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NOTE ON ZEROS OF THE CHARACTERISTIC POLYNOMIAL OF BALANCED TREES

[1]Pavol Híc and [2]Roman Nedela

Abstract. A graph G is called integral if all the zeros of the characteristic polynomial $P(G;\lambda)$ are integers. A tree T is called balanced if the vertices at the same distance from the centre of T have the same degree. In the present paper we investigate the properties of the zeros of characteristic polynomials of balanced trees.

1.INTRODUCTION

A graph G is called *integral* if it has an integral spectrum, i.e. if all the zeros of the characteristic polynomial $P(G;\lambda)$ are integers. The identification of all integral graphs seems to be intractable. However, that of various families of integral graphs was investigated in [1, 3, 4, 5]. In [3] integral balanced trees were studied. A tree T is called *balanced* if the vertices at the same distance from the centre of T have the same degree. According to the parity of the diameter of a tree balanced trees split into two families. We shall code a balanced tree of diameter $2k$ by the sequence $(n_k, n_{k-1}, \dots, n_1)$, where n_j $j = 1, \dots, k$ denotes the number of succesors of a vertex at distance $k-j$ from the centre. In [3] it is proved that all zeros of the characteristic polynomial of the balanced tree with the sequence $(n_k, n_{k-1}, \dots, n_1)$ are zeros of the following recursively defined polynomial $P_k(x)$:

Definition 1.

$$P_0(x) = x$$

$$P_1(x) = x^2 - n_1$$

$$P_j(x) = x.P_{j-1}(x) - n_j.P_{j-2}(x)$$

where $j = 2, \dots, k$.

This fundamental observation allows us to reduce the study of spectra of balanced trees to the study of properties of polynomials $P_k(x)$. The aim of this note is to prove some basic results on the sequence $\{P_k(x)\}$ $k = 0, 1, \dots$. Results proved here are used in [3].

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2.RESULTS

In what follows we always assume that a sequence $\{n_j\}$ $j = 1, 2 \ldots$ of positive integers is given. It is easy to verify by induction on k, that for the terms of the sequence $\{P_k(x)\}$ of polynomials defined by Definition 1 the following statements hold:

Proposition 1.
a. $P_k(0) > 0$, *for* $k \equiv 3 \pmod 4$;
b. $P_k(0) < 0$, *for* $k \equiv 1 \pmod 4$;
c. $P_k(0) = 0$, *for* $k \equiv 0$ *or* $2 \pmod 4$;
d. $P_k(x)$ *is decreasing in point 0 for* $k \equiv 2 \pmod 4$;
e. $P_k(x)$ *is increasing in point 0 for* $k \equiv 0 \pmod 4$.

Now, let x_i be the smallest positive zero of polynomial $P_i(x)$ (i=1,2, $\ldots$). Denote by $\{x_k\}$ the sequence of the smallest positive zeros corresponding to the sequence $\{P_k(x)\}$. The following theorem shows that the above notation is correct.

Theorem 1. *For every* $i \geq 1$ *there exists a positive zero of the polynomial* $P_i(x)$. *Moreover, using the above notation the following statements hold:*
a. $\{x_{2k+1}\}$ *is decreasing;*
b. $\{x_{2k}\}$ *is decreasing;*
c. $x_{2k+2} > x_{2k+1}$, *for k=0,1,* $\ldots$

Proof. a. We shall proceed by induction on k. If k=0, then from $P_1(x) = x^2 - n_1$ we have $x_1 = \sqrt{n_1}$. If k=1, then $P_3(x) = x^4 - (n_1 + n_2 + n_3)x^2 + n_1.n_3$. By Proposition 1.a we get for $x \in < 0, x_1 >$

$$P_3(0) > 0, \tag{1}$$

$$P_3(x_1) = x_1 P_2(x_1) - n_3 P_1(x_1) = x_1[x_1 P_1(x_1) - n_2 x_1] = -n_2 x_1^2 < 0. \tag{2}$$

Using (1) and (2) we deduce that there exists $y \in (0, x_1)$ for which $P_3(y) = 0$. It follows $x_3 < x_1$.

Now, let $x_1 > x_3 > \cdots > x_{2k-1} > 0$. We shall investigate the polynomial $P_{2k+1}(x)$. According to whether $2k + 1 \equiv 3$ or $1 \pmod 4$ we distinguish two cases (see Proposition 1):
Case 1. $P_{2k+1}(0) > 0$;
Case 2. $P_{2k+1}(0) < 0$.
We shall deal only with the Case 1 . The proof in the case 2 can be done similarly.

If $P_{2k+1}(0) > 0$, then by Proposition 1, $P_{2k-1}(0) < 0$ and it follows, that for every $x \in (0, x_{2k-1})$ we have $P_{2k-1}(x) < 0$ because of x_{2k-1} is the smallest positive zero of

$P_{2k-1}(x)$. Hence,

$$\begin{aligned} P_{2k+1}(x_{2k-1}) &= x_{2k-1}P_{2k}(x_{2k-1}) - n_{2k+1}P_{2k-1}(x_{2k-1}) = \\ &= x_{2k-1}P_{2k}(x_{2k-1}) = \\ &= x_{2k-1}[x_{2k-1}P_{2k-1}(x_{2k-1}) - n_{2k}P_{2k-2}(x_{2k-1})] = \\ &= -x_{2k-1}n_{2k}P_{2k-2}(x_{2k-1}). \end{aligned}$$

Further, substituting $x = x_{2k-1}$ into the equality

$$P_{2k-1}(x) = xP_{2k-2}(x) - n_{2k-1}P_{2k-3}(x)$$

we get

$$0 = x_{2k-1}P_{2k-2}(x_{2k-1}) - n_{2k-1}P_{2k-3}(x_{2k-1})$$

and

$$n_{2k-1}P_{2k-3}(x_{2k-1}) = x_{2k-1}P_{2k-2}(x_{2k-1}).$$

By Proposition 1 and the fact $x_{2k-1} \in (0, x_{2k-3})$ the left part of the last equation is positive and it follows

$$P_{2k-2}(x_{2k-1}) > 0.$$

Hence,

$$P_{2k+1}(x_{2k-1}) = -x_{2k-1}n_{2k}P_{2k-2}(x_{2k-1}) < 0.$$

Since $P_{2k+1}(0) > 0$, there exists $x_{2k+1} \in (0, x_{2k-1})$ which is a zero of $P_{2k+1}(x)$.

b. We shall proceed by induction on k. If k=1, then from $P_2(x) = x^3 - (n_1 + n_2).x$ it follows $x_2 = \sqrt{n_1 + n_2}$. If k=2, then $P_4(x) = x.P_3(x) - n_4.P_2(x)$. By Proposition 1.c and 1.e for $x \in < 0, x_2 >$ the polynomial $P_4(x)$ satisfies the following properties:

$$P_4(x^+) > 0, \text{ for some } x^+ \in O_{\epsilon^+}(0), \tag{3}$$

$$\begin{aligned} P_4(x_2) &= x_2P_3(x_2) - n_4P_2(x_2) = x_2[x_2P_2(x_2) - n_3P_1(x_2)] = \\ &= -x_2n_3P_1(x_2) = -x_2n_3(x_2^2 - n_1) < 0. \end{aligned} \tag{4}$$

Here $O_{\epsilon^+}(0)$ denotes a sufficiently small right open neighbourhood of 0. Using (3) and (4) we see that there exists $x_4 \in (0, x_2)$ such that x_4 is a zero of $P_4(x)$.

Now, let the statement hold for every $n < k$ i.e.

$$0 < x_{2k-2} < x_{2k-4} < \cdots < x_4 < x_2.$$

Consider the polynomial $P_{2k}(x)$. According to Proposition 1.d and 1.e we have to distinguish two cases:

$$k \text{ is odd and } P_{2k}(x^+) < 0, \text{ for } x^+ \in O_{\epsilon^+}(0); \tag{5}$$

(6) $$k \text{ is even and } P_{2k}(x^+) > 0, \text{ for } \ x^+ \in O_{\epsilon^+}(0).$$

We shall examine only Case (6). Case (5) can be handled in a similar way. Substituting $x = x_{2k-2}$ into the equation

$$P_{2k}(x) = xP_{2k-1}(x) - n_{2k}P_{2k-2}(x)$$

we have
(7)

$$\begin{aligned} P_{2k}(x_{2k-2}) &= x_{2k-2}P_{2k-1}(x_{2k-2}) = \\ &= x_{2k-2}[x_{2k-2}P_{2k-2}(x_{2k-2}) - n_{2k-1}P_{2k-3}(x_{2k-2})] = \\ &= -x_{2k-2}n_{2k-1}P_{2k-3}(x_{2k-2}) \end{aligned}$$

On the other hand, using the substitution $x = x_{2k-2}$ in the equation

$$P_{2k-2}(x) = xP_{2k-3}(x) - n_{2k-2}P_{2k-4}(x)$$

we have

(8) $$x_{2k-2}P_{2k-3}(x_{2k-2}) = n_{2k-2}P_{2k-4}(x_{2k-2})$$

Hence, using Proposition 1.d and 1.e $P_{2k-4}(x_{2k-2}) > 0$. Combining (7) and (8) we obtain

$$P_{2k}(x_{2k-2}) < 0.$$

According to $P_{2k}(x^+) > 0$ for $x^+ \in O_{\epsilon^+}(0)$ there exists $x_{2k} \in (0, x_{2k-2})$ such that x_{2k} is a zero of the polynomial $P_{2k}(x)$.

c. The statement is trivial for k=0, since $x_2 = \sqrt{n_1 + n_2} > \sqrt{n_1} = x_1$. Now, let the statement hold for every $n < k$; i.e.

(9) $$x_{2k} > x_{2k-1}$$

Suppose $n = k + 1$. We shall restrict ourselves to the case $P_{2k+2}(x^+) > 0$. The case $P_{2k+2}(x^+) < 0$ can be handled similarly. By Definition 1 we have

$$P_{2k+2}(x) = xP_{2k+1}(x) - n_{2k+2}P_{2k}(x)$$

According to Theorem 1.a and (9)

$$x_{2k+1} < x_{2k-1} < x_{2k}.$$

By Proposition 1.d and by the assumption $P_{2k+2}(x^+) > 0$. It follows $P_{2k}(x) < 0$ for every $x \in (0, x_{2k+1})$. On the other hand, $P_{2k+1}(x) > 0$ for every $x \in (0, x_{2k+1})$. Hence, $P_{2k+2}(x) > 0$, for $x \in (0, x_{2k+1})$ and it follows that the smallest zero x_{2k+2} of the polynomial $P_{2k+2}(x)$ is greater than x_{2k+1}. $\square$

Corollary 1. *If the polynomial $P_{2k+1}(x)$ has only integer zeros then 1 is not the zero of $P_{2k}(x)$.*

Proof. Let $x_{2k} = 1$ be the smallest zero of $P_{2k}(x)$. Using Theorem 1. a; and c;

$$1 = x_{2k} > x_{2k-1} > x_{2k+1} > 0,$$

for every $k = 1, 2, \ldots$. However, this contradicts the fact that x_{2k+1} is integer. □

A sequence $\{n_i\}_{i \in I}$, where I is an interval (finite or infinite) of integers ≥ 1 is called *integral* if the corresponding polynomials $P_j(x)$ $j = 1, \ldots$ have only integral zeros.

Corollary 2. *There is no infinite integral sequence.*

Corollary 3. *Every integral sequence $(n_k, n_{k-1}, \ldots, n_1)$ has a length $\leq min\{2\sqrt{n_1}, \sqrt{n_1 + n_2}\}$.*

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[1]Department of Mathematics, Informatics and Physics,

Faculty of Education,

Trnava University, Hornopotočná 23, 918 43 Trnava, Slovakia,

E-mail address: phic@ uvt.mtf.stuba.sk

[2]Department of Mathematics, Faculty of Natural Sciences,

Matej Bel University, Tajovského 40, 974 00 Banská Bystrica, Slovakia

E-mail address: nedela@ bb.sanet.sk

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