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THE EDGE DISTANCE IN SOME FAMILIES OF GRAPHS II

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Abstract. The edge distance between graphs is defined by the equality $d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + ||V_1| - |V_2||$ where $|A|$ is the cardinality of A and $E_{1,2}$ is the edge set of a maximal common subgraph of G_1 and G_2. Further, $\operatorname{diam} F_{p,q} = max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$ where $F_{p,q}$ denotes the set of all graphs with p vertices and q edges. In the paper we prove that for $p \in \{7, 8, 9\}$ $\operatorname{diam} F_{p,p+2} = 2p - 6$ and for $p \geqq 19$ and $p + 3 \leqq q \leqq \frac{3p}{2}$ $\operatorname{diam} F_{p,q} = 2q - 12$.

1. Preliminaries

A graph $G = (V, E)$ consists of a non-empty finite vertex set V and an edge set E. In this paper we consider undirected graphs without loops and multiple edges. A subgraph H of the graph G is a graph obtained from G by deleting some edges and vertices; notation: $H \subseteq G$. By $\triangle(G)$ we denote the maximal degree of vertices of the graph G. A graph G is a common subgraph of graphs G_1, G_2 if there exist graphs H_1, H_2 such that $H_1 \subseteq G_1$, $H_2 \subseteq G_2$ and $H_1 \cong G$, $H_2 \cong G$.

A maximal common subgraph is a common subgraph which contains the maximal number of edges.

The edge distance of the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is defined (see [3]) by

$$d(G_1, G_2) = |E_1| + |E_2| - 2|E_{1,2}| + ||V_1| - |V_2|| \tag{1}$$

where $|E_1|$, $|E_2|$, $|V_1|$, $|V_2|$ are the cardinalities of the edge sets and the vertex sets, respectively and $|E_{1,2}|$ is the number of edges of a maximal common subgraph $G_{1,2}$ of the graphs G_1 and G_2.

Throughout this paper, by $F_{p,q}$ we denote the set of all graphs with p vertices and q edges. Further, $\operatorname{diam} F_{p,q} := \max\{d(G_1, G_2); G_1, G_2 \in F_{p,q}\}$. If $\operatorname{diam} F_{p,q} = d(G, H)$ and $c_{p,q}$ is the number of edges of a maximal common subgraph of the graphs G, H then

$$\operatorname{diam} F_{p,q} = 2q - 2c_{p,q}. \tag{2}$$

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Denote by v a firmly chosen vertex of a maximal degree in the considered graph G and by $v_1, v_2, \dots, v_k$ the vertices adjacent to v (here $k = \triangle(G)$). Denote $U := \{v_1, v_2, \dots, v_k\}$ and $U' := V - \{v, v_1, \dots, v_k\}$. The subgraph of the graph G induced by the vertex set X ($X \subset V$) we denote by $G(X)$ and the set of its edges by $E(G(X))$ or briefly by $E(X)$. The subgraph of the graph G which contains all edges with one vertex in the set U and the other in the set U' is denoted by $G(U, U')$ and the set of its edges by $E(U, U')$.

This paper is a continuation of the articles [1] and [2] but it can be read independently on them.

2. Diameter of $F_{p,p+2}$

In [1] diam $F_{p,p+2}$ is determined for all p except of $p \in \{7, 8, 9\}$. In this section of the paper we will show that diam $F_{p,p+2} = 2p - 6$ for $p \in \{7, 8, 9\}$.

Lemma 2.1. *Let $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8\}$. If the graph G_1 without its isolated vertices is a subgraph of the graph K_5 then $|E_{1,2}| \geqq 5$.*

Proof. It si sufficient to show that G_2 has a subgraph with 5 vertices and with at least 5 edges. It is easy to check this fact by distinguishing the following cases:

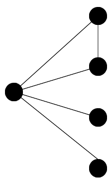
a) $\triangle(G_2) = 3$

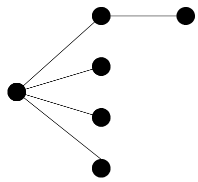
- (i) $|E(U)| \geqq 2$
- (ii) $|E(U)| = 1$
  If $|E(U, U')| = 0$ then $G_2(U')$ is the complete graph K_4.
- (iii) $|E(U)| = 0$
  If every vertex from U' has degree at most 1 in $G_2(U, U')$ then $G_2(U')$ has at least 3 edges.

b) $\triangle(G_2) \geqq 4$

- (i) $|E(U)| \geqq 1$
- (ii) There is a vertex in U' whose degree in $G_2(U, U')$ is at least 2
- (iii) If none of the previous two cases is valid then $G_2(U')$ is the complete graph K_3 and $|E(U, U')| = 3$. □

Lemma 2.2. *Let $G \in F_{p,p+2}$, $p \in \{7, 8, 9\}$ and $\triangle(G) \geqq 4$. Then G contains at least one of the graphs H_1, H_2 (Fig. 2.1).*



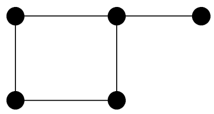


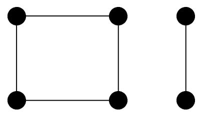
H_1 H_2

Fig. 2.1

Proof. Suppose that $|E(U)| = 0$ and simultaneously $|E(U,U')| = 0$. Then $|E(U')| = |U'| + 3$ which is impossible since $|U'| \leqq 4$. □

Lemma 2.3. *Let $G \in F_{p,p+2}$, $p \in \{7,8,9\}$, $\triangle(G) = 4$ and $|E(U)| = |E(U')| = 0$. Then G contains the graphs H_3 and H_4 (Fig. 2.2).*





H_3 H_4

Fig. 2.2

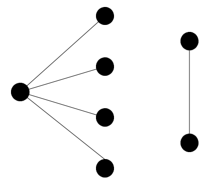
Proof. Since $|E(U,U')| = |U'| + 3$, at least one of the following holds:

(i) there are at least 2 vertices of degree at least 2 in U'

(ii) there is a vertex of degree at least 3 and another vertex of non-zero degree in U' □

Lemma 2.4. *If $G_1, G_2 \in F_{p,p+2}$, $p \in \{7,8,9\}$ and $\triangle(G_1) = \triangle(G_2) = 4$ then $|E_{1,2}| \geqq 5$.*

Proof. In view of Lemma 2.2 it is sufficient to consider the case when exactly one of the graphs G_1, G_2 contains the graph H_1 and exactly one of them contains the graph H_2. Without loss of generality we can assume that the graph G_1 contains the graph H_1 and the graph G_2 contains the graph H_2. According to Lemma 2.1 we can also assume that the graph G_1 without its isolated vertices is not a subgraph of the graph K_5. According to the above facts, we have $|E(G_1(U'))| \geqq 1$. If $|E(G_2(U'))| \geqq 1$ then a common subgraph is the graph H_5 (Fig. 2.3).



H_5

Fig. 2.3

So let $|E(G_2(U'))| = 0$. According to Lemma 2.3 the graph G_2 contains the graphs H_3 and H_4. If $|E(G_1(U))| \geqq 3$ then the graph G_1 contains the graph H_3. In the opposite case we have $|E(G_1(U'))| \geqq |U'| + 1$ whence $|U'| = 4$. Then the graph G_1 contains the graph H_4 and the proof is finished. □

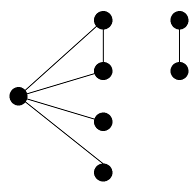
Lemma 2.5. *Let* $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$, $\triangle(G_1) \geqq 4$ *and* $\triangle(G_2) \geqq 5$. *Then* $|E_{1,2}| \geqq 5$.

Proof. The statement of the lemma is trivial if $\triangle(G_1) > 4$. Two cases are possible:

a) $|E(G_1(U, U'))| \geqq 1$,
b) $|E(G_1(U, U'))| = 0$.

In the case a) a common subgraph is the graph H_2. Obviously, if the graph G_2 did not contain the graph H_2 then it would be $|E(G_2(U))| = 0$ and $|E(G_2(U, U'))| = 0$. This yields $|E(G_2(U'))| > |U'|$ and it is impossible.

In the case b) we can assume according to Lemmas 2.1 and 2.2 that G_1 contains the graph H_6 (Fig. 2.4). Clearly, the statement holds if $|E(G_2(U))| \geqq 1$ or $|E(G_2(U, U'))| \geqq 1$. Now it is sufficient to realize that at least one of these inequalities must be valid for the graph G_2. □



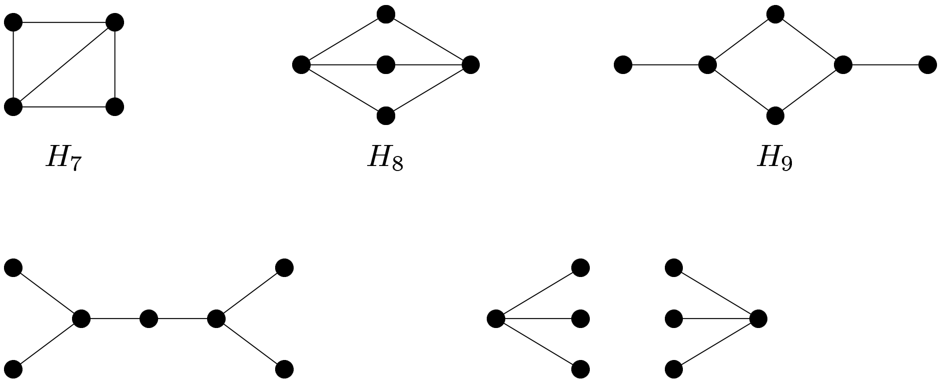
H_6

Fig. 2.4

Let $F = \{G \in F_{7,9} \cup F_{8,10} \cup F_{9,11}; \triangle(G) = 3\}$. Let us consider the next subsets of F:

F_1 contains all graphs which have the subgraph H_7 (Fig. 2.5),
F_2 contains all graphs from $F - F_1$ which have the subgraph H_8 (Fig. 2.5),

F_3 contains all graphs from $F - F_1$ which have the subgraph H_9 (Fig. 2.5),
F_4 contains all graphs from $F - F_1$ which have the subgraph H_{10} (Fig. 2.5),
F_5 contains all graphs from $F - F_1$ which have the subgraph H_{11} (Fig. 2.5).



H_7 H_8 H_9

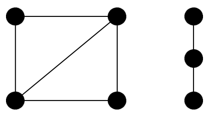
H_{10} H_{11}

Fig. 2.5

Lemma 2.6. *$F_1 \cup F_2 \cup F_3 \cup F_4 \cup F_5 = F$.*

Proof. Obviously, each graph $G \in F$ has at least 4 vertices of degree 3. If there are no two non-adjacent vertices of degree 3 then G contains H_7. If there are two non-adjacent vertices of degree 3 then G must contain at least one of the graphs H_8, H_9, H_{10} and H_{11}. □

Lemma 2.7. *If $G \in F_1$ then G has the subgraph H_{12} in Fig. 2.6.*

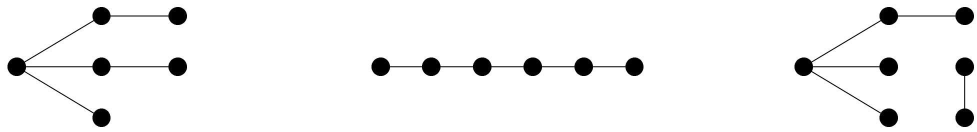


H_{12}

Fig. 2.6

Proof. We know that the graph G has the subgraph H_7 and apart from the vertices of this subgraph G it has other $p - 4$ vertices. The subgraph of the graph G induced by these $p - 4$ vertices has at least $p - 5$ edges and consequently it has a vertex of degree at least 2. □

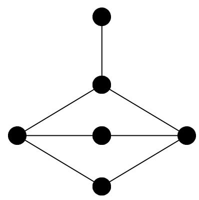
Lemma 2.8. *If $G \in F_2$ then it contains the graphs H_3, H_4, H_{13} and H_{14} (Figs. 2.2, 2.7). Moreover, if $p \neq 7$ then G contains also the graph H_{15} (Fig. 2.7).*



H_{13} H_{14} H_{15}

Fig. 2.7

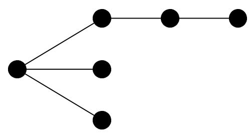
Proof. The component of the graph G which contains the graph H_8 must also contain a vertex which does not belong to H_8, i.e. G contains the graph H_{16} (Fig. 2.8). Obviously, if $p \neq 7$ then G has an edge which is not incident with any vertex of the subgraph H_8. □



H_{16}

Fig. 2.8

Lemma 2.9. *If $G \in F_3$ then it contains the graphs H_3, H_4, H_{13} and H_{17} (Figs. 2.2, 2.7, 2.9).*



H_{17}

Fig. 2.9

Proof. Obviously, G contains the graphs H_3 and H_{17}. Since G contains the graph H_9 and has at least 9 edges, it has an edge which is not incident to any vertex of the circle in the considered subgraph H_9. Therefore G contains H_4. Apart from the edges of H_9 there must exist another edge in G which is incident with at least one vertex of H_9. It follows immediately that G contains H_{13}. □

Lemma 2.10. *If $G \in F_4 \cup F_5$ then G contains the graphs H_{13}, H_{17}, H_{18} and H_{19}. (Figs. 2.7, 2.9, 2.10).*

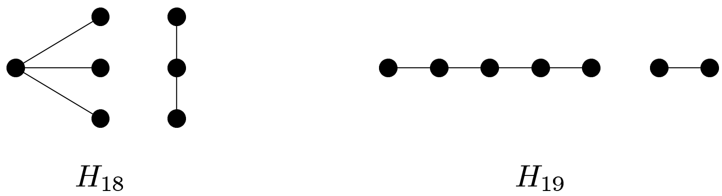


Fig. 2.10

Proof.

a) $G \in F_4$
Obviously, G has the subgraphs H_{17} and H_{18}. Further, the graph G has at least 3 edges except of the edges of the subgraph H_{10}. Each of these edges is incident with at least one vertex of the subgraph H_{10}. It follows that G has the subgraphs H_{13} and H_{19}.

b) $G \in F_5$
Obviously, G has the subgraph H_{18}. Since G can not have two non-trivial components it contains the subgraph H_{17}. If the graph G contains at least one of the subgraphs H_{20}, H_{21} (Fig. 2.11) then it also has the graphs H_{13} and H_{19}.

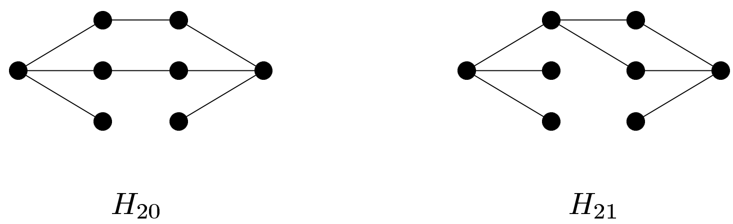
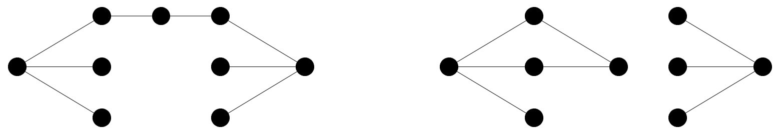


Fig. 2.11

In the opposite case p=9 and G contains at least one of the subgraph H_{22}, H_{23} (Fig. 2.12). Hence G has the subgraph H_{19} and it is easy to verify that it also has the subgraph H_{13}. □



H_{22} H_{23}

Fig. 2.12

Lemma 2.11. *If $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$ and $\triangle(G_1) = \triangle(G_2) = 3$ then $|E_{1,2}| \geqq 5$.*

Proof. The statement follows straightforwardly from Lemmas 2.6 - 2.10. □

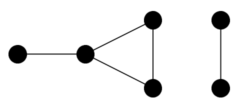
Lemma 2.12. *If $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$, $\triangle(G_1) = 3$, $\triangle(G_2) = 4$ then $|E_{1,2}| \geqq 5$.*

Proof. According to Lemma 2.1 we can assume that G_2 without its isolated vertices is not a subgraph of K_5. We distinguish several cases:

a) $G_1 \in F_1$
According to Lemma 2.7 G has the subgraph H_{12}. We distinguish 4 cases for the graph G_2:

(i) $|E(U)| \geqq 3$
A common subgraph is the graph H_7.

(ii) $|E(U)| = 2$
If the considered edges are adjacent then a common subgraph is H_7. If they are not adjacent then a common subgraph is H_{24} (Fig. 2.13).



H_{24}

Fig. 2.13

(iii) $|E(U)| = 1$
If $|E(U')| \geqq 1$ then a common subgraph is H_{24}. If $|E(U')| = 0$ then there exists a vertex of degree at least 2 in U' and at least one of the graphs H_7 and H_{24} is a common subgraph.

(iv) $|E(U)| = 0$
A common subgraph is H_{18}.

b) $G_1 \in F_2$
We show that G_2 contains at least one of the graphs from Lemma 2.8. We can assume that $|E(U)| \leqq 2$ and each vertex from U' has degree at most 1 in $G_2(U, U')$

(in opposite case a common subgraph is H_3). From this it follows that $|E(U')| \geqq 1$. If $p = 7$ then a common subgraph is H_{14}. So let $p \neq 7$. If $|E(U)| \geqq 1$ then a common subgraph is H_{15}. If $|E(U)| = 0$ then there exists a vertex of degree at least 2 in $G_2(U')$ and if moreover $|E(U, U')| \geqq 2$ then again H_{15} is a common subgraph. In the opposite case $p = 9$, $|E(U')| = 6$ and a common subgraph is H_4.

c) $G_1 \in F_3$
We show that G_2 contains at least one of the graphs from Lemma 2.9. If $|E(U, U') \cup E(U')| > |U'|$ then at least one vertex from U' has degree at least 2 in G_2. Hence it is easy to verify that at least one of the graphs H_3, H_4, H_{17} is a common subgraph. In the opposite case $|E(U)| \geqq 3$ and a common subgraph is H_3.

d) $G_1 \in F_4 \cup F_5$
We show that G_2 contains at least one of the graphs from Lemma 2.10 (i.e. H_{13}, H_{17}, H_{18}, H_{19}).

(i) $|E(U)| \geqq 1$
If $|E(U, U')| \geqq 1$ then a common subgraph is H_{13} or H_{17}. If $|E(U, U')| = 0$ then according to Lemma 2.1 we can assume that $|E(U')| \geqq 1$ and a common subgraph is H_{18} or H_{19}.

(ii) $|E(U)| = 0$
It is easy to check that G_2 contains the graph H_{18}. □

Lemma 2.13. *If* $G_1, G_2 \in F_{p,p+2}$, $p \in \{7, 8, 9\}$, $\triangle(G_1) = 3$ *and* $\triangle(G_2) \geqq 5$ *then* $|E_{1,2}| \geqq 5$.

Proof. We distinguish several cases:

a) $G_1 \in F_1$
We show that G_2 contains a subgraph of the graph H_{12} having 5 edges.

(i) $|E(U)| \leqq 2$
In this case $|E(U, U') \cup E(U')| \geqq |U'| + 1$ and therefore some vertex from U' has degree at least 2 in G_2. A common subgraph is H_{18}.

(ii) $|E(U)| \geqq 3$
If there exist two adjacent edges in $G_2(U)$ then a common subgraph is H_7. If there are no such edges then the common subgraph is H_{24}.

b) $G_1 \in F_2 \cup F_3$
According to Lemmas 2.8 and 2.9 it is sufficient to show that G_2 contains at least one of the graphs H_3 and H_{13}. Obviously, this is true if $|E(U)| \geqq 2$ or if some vertex from U' has degree at least 2 in $G_2(U, U')$. In the opposite case we have $|E(U')| \geqq 2$ and it follows that $p = 9$ and $|E(U, U')| \geqq 2$. Now it is easy to verify that G_2 contains H_3 or H_{13}.

c) $G_1 \in F_4 \cup F_5$
We show that G_2 contains at least one of the graphs H_{13}, H_{17} and H_{18} (see Lemma 2.10).

(i) $|E(U)| \geqq 2$
In this case G_2 contains at least one of the graphs H_{13}, H_{17}.

(ii) $|E(U)| \leqq 1$

In this case at least one vertex in U' has degree at least 2. Hence G_2 contains the graph H_{18}. $\square$

Theorem 2.14. $\operatorname{diam} F_{p,p+2} = 2p - 6$ *for* $p \in \{7, 8, 9\}$.

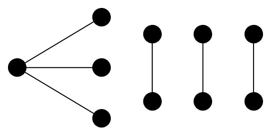
Proof. By Lemmas 2.4, 2.5, 2.11, 2.12 and 2.13 it suffices to find two graphs $G_1, G_2 \in F_{p,p+2}$ with $|E_{1,2}| = 5$. Such graphs are depicted in Fig. 2.14 (one component of G_1 is a circle). $\square$

Fig. 2.14

3. Diameter of $F_{p,p+3}$

In this section of the paper we will determine $\operatorname{diam} F_{p,p+3}$ for $p \geqq 19$.

Lemma 3.1. *If* $G \in F_{p,p+3}$, $p \geqq 17$ *and* $\triangle(G) = 3$ *then* G *contains the graph* H_{25} *(Fig. 3.1).*

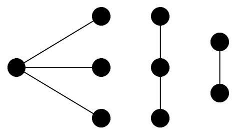


H_{25}

Fig. 3.1

Proof. Let v be a vertex of degree 3 in the graph G. If $p \geqq 7$ then there exists an edge w_1w_2 such that $w_i \notin \{v, v_1, v_2, v_3\}$, $i = 1, 2$. If $p \geqq 12$ then there exists an edge w_3w_4 such that $w_i \notin \{v, v_1, v_2, v_3, w_1, w_2\}$, $i = 3, 4$. If $p \geqq 17$ then there exists an edge w_5w_6 such that $w_i \notin \{v, v_1, v_2, v_3, w_1, w_2, w_3, w_4\}$, $i = 5, 6$. Thus G contains the graph H_{25}. □

Lemma 3.2. *If $G \in F_{p,p+3}$, $p \geqq 14$ and $\triangle(G) = 3$ then G contains the graph H_{26} (Fig. 3.2).*

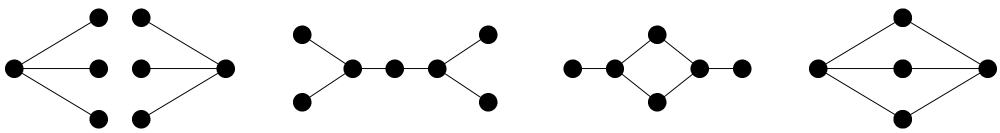


H_{26}

Fig. 3.2

Proof. If $p - 4 < 2(p - 6)$ i.e. $p > 8$ then $G(U')$ has a vertex w_1 of degree at least 2. Let w_2, w_3 are the vertices adjacent to w_1 in $G(U')$. If $p \geqq 14$ then there exists an edge such that neither of its vertices belongs to $\{v, v_1, v_2, v_3, w_1, w_2, w_3\}$ i.e. G contains the graph H_{26}. □

Lemma 3.3. *Let $G \in F_{p,p+3}$ and $\triangle(G) = 3$. Then G contains at least one of the graphs H_{27}, H_{28}, H_{29} and H_{30} (Fig. 3.3).*

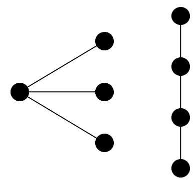


H_{27} H_{28} H_{29} H_{30}

Fig. 3.3

Proof. G has at least 6 vertices of degree 3. We thus get that there are two non-adjacent vertices of degree 3 in G. All four possible cases for these two vertices are depicted in the Fig. 3.3. □

Lemma 3.4. *Let $G \in F_{p,p+3}$ and $\triangle(G) = 3$. If G has at least two components with more edges than vertices then it contains the graphs H_{31} (Fig. 3.4) and H_{27}.*



H_{31}

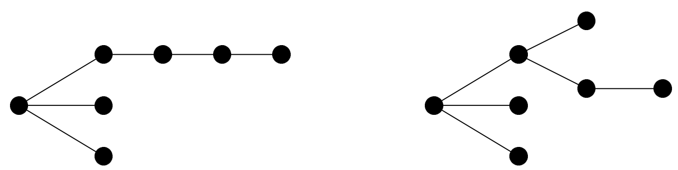
Fig. 3.4

Proof. Obviously, each of the considered components of the graph G contains a vertex of degree 3. Further it is sufficient to realize that such component has more than 3 edges. □

Lemma 3.5. *Let $G \in F_{p,p+3}$, $\triangle(G) = 3$ and G have only one component H having more edges than vertices.*

(a) *If H has the subgraph H_{27} then it contains at least one of the graphs H_{32} (Fig. 3.5) and H_{31}.*

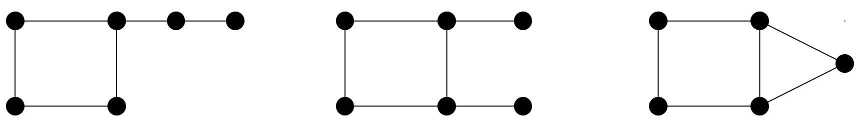
(b) *If H has the subgraph H_{28} then it contains at least one of the graphs H_{32}, H_{33} (Fig. 3.5).*



H_{32} H_{33}

Fig. 3.5

(c) *If H has the subgraph H_{29} then it contains at least one of the graphs H_{34}, H_{35}, H_{36} (Fig. 3.6).*

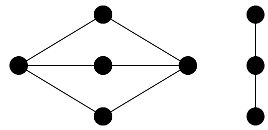


H_{34} H_{35} H_{36}

Fig. 3.6

Proof. The graph H has at least 3 edges more than vertices.
(a) and (b): It is sufficient to realize that H contains more than 6 edges.
(c): If H has exactly 6 vertices then it contains the graph H_{36}. If H has at least 7 vertices then it contains at least one of the graphs H_{34}, H_{35}. □

Lemma 3.6. *Let $G \in F_{p,p+3}$, $p \geqq 8$ and $\triangle(G) = 3$. If G contains the graph H_{30} then it contains the graph H_{37} (Fig. 3.7).*

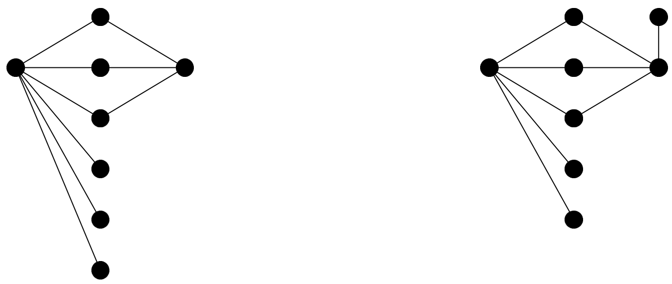


H_{37}

Fig. 3.7

Proof. There are another $p - 5$ vertices in G besides the vertices of H_{30}. The subgraph H of the graph G induced by these $p - 5$ vertices has at least $p - 6$ edges. If H does not contain any vertex of degree 2 then $p - 5 \geqq 2(p - 6)$, i.e. $p \leqq 7$. □

Lemma 3.7. *If G contains at least one of the graphs H_{38}, H_{39} (Fig. 3.8) then it contains the graphs H_{27}, H_{28}, H_{29} and H_{30}.*

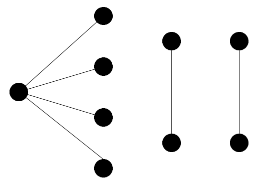


H_{38} H_{39}

Fig. 3.8

Proof. The statement is obvious. □

Lemma 3.8. *If $\triangle(G) = 4$ and $q \geqq 21$ then G contains the graph H_{40} (Fig. 3.9).*

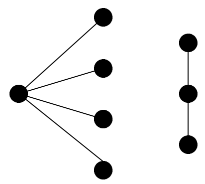


H_{40}

Fig. 3.9

Proof. $|E(U')| \geqq 5$ and hence $G(U')$ contains two independent edges. $\square$

Lemma 3.9. *If $G \in F_{p,p+3}$, $\triangle(G) = 4$ and $p \geqq 19$ then G contains the graph H_{41} (Fig. 3.10).*



H_{41}

Fig. 3.10

Proof. Obviously, if $|E(U)| + |E(U,U')| \leqq 10$ and G does not contain H_{41} then $p - 5 \geqq 2(p - 11)$ i.e. $p \leqq 17$. We can thus assume that $|E(U)| + |E(U,U')| \geqq 11$. Since $\triangle(G) = 4$ it holds $|E(U)| + |E(U,U')| \leqq 12$. We distinguish two cases:

(i) $|E(U)| + |E(U,U')| = 12$
In this case $|E(U)| = 0$ and every vertex from U has degree 3 in $G(U,U')$. We can assume that there are at most 2 vertices from U' of degree 0 in $G(U')$. In fact, in the opposite case it holds (if G does not contain H_{41}) $p-8 \geqq 2(p-13)$, i.e. $p \leqq 18$. If some vertex from U' has degree 2 or 3 in $G(U,U')$ then G contains the graph in Fig. 3.11.

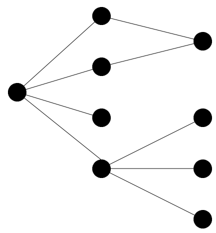


Fig. 3. 11

It follows that G contains the graph H_{41}. So, let no vertex from U' has degree 2 or 3 in $G(U,U')$. There are at most 3 vertices from U' which have degree 4 in $G(U,U')$. Let k be the number of them.

a) k=3
Since there are at most 2 vertices of degree 0 in $G(U')$ this case is impossible.

b) k=2
In this case there are exactly 4 vertices in U' of degree 1 in $G(U,U')$. At least 2 of these 4 vertices have degree at least 1 in $G(U')$ and it follows that G contains H_{41}.

c) $k \leqq 1$
The statement is obviuos.

(ii) $|E(U)| + |E(U,U')| = 11$
In this case $|E(U)| \leqq 1$. If there was an isolated vertex in $G(U')$ and G did not contain the graph H_{41} then it would hold $p - 6 \geqq 2(p - 12)$, i.e. $p \leqq 18$. We can thus assume that no vertex in $G(U')$ is isolated. It follows that no vertex from U' has degree 4 in $G(U,U')$. The statement of the lemma holds if no vertex from U' has degree 2 or 3 in $G(U,U')$. If some vertex $u \in U'$ has degree 3 in $G(U,U')$ then it is sufficient to realize that the vertex from U not adjacent to the vertex u has degree at least 2 in $G(U,U')$. Since the vertex u is not isolated in $G(U')$ then G contains the graph H_{41}.
If some vertex from U' has degree 2 in $G(U,U')$ and is adjacent to vertices $v_1, v_2 \in U$ then it is sufficient to take into account that at least one of the vertices $v_3, v_4 \in U$ has degree 4 and is not adjacent to the vertex v_i for $i = 1, 2$. □

Lemma 3.10. *Let* $G_1, G_2 \in F_{p,p+3}$, $p \geqq 14$ *and* $\triangle(G_1) = \triangle(G_2) = 3$. *Then* $|E_{1,2}| \geqq 6$.

Proof. The statement is a consequence of Lemma 3.2. □

Lemma 3.11. *Let* $G_1, G_2 \in F_{p,p+3}$, $p \geqq 18$ *and* $\triangle(G_1) = \triangle(G_2) = 4$. *Then* $|E_{1,2}| \geqq 6$.

Proof. The statement is a consequence of Lemma 3.8. □

Lemma 3.12. *Let $G_1, G_2 \in F_{p,p+3}$, $p \geqq 19$ and $\triangle(G_1) = 3$, $\triangle(G_2) \geqq 4$. Then $|E_{1,2}| \geqq 6$.*

Proof. First realize that if we prove that G_2 contains at least one of the graphs H_{25}, H_{26} then the statement of the lemma holds by Lemmas 3.1 and 3.2. We distinguish several cases for G_2:

a) There are 2 independent edges in $G_2(U')$

a_1) $\triangle(G_2) \geqq 5$

(i) If $|E(U)| \neq 0$ or $|E(U,U')| \neq 0$ then G_2 contains at least one of the graphs H_{25}, H_{26}.

(ii) If $|E(U)| = 0$ and $|E(U,U')| = 0$ then $|E(U')| \geqq 8$ and hence G_2 contains at least one of the graphs H_{25}, H_{26}.

a_2) $\triangle(G_2) = 4$
If $|E(U,U')| \neq 0$ or $|E(U')| \geqq 7$ then G_2 contains at least one of the graphs H_{25}, H_{26}; in the opposite case it holds $q \leqq 16$, a contradiction.

b) $|E(U')| \geqq 2$ and any two edges in $G_2(U')$ are adjacent edges in U'
In this case $G_2(U')$ contains the graph in Fig. 3.12

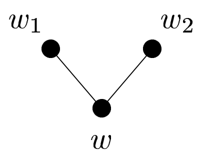


Fig. 3.12

b_1) there is an edge in $G_2(U,U')$ which is not incident with any vertex from $\{w_1, w_2, w\}$
In this case G_2 contains H_{26}.

b_2) $\triangle(G_2) \geqq 5$ and b_1) does not hold
We distinguish 3 subcases:

(i) $|E(U)| \neq 0$
In this case G_2 contains H_{26}.

(ii) $|E(U)| = 0$ and the vertex w has degree at least 3 in $G_2(U,U')$
The statement holds by Lemmas 3.7 and 3.3.

(iii) $|E(U)| = 0$ and the vertex w has degree at most 2 in $G_2(U,U')$
If $|E(U')| = 2$ then degree of w_1 or w_2 is at least 2 in $G_2(U,U')$ (since $|E(U,U')| \geqq 5$) and hence G_2 contains H_{26}. If $|E(U')| \geqq 3$ then again G_2 contains H_{26} (since $|E(U,U')| \geqq 4$).

b_3) $\triangle(G_2) = 4$ and b_1) does not hold
In this case $q \leqq 20$, a contradiction.

c) $|E(U')| = 1$

$c_1)$ $\triangle(G_2) \geqq 6$

(i) $|E(U)| \geqq 2$
If there are two adjacent edges in U then G_2 contains H_{26}. Now let us consider the opposite case. If $\triangle(G_2) \geqq 7$ then G_2 contains H_{25}. If $\triangle(G_2) = 6$ then $|E(U, U')| \geqq 2$ and hence G_2 contains H_{25} or H_{26}.

(ii) $|E(U)| \leqq 1$
In this case $|E(U, U')| \geqq |U'| + 2$ and if G_2 does not contain H_{26} then it contains at least one of the graphs in Fig. 3.13.

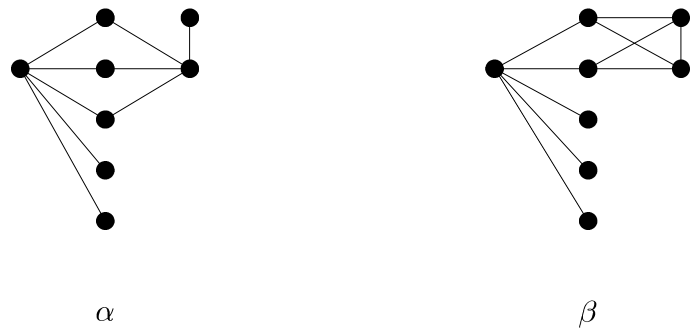


Fig. 3.13

If G_2 contains the graph α then the lemma holds by Lemmas 3.7 and 3.3. If G_2 contains the graph β then G_2 contains each of the graphs H_{27}, H_{28}, H_{29}, H_{30} and the statement holds by Lemma 3.3.

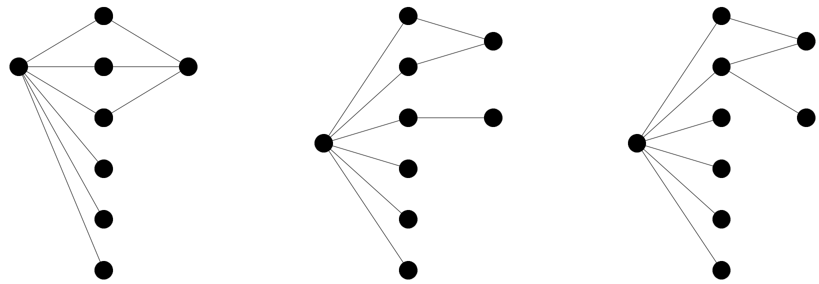
$c_2)$ $\triangle(G_2) = 5$
If there are at least two edges in $G_2(U, U')$ which are not adjacent to the edge of the graph $G_2(U')$ then G_2 contains at least one of the graphs H_{25}, H_{26}. In the opposite case at least 3 edges from $G_2(U, U')$ are incident with the same vertex of the edge of $G_2(U')$. Then the statement of the lemma follows from Lemmas 3.7 and 3.3.

$c_3)$ $\triangle(G_2) = 4$
This case is not possible for $q \geqq 18$.

d) $|E(U')| = 0$

$d_1)$ $\triangle(G_2) \geqq 6$

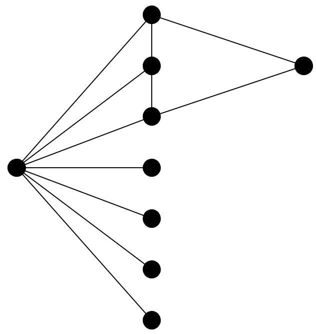
(i) there is a vertex in U' which has degree at least 2 and $|E(U, U')| \geqq 3$
Then G_2 contains at least one of the graphs in Fig. 3.14. In the case α) the statement of the lemma follows from Lemmas 3.7 and 3.3. In the case β) G_2 contains H_{26}. In case γ it holds $|E(U)| + |E(U, U')| \geqq 6$ and hence G_2 again contains H_{26}.



α β γ

Fig. 3.14

(ii) there is a vertex in U' which has degree at least 2 and $|E(U, U')| = 2$ In this case $|E(U)| \geqq 3$ and if $\triangle(G_2) = 6$ then G_2 contains each of the graphs H_{27}, H_{28}, H_{29}, H_{30} (since $q \geqq 22$) and the statement of the lemma holds by Lemma 3.3. If $\triangle(G_2) \geqq 7$ then G_2 contains H_{26} or the graph H_{42} (Fig. 3.15).



H_{42}

Fig. 3.15

The graph H_{42} contains the graphs H_{27}, H_{28} and H_{29}. If the graph G_1 does not contain any of the graphs H_{27}, H_{28} and H_{29} then it contains H_{30} by Lemma 3.3 and H_{37} by Lemma 3.6. The graphs H_{42} and H_{37} have a common subgraph which is depicted in Fig. 3.16.

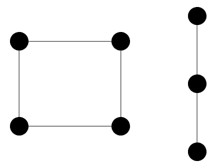


Fig. 3.16

(iii) every vertex from U' has degree at most 1 and there exists a vertex from U of degree at least 2 in $G_2(U, U')$
In this case $|E(U)| \geqq 4$. If G_2 does not contain H_{26} then the statement of the lemma follows from Lemmas 3.7 and 3.3.

(iv) there are no adjacent edges in $G_2(U, U')$ and $|E(U, U')| \geqq 2$
In this case $|E(U)| \geqq 4$. If $|E(U, U')| \geqq 3$ then G_2 contains H_{25}. If $|E(U, U')| = 2$ and $\triangle(G_2) \geqq 7$ then obviously a common subgraph is H_{25} or H_{26}. If $|E(U, U')| = 2$ and $\triangle(G_2) = 6$ then G_2 contains each of the graphs H_{27}, H_{28}, H_{29} and H_{30} (since $q \geqq 22$). Now use Lemma 3.3.

(v) $|E(U, U')| \leqq 1$
In this case we have $|E(U)| \geqq 4$. We distinguish three subcases:
1) $\triangle(G_2) \geqq 9$. If there is a vertex of degree at least 3 in $G_2(U)$ then the statement of the lemma holds by Lemmas 3.7 and 3.3. If there is no vertex of degree at least 3 in $G_2(U)$ and G_2 contains neither the graph H_{25} nor the graph H_{26} then G_2 contains the graph in Fig. 3.17.

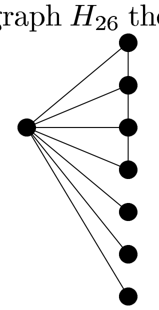


Fig. 3.17

The statement of the lemma follows from Lemmas 3.3, 3.4 and 3.5 (the component H from Lemma 3.5 contains at least one of the graphs H_{27}, H_{28}, H_{29} and H_{30}).
2) $\triangle(G_2) \in \{7, 8\}$
If $q \geqq 18$ then there exists a vertex of degree at least 3 in $G_2(U)$ and

hence G_2 contains the graph α in Fig. 3.14 and the statement of the lemma holds by Lemmas 3.7 and 3.3.
3) $\triangle(G_2) = 6$
This case is not possible for $q \geqq 22$.

d_2) $\triangle(G_2) = 5$
Since $|E(U')| = 0$ and $q \geqq 22$ there is a vertex from U which has degree at least 3 in $G_2(U, U')$. Now, if we realize that there exists an edge in $G_2(U, U')$ which is not incident with the considered vertex of degree at least 3 then we get that G_2 contains H_{26}.

d_3) $\triangle(G_2) = 4$
This case is not possible for $q \geqq 17$. □

Lemma 3.13. *Let $G_1, G_2 \in F_{p,p+3}$, $\triangle(G_1) = 4$, $\triangle(G_2) \geqq 5$ and $p \geqq 19$. Then $|E_{1,2}| \geqq 6$.*

Proof. In wiew of Lemmas 3.8 and 3.9 it is sufficient to show that G_2 contains at least one of the graphs H_{40}, H_{41}.

a) $E(U') \neq 0$
If $\triangle(G_2) \geqq 6$ then the statement of the lemma is obvious. If $\triangle(G_2) = 5$ then it is sufficient to use the fact that $|E(U, U')| + |E(U')| > 1$.

b) $|E(U')| = 0$
If $\triangle(G_2) \geqq 6$ and $|E(U, U')| \geqq 2$ then the statement obviously holds. So, it is sufficient to consider two cases:

(i) $\triangle(G_2) \geqq 6$ and $|E(U, U')| \leqq 1$.
Obviously, the statement holds if $\triangle(G_2) \geqq 8$. If $\triangle(G_2) = 7$ then there exists a vertex of degree at least 2 in $G_2(U)$ and hence G_2 contains the graph H_{41}. The case $\triangle(G_2) = 6$ is impossible since $q \geqq 22$.

(ii) $\triangle(G_2) = 5$.
There exists a vertex from U of degree at least 2 in $G_2(U, U')$. □

Lemma 3.14. *Let $G_1, G_2 \in F_{p,p+3}$, $\triangle(G_1) \geqq 5$, $\triangle(G_2) \geqq 5$, $p \geqq 19$. Then $|E_{1,2}| \geqq 6$.*

Proof. Obviously, the statement holds if $\triangle(G_1) \geqq 6$ and $\triangle(G_2) \geqq 6$. We distinguish two cases:

a) $\triangle(G_1) = \triangle(G_2) = 5$
It is sufficient to consider the case that none of the graphs H_{43}, H_{44} (Fig. 3.18) is a common subgraph of the graphs G_1 and G_2. So we can assume that $|E(G_1(U, U'))| = 0$ and $|E(G_2(U'))| = 0$.

H_{43} H_{44}

Fig. 3.18

We show that a common subgraph is the graph H_{41}. If the graph G_1 did not contain the graph H_{41} then it would hold $p-6 \geqq 2(p-12)$, i.e. $p \leqq 18$. The graph G_2 contains the graph H_{41} since $|E(U,U'| \geqq 14$.

b) $\triangle(G_1) = 5$ and $\triangle(G_2) \geqq 6$.
If the graph G_1 does not contain H_{44} then it contains each of the graphs H_{43}, H_{41}. The graph G_2 contains the graph H_{44} (the case $\triangle(G_2) > 6$ is trivial and if $\triangle(G_2) = 6$ then $|E(U,U')| + |E(U')| \geqq 1$). If G_2 does not contain H_{43} then $|E(U)| = 0$ and $|E(U,U')| = 0$. This implies $|E(U')| = |U'| + 4$ and hence G_2 contains the graph H_{41}. □

Theorem 3.15. diam $F_{p,p+3} = 2p - 6$ *for* $p \geqq 19$.

Proof. In view of Lemmas 3.10 - 3.14 it suffices to find two graphs $G_1, G_2 \in F_{p,p+3}$ with $|E_{1,2}| = 6$. Such a graph G_1 is depicted in Fig. 3.19 and G_2 is an arbitrary graph for which $\triangle(G) = 3$. □

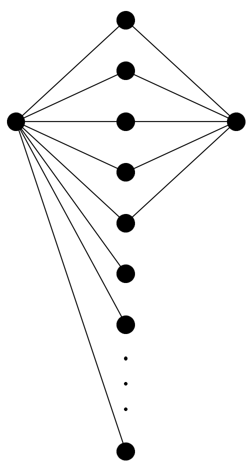


Fig. 3.19

4. Some other results about $\operatorname{diam} F_{p,q}$

Theorem 4.1.

a) *(i)* $\operatorname{diam} F_{5,8} = 2$
(ii) $\operatorname{diam} F_{6,9} = 6$
(iii) $\operatorname{diam} F_{8,11} = 12$
b) *If* $p+3 \leqq q \leqq \frac{3p}{2}$ *and* $7 \leqq p \leqq 18$ *then* $\operatorname{diam} F_{p,q} \in \{2q-12, 2q-10\}$.
c) *If* $p+3 \leqq q \leqq \frac{3p}{2}$ *and* $p \geqq 19$ *then* $\operatorname{diam} F_{p,q} = 2q-12$.

Proof.

a) (i) According to Theorems 5 and 2 from [3] we get

$$\operatorname{diam} F_{5,8} = \operatorname{diam} F_{5,2} = 2.$$

(ii) According to Theorem 5 from [2] we have $c_{6,6} = 3$. Now by using Theorem 5 from [3] we get

$$\operatorname{diam} F_{6,9} = \operatorname{diam} F_{6,6} = 2.6 - 2.3 = 6.$$

(iii) According to Lemma 2.14 we have $c_{8,10} = 5$. Since $c_{8,11} \geqq c_{8,10}$, it is sufficient to find two graphs $G_1, G_2 \in F_{8,11}$ with $|E_{1,2}| = 5$. Such graphs are depicted in Fig. 4.1.

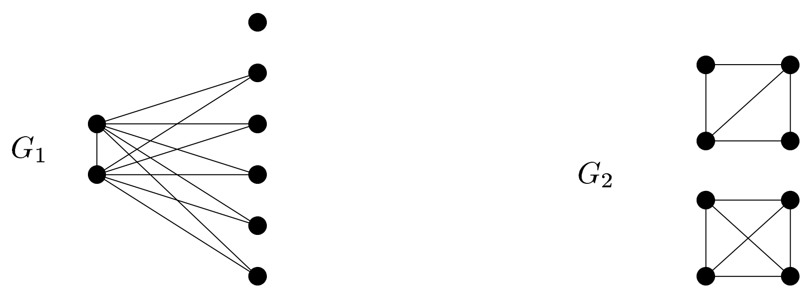


Fig. 4.1

b) According to Lemma 2.14 and Theorem 14 from [1] we have $c_{p,p+2} = 5$ for $7 \leqq p \leqq 18$. It implies $c_{p,q} \geqq 5$ for $q \geqq p+3$. To show that $c_{p,q} \leqq 6$ it is sufficient to find two graphs $G_1, G_2 \in F_{p,q}$ with $|E_{1,2}| = 6$. The graph G_1 is depicted in Fig. 4.2 and G_2 is an arbitrary graph for which $\triangle(G) = 3$.

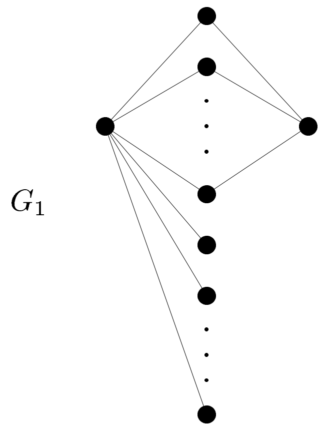


Fig. 4.2

c) By Theorem 3.15 we get $c_{p,q} \geqq 6$. To show that $c_{p,q} = 6$ it is sufficient to find two graphs $G_1, G_2 \in F_{p,q}$ with $|E_{1,2}| = 6$. The graph G_1 is depicted in Fig. 4.2 and G_2 is an arbitrary graph for which $\triangle(G) = 3$. $\square$

References

[1] P.Hrnčiar, G.Monoszová, *The edge distance in some families of graphs*, Acta Univ. M.Belii, Ser. Math. **2** (1994), 29-38.

[2] P.Hrnčiar, A.Haviar, G.Monoszová, *Some characteristics of the edge distance between graphs* (to appear in Czech. Math. J.).

[3] M.Šabo, *On a maximal distance between graphs*, Czech.Math.Jour. 41, 1991, pp. 265-268.

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