ x = x^{++}\}$ give Boolean algebras $(B(\boldsymbol{L}); \triangledown, \wedge, ^\star, 0, 1)$ and $(\overline{B}(\boldsymbol{L}); \vee, \triangle, ^+, 0, 1)$ where $x \triangledown y = (x \vee y)^{\star\star}$ and $x \triangle y = (x \wedge y)^{\star\star}$. If $\boldsymbol{L}$ is a quasi-modular double S-algebra, then $B(\boldsymbol{L})$ $(= \overline{B}(\boldsymbol{L}))$ is a subalgebra of $\boldsymbol{L}$ (cf. [Ka-Me 1983; 6.8] and [Ka 1974]) and $x^{\star +} = x^{\star\star}$, $x^{+\star} = x^{++}$.

The sets $D(\boldsymbol{L}) = \{x \in L;\ x^\star = 0\}$ and $\overline{D}(\boldsymbol{L}) = \{x \in L;\ x^+ = 1\}$ form a filter and an ideal of $\boldsymbol{L}$, respectively. The set $K(\boldsymbol{L}) = D(\boldsymbol{L}) \cap \overline{D}(\boldsymbol{L})$ is the *core* of $\boldsymbol{L}$. The class of quasi-modular double S-algebras with non-empty core includes bounded lattices with a new zero and unit adjoined, Post algebras of order $n > 2$, injective double Stone algebras, etc.

Congruences on double p-algebras are lattice congruences preserving the operations * and $^+$. The congruence Φ of a double p-algebra defined by

$x \equiv y(\Phi)$ if and only if $x^\star = y^\star$ and $x^+ = y^+$

is called the *determination congruence.* A double p-algebra is *regular* (i.e. two congruence relations having a congruence class in common coincide) if and only if $\Phi = \omega$ (see [V 1972]). Regular double p-algebras form a variety defined by the identity $(x \wedge x^+) \vee (y \vee y^\star) = y \vee y^\star$. Further, a regular double p-algebra $\boldsymbol{L}$ is distributive (see [Ka 1973b]). In [B 1976] regular double p-algebras were shown to be congruence permutable, hence the variety of regular double p-algebras is arithmetical. A (quintuple) construction of regular double p-algebras was presented in [Ka 1974].

A special subclass (not a subvariety) of the variety of regular double Stone algebras (i.e. three-valued Lukasiewicz algebras) form *Post algebras of order* 3, which are defined by the condition $|K(\boldsymbol{L})| = 1$ (see e.g. [B 1983]).

Let $\boldsymbol{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ be a quasi-modular double p-algebra with a non-empty bounded core $K(L) = [k, l]$. Since $\boldsymbol{L}$ satisfies the identities $x = x^{\star\star} \wedge (x \vee x^\star)$ and $x = x^{++} \vee (x \wedge x^+)$, it obviously satisfies the equations $x = x^{\star\star} \wedge (x \vee k)$ and $x = x^{++} \vee (x \wedge l)$. Thus $\boldsymbol{L}$ satisfies the equation

(1) $\quad x = x^{++} \vee (x^{\star\star} \wedge (x \vee k) \wedge l)$.

In [Mu-En 1986; Theorem 5] it was shown that the filter $D(\boldsymbol{L})$ of a quasi-modular p-algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, 0, 1)$ is a neutral element in the lattice $F(\boldsymbol{L})$ of all filters of $\boldsymbol{L}$. So if $D(L) = [k)$, then for all $x, y \in L$, $([x) \vee [y)) \wedge [k) = ([x) \wedge [k)) \vee ([y) \wedge [k))$ holds in $F(\boldsymbol{L})$. Consequently, $(x \wedge y) \vee k = (x \vee k) \wedge (y \vee k)$ for all $x, y \in L$. Thus in a quasi-modular double p-algebra $\boldsymbol{L}$ with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, the elements k, l are distributive.

For these and other properties of double p-algebras as well as for the standard rules of computation in double p-algebras we refer to [B 1976] or [Ka 1973b].

In the second part of this preliminary section we present a collection of results concerning (local) affine completeness of some classes of algebras which will frequently be used in our investigations.

We start with basic Grätzer's results.

2.1 Theorem ([G 1962]). *Any Boolean algebra is affine complete.*

Let us recall that a function $f : L^n \to L$ on a lattice $\boldsymbol{L}$ is *order-preserving* if $x_i \le y_i$ $(x_i, y_i \in L,\ i = 1, \ldots, n)$ implies $f(x_1, \ldots, x_n) \le f(y_1, \ldots, y_n)$. It is well-known that every polynomial function on a lattice is order-preserving.

2.2 Theorem ([G 1964; Corollaries 1,3]). *Let $\boldsymbol{L}$ be a bounded distributive lattice. The following conditions are equivalent.*
(1) $\boldsymbol{L}$ is affine complete;
(2) every compatible function on $\boldsymbol{L}$ is order-preserving;
(3) $\boldsymbol{L}$ contains no proper Boolean interval.

When omiting the distributivity of $\boldsymbol{L}$, one can prove (at least) the following:

2.3 Proposition. *If a lattice $\boldsymbol{L}$ contains a Boolean interval $[a, b]$ $(a < b)$, then $\boldsymbol{L}$ is not affine complete.*

Proof. Define a function $f : L \to [a, b]$ by the rule $f(x) = ((x \vee a) \wedge b)'$, where $'$ denotes the complement in the Boolean interval [a,b]. For any non-trivial congruence $\Phi \in \mathrm{Con}(L)$ and $x \equiv y\ (\Phi)$ $(x,\ y \in L)$ we have $((x \vee a) \wedge b)' \equiv ((y \vee a) \wedge b)'\ (\Phi)$, i.e. f is a compatible function of $\boldsymbol{L}$. But f is not order-preserving because $f(a) = b,\ f(b) = a$, therefore f cannot be represented by a lattice polynomial. Hence $\boldsymbol{L}$ is not affine complete. □

2.4 Corollary. *A finite lattice $\boldsymbol{L}$ is affine complete if and only if $|L| = 1$.* □

If the property we study is the local affine completeness (in the sense of [P 1972]), then the trivial lattices are the only members of the variety of all lattices having this property:

2.5 Proposition. *A lattice $\boldsymbol{L}$ is locally affine complete if and only if $|L| = 1$.*

Proof. Let $\boldsymbol{L}$ be locally affine complete and let $a, b \in L,\ a < b$. The function $f = \{(a, b), (b, a)\}$ is a finite partial compatible function on $\boldsymbol{L}$, thus by hypothesis it can be interpolated on $\{a, b\}$ by a polynomial of $\boldsymbol{L}$, which is an order-preserving function. But we have $f(a) = b,\ f(b) = a$, a contradiction. □

On the other hand, there are varieties of which all members are locally affine complete. The following result (see [P 1979] or [P 1982] or [P 1991]) characterizes them as arithmetical, i.e. congruence-distributive and congruence-permutable (meaning that the congruence lattice of each algebra in such variety is distributive and every two congruences permute):

2.6 Theorem. *A variety V is arithmetical if and only if for each algebra $\boldsymbol{A} \in V$, a finite partial function f on $\boldsymbol{A}$ can be interpolated by a polynomial function of $\boldsymbol{A}$ just in the case f is Con($\boldsymbol{A}$)-compatible.*

This also yields that every finite algebra in an arithmetical variety is affine complete.

The following technical lemma will be used several times in the sequel (and we repeat its proof from [Ha 1992]):

2.7 Lemma. *Let $\boldsymbol{D} = (D, \vee, \wedge, f_1, \dots, f_k, 0, 1)$ be any algebra such that its reduct $(D, \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the algebra $\boldsymbol{D}$ is a subdirect product of 2-element algebras. Let $f', g' : D^n \to D$ be partial compatible functions with domains F and G ($F, G \subset D^n$), respectively, let $S := F \cap G$ and let $S \cap \{0,1\}^n \neq \varnothing$. For any $(0,1)$-homomorphism $h : D \to \{0,1\}$ between the algebra $\boldsymbol{D}$ and a 2-element algebra $\mathbf{2} = \{0,1\}$, denote $h(S) := \{(h(x_1), \dots, h(x_n)) \in \{0,1\}^n;\ (x_1, \dots, x_n) \in S\}$ and let $h(S) = h(S \cap \{0,1\}^n)$ hold. Then $f' \equiv g'$ identically on S if and only if $f' \equiv g'$ identically on $S \cap \{0,1\}^n$.*

Proof. Let $f' \equiv g'$ identically on $S \cap \{0,1\}^n$. Suppose on the contrary that there exists an n-tuple $(d_1, \dots, d_n) \in S$ such that $f'(d_1, \dots, d_n) = a \neq b = g'(d_1, \dots, d_n)$. Since $a \neq b$ in $\boldsymbol{D}$ which is a subdirect product of 2-element algebras, there exists a 'projection map' $h : D \to \{0,1\}$, which is a $(0,1)$-homomorphism between the algebra $\boldsymbol{D}$ and some algebra $\mathbf{2} = \{0,1\}$, such that $h(a) \neq h(b)$. Define functions $f'_2,\ g'_2 : h(S) \to \{0,1\}$ by the following rules:

$f'_2(h(x_1), \dots, h(x_n)) = h(f'(x_1, \dots, x_n))$,

$g'_2(h(x_1), \dots, h(x_n)) = h(g'(x_1, \dots, x_n))$ where $(x_1, \dots, x_n) \in S$.

Obviously, $f'_2,\ g'_2$ are well-defined, since f', g' preserve the kernel congruence of the homomorphism h. Obviously, $f'_2 \equiv g'_2$ identically on $h(S)$, because $h(S) = h(S \cap \{0,1\}^n)$, $h(0) = 0,\ h(1) = 1$ and $f' \equiv g'$ identically on $S \cap \{0,1\}^n$. Therefore

$h(a) = h(f'(d_1, \dots, d_n)) = f'_2(h(d_1), \dots, h(d_n)) = g'_2(h(d_1), \dots, h(d_n)) =$
$h(g'(d_1, \dots, d_n)) = h(b)$, a contradiction. Hence $f' \equiv g'$ identically on S and the proof is complete. $\square$

In order to abbreviate some expressions, we shall often use the notation $\tilde{\mathrm{x}}$ for an n-tuple $(x_1, \dots, x_n)$, and $f(\tilde{\mathrm{x}})$ for $f(x_1, \dots, x_n)$ in the next section. Further, $\tilde{\mathrm{x}}^\star$ and $\tilde{\mathrm{x}}^+$ will denote $(x_1^\star, \dots, x_n^\star)$ and $(x_1^+, \dots, x_n^+)$, respectively, $(\tilde{\mathrm{x}} \vee k) \wedge l$ will abbreviate $((x_1 \vee k) \wedge l, \dots, (x_n \vee k) \wedge l)$, etc.

3. Afinne completeness.

We start this section with Beazer's characterization of affine complete double Stone algebras with a non-empty bounded core and its consequences.

3.1 Theorem ([B 1983; Theorem 5]). *Let $\boldsymbol{L}$ be a double Stone algebra having a non-empty bounded core $K(\boldsymbol{L})$. The following conditions are equivalent.*

(1) $\boldsymbol{L}$ is affine complete;

(2) $K(\boldsymbol{L})$ is an affine complete distributive lattice;

(3) No proper interval of $K(\boldsymbol{L})$ is Boolean.

3.2 Corollary ([B 1983, Corollary 6]). *Any Post algebra $\boldsymbol{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ of order 3 is an affine complete double Stone algebra.*

3.3 Corollary ([B 1983, Corollary 7]). *A finite double Stone algebra having a non-empty core is affine complete if and only if it is a Post algebra of order 3.*

In the first part we generalize 3.3 to a larger class of double S-algebras.

3.4 Theorem. *Let $\boldsymbol{L}$ be a quasi-modular double p-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. If $\boldsymbol{L}$ is affine complete then $K(\boldsymbol{L})$ is an affine complete lattice.*

Proof. Let $\boldsymbol{L}$ be affine complete. Similarly as in [B 1983], for any compatible function $f_K : K(\boldsymbol{L})^n \to K(\boldsymbol{L})$ we define a function $f : L^n \to L$ by

$$f(x_1, \ldots, x_n) = f_K((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l).$$

Obviously, $f \restriction K(L)^n = f_K$ and f preserves the congruences of $\boldsymbol{L}$. Thus by hypothesis, f can be represented by a polynomial $p_0(x_1, \ldots, x_n)$ of $\boldsymbol{L}$. From now we proceed as follows: we apply the formulas $(x \wedge y)^\star = x^\star \bigtriangledown y^\star$, $(x \vee y)^\star = x^\star \wedge y^\star$, $(x \wedge y)^+ = x^+ \vee y^+$ and $(x \vee y)^+ = x^+ \bigtriangleup y^+$ in $p_0(\tilde{\mathrm{x}})$ everywhere it is possible (see Example 3.5 below) and we obtain a polynomial $p_1(x_1, \ldots, x_n)$ of the partial algebra $(L; \vee, \wedge, \bigtriangledown, \bigtriangleup, ^\star, ^+, 0, 1)$ with two partial operations $\bigtriangledown$ and $\bigtriangleup$ defined only for elements of $B(\boldsymbol{L})$ and $\overline{B}(\boldsymbol{L})$, respectively.

Let $\tilde{\mathrm{x}} \in K(\boldsymbol{L})^n$. Then $f_K(\tilde{\mathrm{x}}) = f(\tilde{\mathrm{x}}) = p_1(\tilde{\mathrm{x}})$, and moreover, in $p_1(\tilde{\mathrm{x}})$ we can put $x_i^+ = 1$, $x_i^\star = 0$ for all $i = 1, \ldots, n$. Hence each part of the form $(...)^\star$ or $(...)^+$ in $p_1(\tilde{\mathrm{x}})$ can be rewritten as a constant symbol equal to 0 or 1 if in the brackets were variables only, or as a constant of $B(\boldsymbol{L})$ or $\overline{B}(\boldsymbol{L})$ if there was at least one constant symbol of $\boldsymbol{L}$ in the brackets (see again 3.5). Rewriting the polynomial $p_1(\tilde{\mathrm{x}})$ in this way, we obtain a polynomial $p_2(\tilde{\mathrm{x}})$ of the lattice $(L; \vee, \wedge, 0, 1)$. If $a_1, \ldots, a_m$ are all constant symbols in $p_2(\tilde{\mathrm{x}})$, then $p_2(\tilde{\mathrm{x}})$ can be expressed as a term $t(\tilde{\mathrm{x}}, \tilde{\mathrm{a}})$ of the algebra $(L; \vee, \wedge, 0, 1, a_1, \ldots, a_m)$. Now using the lattice homomorphism $\varphi : L \to [k, l]$, $\varphi(\tilde{\mathrm{x}}) = (x \vee k) \wedge l$ (note that in Section 2 we showed that the elements k, l are distributive), we get

$$f_K(\tilde{\mathrm{x}}) = \varphi(t(\tilde{\mathrm{x}}, \tilde{\mathrm{a}})) = t(\varphi(x_1), \ldots, \varphi(x_n), \varphi(a_1), \ldots, \varphi(a_m))$$

hence $f_K(\tilde{\mathrm{x}})$ can be represented by a polynomial of the lattice $K(\boldsymbol{L})$. The proof is complete. □

3.5 Example. We illustrate the method described in the proof of Theorem 3.4 on a simple example. Let $\boldsymbol{L}$ be a quasi-modular double p-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, $f_K(x_1, x_2, x_3)$ be a compatible function of the lattice $K(\boldsymbol{L})$ and let

$$p_0(x_1, x_2, x_3) = [(x_1 \wedge a^\star) \vee (x_2^+ \wedge b)]^+ \wedge x_3$$

be a polynomial of $\boldsymbol{L}$ representing the function $f(x_1, x_2, x_3) : L^3 \to L$ associated to the function f_K as in the proof of Theorem 3.4. In the first step, we get the polynomial

$$p_1(x_1, x_2, x_3) = [(x_1 \wedge a^\star)^+ \bigtriangleup (x_2^+ \wedge b)^+] \wedge x_3 = [(x_1^+ \vee a^{\star +}) \bigtriangleup (x_2^{++} \vee b^+)] \wedge x_3.$$

In the second step, by putting $x_1^+ = 1$, $x_2^{++} = 0$ we obtain a polynomial $p_2(x_1, x_2, x_3) = b^+ \wedge x_3$, which is a term $t(x_1, x_2, x_3, b^+)$ of the algebra $(L; \wedge, \vee, 0, 1, b^+)$. Finally, we get

$$f_K(x_1, x_2, x_3) = t(\varphi(x_1), \varphi(x_2), \varphi(x_3), \varphi(b^+)) = [(b^+ \vee k) \wedge l] \wedge [(x_3 \vee k) \wedge l],$$

hence $f_K(x_1, x_2, x_3)$ is a polynomial function of the lattice $K(\boldsymbol{L})$. □

3.6 Corollary. *Let $\boldsymbol{L}$ be a quasi-modular double p-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. If $K(\boldsymbol{L})$ contains a proper Boolean interval then $\boldsymbol{L}$ is not affine complete.*

Proof. The result follows from Theorem 3.4 and Proposition 2.3. □

3.7 Corollary. *A finite quasi-modular double p-algebra with a non-empty core is affine complete if and only if it is a regular double p-algebra.*

Proof. The necessity follows from 3.4 and 2.4. Since the variety of all regular double p-algebras is arithmetical, all its finite members are affine complete by 2.6. □

The next result generalizes 3.3 to a larger class of double S-algebras:

3.8 Corollary. *Let $\boldsymbol{L}$ be a quasi-modular double S-algebra with a non-empty finite core. Then $\boldsymbol{L}$ is affine complete if and only if $\boldsymbol{L}$ is a Post algebra of order 3.*

Proof. Affine completeness of $\boldsymbol{L}$ yields $|K(\boldsymbol{L})| = 1$ by 3.4 and 2.4. Hence $\boldsymbol{L}$ is a regular double p-algebra, so $\boldsymbol{L}$ is distributive. Thus $\boldsymbol{L}$ is a Post algebra of order 3. The converse follows from 3.2. □

3.9 Example. Take the lattice $\boldsymbol{M}_\infty$ of height 2 having an infinite number of atoms and any Boolean algebras $\boldsymbol{B}_1, \boldsymbol{B}_2$. The lattice $\boldsymbol{L}_1 = \boldsymbol{B}_1 \oplus \boldsymbol{M}_\infty \oplus \boldsymbol{B}_2$ ($\oplus$ means linear sum) obviously gives a quasi-modular double p-algebra with the core $K(\boldsymbol{L}_1) = \boldsymbol{M}_\infty$. (For $\boldsymbol{B}_1 = \boldsymbol{B}_2 = \mathbf{1}$, the quasi-modular double S-algebra $\boldsymbol{L}_1 = \mathbf{1} \oplus \boldsymbol{M}_\infty \oplus \mathbf{1}$ is depicted in Figure 1a.) By 3.6, $\boldsymbol{L}_1$ is not affine complete.

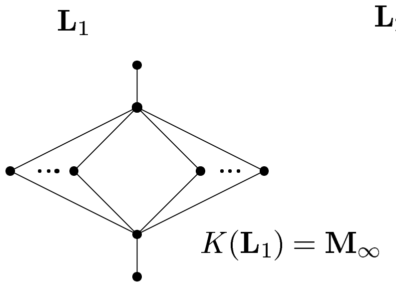

Figure 1a

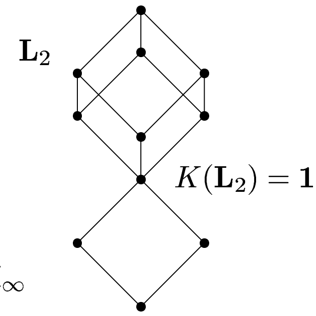

Figure 1b

Now let $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$ be finite Boolean algebras and $\boldsymbol{D}$ be a finite distributive lattice. Construct a quasi-modular double p-algebra $\boldsymbol{L}_2$ such that the zero of $\boldsymbol{B}_1$ will be identified with the unit of $\boldsymbol{D}$ and the zero of $\boldsymbol{D}$ will be identified with the unit of $\boldsymbol{B}_2$. By 3.7, $\boldsymbol{L}_2$ is affine complete if and only if $|D| = 1$. Hence the regular double p-algebra $\boldsymbol{L}_2$ in Figure 1b ($\boldsymbol{B}_1 = 2^3$, $\boldsymbol{B}_2 = 2^2$ and $\boldsymbol{D} = \mathbf{1}$) is affine complete. □

R. Beazer's technique employed in 3.1 was based on the fact that subdirectly irreducible double Stone algebras are very simple - the chains with at most four elements. This and also a 'good behaviour' of the operations * and $^+$ in double Stone algebras enabled him to find an exact form of the polynomials representing compatible functions. However, if we turn to a larger class of double S-algebras, which contains various subdirectly irreducible algebras, the situation becomes more complex and Beazer's method seems to be non-applicable.

Therefore we employ a technique based on the fact that in the class of quasi-modular double S-algebras with a non-empty bounded core, every element can be decomposed on two 'closed' elements and an element of the core - see the equation (1) in Section 2. Hence the elements from the range of any compatible function can be decomposed in this way, too. Since the set of all closed elements of a quasi-modular double S-algebra $\boldsymbol{L}$ forms a Boolean subalgebra $B(\boldsymbol{L})$ $(= \overline{B}(\boldsymbol{L}))$ and we assume that the core $K(\boldsymbol{L})$ is a bounded lattice, it would be natural to reduce the property of affine completeness of $\boldsymbol{L}$ into that of $B(\boldsymbol{L})$ $(= \overline{B}(\boldsymbol{L}))$ and $K(\boldsymbol{L})$. (This idea is, in fact, in accordance with a general idea of approaching quasi-modular p-algebras presented in [Ka 1980; p. 559].)

The main problem which arises when realizing the idea of the reduction is how to decompose a compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$ into (well-defined) functions of $B(\boldsymbol{L})$ ($= \overline{B}(\boldsymbol{L})$) and $K(\boldsymbol{L})$, respectively. As we shall see, the first part of this task concerning $B(\boldsymbol{L})$ ($= \overline{B}(\boldsymbol{L})$) can be quite easily managed, while the second is difficult so that we are forced to deal with partial functions of the lattice $K(\boldsymbol{L})$.

In the sequel, by $\boldsymbol{L} = (L; \vee, \wedge, ^\star, ^+, 0, 1)$ we always mean a quasi-modular double S-algebra having a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. In such case, the map $\varphi : L \to K(\boldsymbol{L})$, $\varphi(x) = (x \vee k) \wedge l$ is a lattice homomorphism. Further, we abbreviate $(\tilde{x} \vee k) \wedge l$ as $\varphi(\tilde{x})$.

To any compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$ we associate a partial function $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ as follows:

(2) $\quad f'_K(\varphi(\tilde{x}), \varphi(\tilde{x}^\star), \varphi(\tilde{x}^{\star\star}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})) = \varphi(f(\tilde{x})) \quad (\tilde{x} \in L^n)$

and f'_K is undefined elsewhere.

3.10 Lemma. *The function f'_K defined above is a well-defined partial compatible function of the lattice $K(\boldsymbol{L})$.*

Proof. To show that f'_K preserves the congruences of $K(\boldsymbol{L})$ where defined, let θ_K be a congruence of $K(\boldsymbol{L})$ and $\varphi(x_i^j) \equiv \varphi(y_i^j)\ (\theta_K)$ for $x_i, y_i \in L$, $i = 1, \ldots, n$, $j \in \{-2, -1, 0, 1, 2\}$ where $x^0 = x$, $x^1 = x^+$, $x^2 = x^{++}$, $x^{-1} = x^\star$, $x^{-2} = x^{\star\star}$. We associate to the congruence θ_K an equivalence relation θ_L on $\boldsymbol{L}$ defined by

(3) $\quad x \equiv y\ (\theta_L)$ if and only if $\varphi(x^j) \equiv \varphi(y^j)\ (\theta_K)$ for all $j \in \{-2, -1, 0, 1, 2\}$.

Since $\boldsymbol{L}$ is a quasi-modular double S-algebra, i.e. $B(\boldsymbol{L})$ ($= \overline{B}(\boldsymbol{L})$) is a sublattice of $\boldsymbol{L}$, one can easily verify that θ_L is a congruence on $\boldsymbol{L}$. Hence we have $x_i \equiv y_i\ (\theta_L)$, thus $f(\tilde{x}) \equiv f(\tilde{y})\ (\theta_L)$ as f is compatible on $\boldsymbol{L}$. Now again by (3) $\varphi(f(\tilde{x})) \equiv \varphi\ (f(\tilde{y}))\ (\theta_K)$, i.e. f'_K preserves the congruences of $K(\boldsymbol{L})$ where defined. To show that f'_K is well-defined, use the same method with $\theta_K = \triangle_{K(\boldsymbol{L})}$, the smallest congruence of K(**L**). $\square$

3.11 Definition. *We shall say that $\boldsymbol{L}$ satisfies an 'extension' property*

(E) *if for any compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$, the partial compatible function $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ defined by (2) can be extended to a total compatible function of the lattice $K(\boldsymbol{L})$.*

We will present two situations when the condition (E) is satisfied (and later on the third in 3.17).

3.12 Proposition. *If $\boldsymbol{L}$ is affine complete then $\boldsymbol{L}$ satisfies (E).*

Proof. Let f'_K be the function associated to a compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$. We define a function $f_1 : \boldsymbol{L}^n \to \boldsymbol{L}$ by $f_1(\tilde{x}) = \varphi(f(\tilde{x}))$. This is evidently compatible on $\boldsymbol{L}$, hence by hypothesis it can be represented by a polynomial $p(x_1, \ldots, x_n)$ of $\boldsymbol{L}$. Using de Morgan laws for * and $^+$, $p(\tilde{x})$ can be rewritten as $l(\tilde{x}, \tilde{x}^\star, \tilde{x}^{\star\star}, \tilde{x}^+, \tilde{x}^{++})$ for some lattice polynomial $l(x_1, \ldots, x_{5n})$ of $\boldsymbol{L}$. Further, using the homomorphism φ, one can show that for all $\tilde{x} \in L^n$

$$f'_K(\varphi(\tilde{x}), \ldots, \varphi(\tilde{x}^{++})) = \varphi(f(\tilde{x})) = f_1(\tilde{x}) = p(\tilde{x}) = l(\tilde{x}, \tilde{x}^\star, \tilde{x}^{\star\star}, \tilde{x}^+, \tilde{x}^{++}) = \\ = \varphi(l(\tilde{x}, \tilde{x}^\star, \tilde{x}^{\star\star}, \tilde{x}^+, \tilde{x}^{++})) = l'(\varphi(\tilde{x}), \ldots, \varphi(\tilde{x}^{++})),$$

where $l'(x_1, \ldots, x_{5n})$ is a polynomial of the lattice K(**L**). Then, of course, l' is the required total compatible extension of the partial function f'_K. $\square$

Let $\boldsymbol{L}$ be a quasi-modular double S-algebra with a non-empty core $K(\boldsymbol{L}) = [k, l]$ such that $K(\boldsymbol{L})$ is a Boolean lattice. Let $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ be the partial function from (2) and let S be its domain. Define a function $q(x_1, \ldots, x_{5n})$ on $K(\boldsymbol{L})$ by the rule

$$q(x_1, \ldots, x_{5n}) = \bigvee_{\tilde{a} \in S \cap \{k,l\}^{5n}} f'_K(a_1, \ldots, a_{5n}) \wedge y_1 \wedge \cdots \wedge y_{5n},$$

where $y_i = \begin{cases} x_i & \text{if } a_i = l \\ x'_i & \text{if } a_i = k. \end{cases}$

Obviously, $f'_K \equiv q$ identically on $S \cap \{k, l\}^{5n}$ and q is compatible on K($\mathbf{L}$). One can verify that $(K(\boldsymbol{L}); \vee, \wedge, k, l)$ with the partial compatible functions f'_K and q satisfy the assumptions of Lemma 2.7. Hence by the conclusion of Lemma 2.7 $f'_K \equiv q$ identically on S, thus the compatible function $q(x_1, \ldots, x_{5n})$ is a total extension of the function f'_K . So we have showed:

3.13 Proposition. *Let $\boldsymbol{L}$ be a quasi-modular double S-algebra with a non-empty core $K(\boldsymbol{L})$ which is a Boolean lattice. Then (E) is fulfilled in $\boldsymbol{L}$.*

Now we present a characterization theorem and its consequences.

3.14 Theorem. *Let $\boldsymbol{L}$ be a quasi-modular double S-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. Then $\boldsymbol{L}$ is affine complete if and only if $K(\boldsymbol{L})$ is an affine complete lattice and $\boldsymbol{L}$ satisfies (E).*

Proof. The necessity follows from Theorem 3.4 and Proposition 3.12. Now let $K(\boldsymbol{L})$ be an affine complete lattice and let $\boldsymbol{L}$ satisfy (E). Let $f : \boldsymbol{L}^n \to \boldsymbol{L}$ be a compatible function on $\boldsymbol{L}$. Since $\boldsymbol{L}$ satisfies the equation (1), we can write

(4) $f(\tilde{x}) = f(\tilde{x})^{++} \vee (f(\tilde{x})^{\star\star} \wedge (f(\tilde{x}) \vee k) \wedge l)$ for any $\tilde{x} = (x_1, \ldots, x_n) \in L^n$.

We shall show that the right side of (4) can be replaced by a polynomial of the algebra $\boldsymbol{L}$.

To replace $f(\tilde{x})^{\star\star}$ (and similarly $f(\tilde{x})^{++}$) in (4) by a polynomial of $\boldsymbol{L}$, we define a partial function $f'_B : B(\boldsymbol{L})^{2n} \to B(\boldsymbol{L})$ on the Boolean algebra $B(\boldsymbol{L})$ (= $\overline{B}(\boldsymbol{L})$) by

$$f'_B(\tilde{x}, \tilde{x}^{+}) = f(\tilde{x})^{\star\star} \quad (f(\tilde{x})^{++}) \quad (\tilde{x} \in L^n)$$

and f'_B is undefined elsewhere. Obviously, f'_B is well-defined since $x_i^\star = y_i^\star$, $x_i^+ = y_i^+$, $i = 1, \ldots, n$ yields $x_i \equiv y_i(\Phi)$ (the determination congruence), which follows $f(\tilde{x}) \equiv f(\tilde{y})(\Phi)$, thus $f(\tilde{x})^{\star\star} = f(\tilde{y})^{\star\star}$ $(f(\tilde{x})^{++} = f(\tilde{y})^{++})$. Further, for any congruence θ_B of B($\mathbf{L}$) we define an equivalence relation θ_L on L by $x \equiv y\ (\theta_L)$ if and only if $x^\star \equiv y^\star\ (\theta_B)$ and $x^+ \equiv y^+\ (\theta_B)$. Since $B(\boldsymbol{L})$ (= $\overline{B}(\boldsymbol{L})$) is a subalgebra of $\boldsymbol{L}$, θ_L is obviously a congruence of $\boldsymbol{L}$ containing θ_B. Using θ_L, one can easily show that f'_B preserves the congruences of $B(\boldsymbol{L})$ where defined. Let S be the domain of f'_B, i.e.

$$S = \{(\tilde{x}^\star, \tilde{x}^+);\ \tilde{x} \in L^n\} \subset B(\boldsymbol{L})^{2n}.$$

Note that if $(\tilde{a}, \tilde{b}) = (a_1, \ldots, a_n, b_1, \ldots, b_n) \in S \cap \{0, 1\}^{2n}$ then $a_i = 1$ implies $b_i = 1$.
One can easily verify that the function f'_B can be interpolated on the set $S \cap \{0, 1\}^{2n}$ by a Boolean polynomial function $b : B(\boldsymbol{L})^{2n} \to B(\boldsymbol{L})$ defined as follows:

$$b(x_1, \ldots, x_{2n}) = \bigvee_{(\tilde{a}, \tilde{b}) \in S \cap \{0,1\}^{2n}} (f'_B(\tilde{a}, \tilde{b}) \wedge x_1^{a_1} \wedge \cdots \wedge x_n^{a_n} \wedge x_{n+1}^{b_1} \wedge \cdots \wedge x_{2n}^{b_n}$$

where $x_i^1 = x_i$, $x_i^0 = x'_i = x_i^\star = x_i^+$. By Lemma 2.7, $f'_B \equiv b$ identically on the whole set S, hence for any $\tilde{x} \in L^n$ we have

$$f(\tilde{x})^{\star\star} \ (f(\tilde{x})^{++}) = f'_B(\tilde{x}^\star, \tilde{x}^+) = b(\tilde{x}^\star, \tilde{x}^+).$$

Therefore $f(\tilde{x})^{\star\star}$ (and similarly $f(\tilde{x})^{++}$) can be replaced in (4) by some polynomial $b_1(\tilde{x}^\star, \tilde{x}^+)$ ($b_2(\tilde{x}^\star, \tilde{x}^+)$) of the algebra $\boldsymbol{L}$.

Now we associate to $f(\tilde{x})$ the partial function $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ defined by (2). By (E) there exists a total compatible function $f_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ which extends f'_K. Affine completeness of $K(\boldsymbol{L})$ yields that f_K can be represented by a lattice polynomial $l(x_1, \dots, x_{5n})$. Hence in (4) we have for any $\tilde{x} \in L^n$,

$$f(\tilde{x}) = b_2(\tilde{x}^\star, \tilde{x}^+) \vee (b_1(\tilde{x}^\star, \tilde{x}^+) \wedge l(\varphi(\tilde{x}), \varphi(\tilde{x}^\star), \varphi(\tilde{x}^{\star\star}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})))$$

where $\varphi(\tilde{x})$ means $(\tilde{x} \vee k) \wedge l := ((x_1 \vee k) \wedge l, \dots, (x_n \vee k) \wedge l)$.

So f is a polynomial function of the algebra $\boldsymbol{L}$ and the proof is complete. □

We shall finally derive the Beazer characterization of double Stone algebras with a non-empty bounded core.

3.15 Lemma. *Let $\boldsymbol{L}$ be a double Stone algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$ and $x, y \in L$. Then for the lattice homomorphism $\varphi : L \to K(\boldsymbol{L})$, $\varphi(x) = (x \vee k) \wedge l$ we have*

$\varphi(x^\star) = \varphi(y^\star)$ *if and only if* $\varphi(x^{\star\star}) = \varphi(y^{\star\star})$ *and*

$\varphi(x^+) = \varphi(y^+)$ *if and only if* $\varphi(x^{++}) = \varphi(y^{++})$.

Proof. Let $\varphi(x^+) = \varphi(y^+)$. The identities $x^+ \wedge x^{++} = 0$ and $x^+ \vee x^{++} = 1$ imply $\varphi(x^+) \wedge \varphi(x^{++}) = k$, $\varphi(x^+) \vee \varphi(x^{++}) = l$ for any $x \in L$. Hence

$\varphi(x^{++}) = (\varphi(y^+) \wedge \varphi(y^{++})) \vee \varphi(x^{++}) = (\varphi(x^+) \vee \varphi(x^{++})) \wedge (\varphi(y^{++}) \vee \varphi(x^{++})) = \varphi(y^{++}) \vee \varphi(x^{++})$.

In the same way one can show $\varphi(y^{++}) = \varphi(y^{++}) \vee \varphi(x^{++})$. The converse statement as well as the proof of the first statement are analogous. □

3.16 Lemma. *Let $\boldsymbol{L}$ be a double Stone algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, $k < l$ and let $x \in L$ such that $\varphi(x^\star), \varphi(x^{\star\star}), \varphi(x^+), \varphi(x^{++}) \in \{k, l\}$ for the lattice homomorphism $\varphi : L \to K(\boldsymbol{L})$, $\varphi(x) = (x \vee k) \wedge l$. Then $\varphi(x^\star) = l$ implies $\varphi(x^{\star\star}) = k$ and analogously, $\varphi(x^+) = l$ implies $\varphi(x^{++}) = k$.*

Proof. Let $\varphi(x^\star) = l$. It is obvious that $\varphi(x^{\star\star}) = l$ would yield $l = \varphi(x^\star) \wedge \varphi(x^{\star\star}) = \varphi(0) = k$, a contradiction. Analogously, if $\varphi(x^+) = l = \varphi(x^{++})$, then $l = \varphi(x^+) \wedge \varphi(x^{++}) = \varphi(0) = k$, using the dual Stone identity $x^+ \wedge x^{++} = 0$. □

3.17 Proposition. *Let $\boldsymbol{L}$ be a double Stone algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$ such that $K(\boldsymbol{L})$ contains no proper Boolean interval. Then $\boldsymbol{L}$ satisfies (E).*

Proof. If $k = l$, then $\boldsymbol{L}$ is a Post algebra of order 3 and trivially, $\boldsymbol{L}$ satisfies (E). So let us further assume that $k < l$.

Let $f'_K : K(\boldsymbol{L})^{5n} \to K(\boldsymbol{L})$ be the partial compatible function associated to a compatible function $f : \boldsymbol{L}^n \to \boldsymbol{L}$. Let $T = \{(\tilde{x} \vee k) \wedge l, \dots, ((\tilde{x}^{++} \vee k) \wedge l);\ \tilde{x} \in L^n\}$ be the domain of f'_K. We shall show that f'_K can be interpolated on the set $T \cap \{k, l\}^{5n}$ by the following polynomial of the lattice K($\mathbf{L}$):

$$(5) \quad q(x_1, \dots, x_{5n}) = \bigvee_{\tilde{b} \in T \cap \{k,l\}^{5n}} (f'_K(b_1, \dots, b_{5n}) \wedge y_1 \wedge \dots \wedge y_{5n}),$$

where $y_i = \begin{cases} x_i, & \text{if } b_i = l \\ l, & \text{if } b_i = k. \end{cases}$

Let $\tilde{x}$ be any (fixed) vector from $T \cap \{k,l\}^{5n}$. If $\tilde{b} \neq \tilde{x}$ and $b_j \neq x_j$ for some $n < j \leq 5n$, then either $b_j = l,\ x_j = k$ and then $f'_K(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n} = k$ or $b_j = k,\ x_j = l$ and then by Lemmas 3.15, 3.16 there exists s, $n < s \leq 5n$ such that $x_s = k, b_s = l$, thus again $f'_K(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n} = k$. Hence it suffices to take into account in (5) only conjunctions $f'_K(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n}$ such that $b_i = x_i$ for all $n < i \leq 5n$ and moreover, $b_i \leq x_i$ for $1 \leq i \leq n$. So

$$q(x_1, \ldots, x_{5n}) = \bigvee_{\tilde{b} \in T \cap \{k,l\}^{5n},\ \tilde{b} \leq \tilde{x}} (f'_K(b_1, \ldots, b_n, x_{n+1}, \ldots, x_{5n}).$$

In next we show that $f'_K(\tilde{b}) \leq f'_K(\tilde{x})$ for any $\tilde{b} \in T \cap \{k,l\}^{5n}$ such that $b_i = x_i$ for $i = n+1, \ldots, 5n$ and $b_i \leq x_i$ for $i = 1, \ldots, n$. For $s = 1, \ldots, n$ denote $u_s = b_s$ if $b_s = x_s$, otherwise $u_s = u$. We get a unary compatible function $g : K(\boldsymbol{L}) \to K(\boldsymbol{L}),\ g(u) = f'_K(u_1, \ldots, u_n, x_{n+1}, \ldots, x_{5n})$ and we want to show that $f'_K(\tilde{b}) = g(k) \leq g(l) = f'_K(\tilde{x})$. Since $g(k) \equiv g(u)\ (\theta_{\text{lat}}(k,u))$ and $g(u) \equiv g(l)\ (\theta_{\text{lat}}(u,l))$ for any $u \in K(\boldsymbol{L})$, we get

$g(u) \vee u = g(k) \vee u$ and

$g(u) \wedge u = g(l) \wedge u.$

This means that for any $u \in [g(l), g(k) \vee g(l)]$, g(u) is the relative complement of u in this interval, which is therefore Boolean. By hypothesis ($K(\boldsymbol{L})$ contains no Boolean interval) this implies $g(k) \leq g(l)$, what was to be proved. Hence

$q(x_1, \ldots, x_{5n}) = f'_K(x_1, \ldots, x_{5n})$ for any $\tilde{x} \in T \cap \{k,l\}^{5n}$, and by applying Lemma 2.7, $q(x_1, \ldots, x_{5n})$ is the required total compatible extension of the partial function f'_K. □

From 3.14 and 3.17 we now get the Beazer result 3.1 and its consequences.

By Theorem 2.6 every finite algebra in an arithmetical variety, is affine complete. Hence, as Beazer concluded in [B 1983], any finite regular double p-algebra is affine complete. Afterwards he raised a question whether or not every infinite regular double p-algebra is affine complete.

Later on, K. Kaarli and A.F. Pixley [Kaa-P 1987] proved that an arithmetical variety of finite type is affine complete if and only if it has definable principal congruences and all its subdirectly irreducible members are finite and have no proper subalgebras. It is well-known that the variety of all regular double p-algebras has infinite subdirectly irreducible members, e.g. infinite Boolean algebras with a new unit adjoined. Moreover, every subdirectly irreducible regular double p-algebra having more than two elements has a proper subalgebra $\{0,1\}$. Hence the variety of all regular double p-algebras is not affine complete, and consequently, it must exist an infinite regular double p-algebra which is not affine complete.

Next we construct an example of an infinite regular double Stone algebra which is not affine complete. This has been motivated by techniques in [P 1993].

3.18 Example. Let $\mathbf{3} = (\{0,a,1\}; \vee, \wedge, ^{\star}, ^{+}, 0, 1)$ be the 3-element double Stone algebra with the core $\{a\}$. Let $\boldsymbol{L}$ be a subalgebra of $\mathbf{3}^{\omega}$ consisting of the sequences $\tilde{x} = (x_1, x_2, x_3, \ldots)$ which are 0 for all but finitely many n or are 1 for all but finitely many n. One can easily check that $\boldsymbol{L}$ is a regular double Stone algebra with an empty core.

Let $f : L \to L$ be defined componentwise as follows:

$$f(\tilde{x})_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ x_n & \text{if } n \text{ is even.} \end{cases}$$

We shall show that $f(\tilde{x}) \equiv f(\tilde{y})(\theta(\tilde{x}, \tilde{y}))$ for any $\tilde{x}, \tilde{y} \in L$. Let $\tilde{x}, \tilde{y} \in L$. By construction of $\boldsymbol{L}$, there is a natural number K such that

$\tilde{x} = (x_1, \ldots, x_K, x, x, x, \ldots), \quad x \in \{0, 1\}$

$\tilde{y} = (y_1, \ldots, y_K, y, y, y, \ldots), \quad y \in \{0, 1\}$.

For $x \in \{0, 1\}$, let $\underline{x}$ denote the constant sequence $(x, x, x, \ldots)$. Congruence-distributivity yields that congruences on finite subdirect products are 'skew-free', hence

$$\theta(\tilde{x}, \tilde{y}) = \theta(x_1, y_1) \times \cdots \times \theta(x_K, y_K) \times \theta(\underline{x}, \underline{y}).$$

Now it is clear that $f(\tilde{x}) \equiv f(\tilde{y})(\theta(\tilde{x}, \tilde{y}))$, thus f is a compatible function on $\boldsymbol{L}$. Suppose that $\boldsymbol{L}$ is affine complete. Then f is a polynomial function of $\boldsymbol{L}$, thus there is an $(m+1)$-ary term t of $\boldsymbol{L}$ and elements $\tilde{c}^1, \ldots, \tilde{c}^m \in L$ such that

$$f(\tilde{x}) = t(\tilde{x}, \tilde{c}^1, \ldots, \tilde{c}^m).$$

For the constants $\tilde{c}^1, \ldots, \tilde{c}^m$ there is a natural number N such that for $i = 1, \ldots, m$

$$\tilde{c}^i = (c_1^i, c_2^i, \ldots, c_N^i, c^i, c^i, c^i, \ldots), \quad c^i \in \{0, 1\}.$$

Take $\tilde{x} \in L$ such that $x_n = x_{n+1} = 0$ for some even $n > N$. Then we get

$$0 = x_n = f(\tilde{x})_n = t(0, c^1, \ldots, c^m) = f(\tilde{x})_{n+1} = 1,$$

a contradiction. Hence f cannot be represented by a polynomial of $\boldsymbol{L}$, so $\boldsymbol{L}$ is not affine complete. □

We finally turn to local affine completeness and we present 'local versions' of the previous results. As we shall see, much easier descriptions can be obtained.

3.19 Theorem. *Let $\boldsymbol{L}$ be a quasi-modular double p-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$. $\boldsymbol{L}$ is locally affine complete if and only if $\boldsymbol{L}$ is a regular double p-algebra.*

Proof. If $\boldsymbol{L}$ is locally affine complete then analogously as in the proof of 3.4 (the only difference is that the functions f_K, f are finite partial compatible functions in this case) one can show that $K(\boldsymbol{L})$ is a locally affine complete lattice, thus by 2.5, $|K(\boldsymbol{L})| = 1$. Hence $\boldsymbol{L}$ is a regular double p-algebra. The converse follows from 2.6. □

Corollary 3.20. *Post algebras of order 3 are the only locally affine complete quasi-modular double S-algebras with a non-empty bounded core.*

Proof. If $\boldsymbol{L}$ is a locally affine complete quasi-modular double S-algebra with a non-empty bounded core $K(\boldsymbol{L}) = [k, l]$, then by 3.19 $\boldsymbol{L}$ is a regular double Stone algebra (i.e. a three-valued Lukasiewicz algebra). Moreover, since $|K(\boldsymbol{L})| = 1$, $\boldsymbol{L}$ is a Post algebra of order 3. □

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