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ON THE S-DISTANCE BETWEEN POSETS

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Abstract. V. Baláž, V. Kvasnička and J. Pospíchal [1] proved that the distances based on maximal common subgraph and minimal common supergraph are identical. Here we shall study an analogy for posets.

Throughout this paper all partially ordered sets are assumed to be finite. In [2] a metric on a system of isomorphism classes of posets, which have the same cardinality, is defined. Without loss of generality we can suppose that all posets are defined on the same set P. We will often write a poset R instead of a poset (P, R).

Let $B(P)$ be the set of all bijective maps of P onto itself. For any $f \in B(P)$ and posets $(P, R), (P, S)$ we denote by $d_f(R, S)$ the number defined by

$$d_f(R,S) = |f(R) \setminus S| + |S \setminus f(R)|, \tag{1}$$

where $f(R) = \{[f(a), f(b)];\ [a, b] \in R\}$ (cf. [2]). Since the posets (P, R) and $(P, f(R))$ are isomorphic, then

$$d_f(R,S) = |R| + |S| - 2|f(R) \cap S|. \tag{2}$$

The *distance* of the posets $(P, R), (P, S)$ is defined by

$$d(R,S) = \min\{d_f(R,S);\ f \in B(P)\}. \tag{3}$$

If we identify isomorphic posets, then (3) defines a *metric* on the set of all (finite, non-isomorphic) posets defined on the same set P.

If a map $f \in B(P)$ is an isotone map of a poset (P, R) onto a poset (P, S), then $f(R) \subseteq S$ and $d(R, S) = d_f(R, S) = |S| - |R|$ (cf. Remark 2 in [2]).

The following lemma is easy to verify (cf. Lemma 1 in [2]).

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Lemma 1. *For any posets $(P,R),(P,S)$ and any maps $f,g \in B(P)$ the following properties are satisfied:*

(i)	$d_f(R,S) = d_g(R,S) \quad iff \quad \lvert f(R)\cap S\rvert = \lvert g(R)\cap S\rvert,$
(ii)	$d_f(R,S) < d_g(R,S) \quad iff \quad \lvert f(R)\cap S\rvert > \lvert g(R)\cap S\rvert,$
(iii)	$\lvert f(R)\cap S\rvert = \lvert R\cap f^{-1}(S)\rvert.$

Let (P,R), (P,S) be posets and let $f \in B(P)$. If $d_f(R,S) = d(R,S)$, f is said to be an *optimal map* of (P,R) onto (P,S) (cf. Definition in [2]). From Lemma 1 it follows that f is an optimal map if and only if $|f(R)\cap S|$ is maximal. Any isotone map $f \in B(P))$ is optimal (Remark 2 in [2]).

Let (P,R), (P,R') be posets. The poset (P,R') is called a *w-subposet* of the poset (P,R) if there is a map $f \in B(P)$ with $f(R') \subseteq R$.

If a poset (P,R') is a w-subposet of posets (P,R), (P,S) then we will say that (P,R') is a *common w-subposet* of (P,R) and (P,S).

Let $\{(P,R_i);\ i\in I\}$ be a set all common w-subposet of posets (P,R), (P,S). If there is $m\in I$ with $|R_i| \le |R_m|$ for each $i \in I$, then we will say that (P,R_m) is a *maximal common w-subposet* (MCWS) of posets (P,R), (P,S).

Let (P,Q) be a MCWS of posets (P,R), (P,S). The *s-distance* between posets (P,R), (P,S) is the number defined by

$$d^s(R,S) = |R| + |S| - 2|Q|. \tag{4}$$

Lemma 2. *Let (P,R), (P,S) be posets. Then $d(R,S) = d^s(R,S)$.*

Proof. If $f \in B(P)$ is an optimal map of (P,R) onto (P,S) then $d(R,S) = d_f(R,S) = |R| + |S| - 2|f(R)\cap S|$. It suffices to show that the poset $(P, f(R)\cap S)$ is an MCWS of posets (P,R), (P,S).

If $[a,b] \in f(R)\cap S$, then $[f^{-1}(a), f^{-1}(b)] \in R$, $f^{-1} \in B(P)$ and $[\mathrm{id}_P(a), \mathrm{id}_P(b)] \in S$, $id_P \in B(P)$. Thus $(P, f(R)\cap S)$ is a common w-subposet of (P,R), (P,S). Suppose on the contrary that the poset $(P, f(R)\cap S)$ is not an MCWS. Let (P,Q) be an MCWS. Then $|f(R)\cap S| < |Q|$ and there are optimal maps $h, g \in B(P)$ with $h(Q) \subseteq R$ and $g(Q) \subseteq S$. From this we have

$$g(Q) = gh^{-1}h(Q) \subseteq gh^{-1}(R)$$

and so

$$g(Q) \subseteq gh^{-1}(R) \cap S$$

which gives

$$|Q| = |g(Q)| \le |gh^{-1}(R)\cap S|.$$

We thus get

$$|f(R)\cap S| < |gh^{-1}(R)\cap S|.$$

Therefore by (ii)

$$d_f(R,S) > d_{gh^{-1}}(R,S), \quad gh^{-1} \in B(P),$$

a contradiction. □

The next theorem follows from Lemma 2 immediately.

Theorem 3. *Let $\mathcal{F}_n$, $n \in N$, be a system of all (non-isomorphic) posets on a set P of the cardinality n. Then the function d^s on the system $\mathcal{F}_n$ given by (4) is a metric.*

Let (P,R), (P,R') be posets. If a poset (P,R') is a w-subposet of a poset (P,R) then we will say that (P,R) is a *w-owerposet* of a poset (P,R'). If a poset (P,R) is a w-overposet of posets (P,S), (P,Q) then (P,R) will be called a *common w-owerposet* of (P,S) and (P,Q). Let $\{(P,R_i);\, i \in I\}$ be a set all common w-overposet of posets (P,S), (P,Q). If there is $m \in I$ with $|R_m| \leq |R_i|$ for each $i \in I$, then we will say that (P,R_m) is a *minimal common w-owerposet* (mCWO) of posets (P,S), (P,Q).

Let (P,M) be an mCWS of posets (P,R), (P,S). We denote by $d^o(R,S)$ the number defined by

$$d^o(R,S) = 2|M| - |R| - |S|. \tag{5}$$

Proposition 4. *Let (P,R), (P,S) be posets. Then $d(R,S) \leq d^o(R,S)$.*

Proof. Let (P,M) be a mCWO of posets (P,R), (P,S). If $f \in B(P)$ is an optimal map of (P,R) onto (P,S), then

$$\begin{aligned} d(R,S) = d_f(R,S) &= |R| + |S| - 2|f(R) \cap S| = \\ &= |f(R)| + |S| - 2|f(R) \cap S| = |f(R)| + |S| - 2(|f(R)| + |S| - |f(R) \cup S|) = \\ &= 2|f(R) \cup S| - |f(R)| - |S| = 2|f(R) \cup S| - |R| - |S|. \end{aligned}$$

Since $|f(R) \cup S|$ is minimal if and only if $|f(R) \cap S|$ is maximal,

$$d(R,S) = 2|f(R) \cup S| - |R| - |S| \leq 2|M| - |R| - |S| = d^o(R,S).$$

Proposition 5. *Let (P,Q) be an mCWO of posets (P,R), (P,S). If $d(R,S) = d_f(R,S)$ and $|Q| = |f(R) \cup S|$, then there exists a map $f' \in B(P)$ with $d_{f'}(R,S) = d(R,S)$ and $(P, f'(R) \cup S)$ is a poset isomorphic to a poset (P,Q).*

Proof. Since (P,Q) is an mCWO of posets (P,R), (P,S) there are isotone maps $g, h \in B(P)$ with $g(R) \subseteq Q$, $h(S) \subseteq Q$ and so

$$d_g(R,Q) = d(R,Q) = |Q| - |R|, \qquad d_h(S,Q) = d(S,Q) = |Q| - |S|.$$

From (iii) we have $d(Q,S) = d_{h^{-1}}(Q,S) = |Q| - |S|$. By assumption $|Q| = |f(R) \cup S|$ we obtain

$$\begin{aligned} d(R,Q) + d(S,Q) =& d_g(R,Q) + d_{h^{-1}}(Q,S) = 2|Q| - |R| - |S| = \\ =& 2|f(R) \cup S| - |R| - |S| = 2(|f(R)| + |S| - |f(R) \cap S| - |R| - |S|) = \\ =& 2(|R| + |S| - |f(R) \cap S| - |R| - |S|) = |R| + |S| - 2|f(R) \cap S| = \\ =& d_f(R,S) = d(R,S). \end{aligned}$$

As in the proof of Theorem 1 in [2] we obtain

$$d_{h^{-1}g}(R,S) \leq d_g(R,Q) + d_{h^{-1}}(Q,S) = d(R,S).$$

Thus

$$d_{h^{-1}g}(R,S) = d(R,S)$$

and so $h^{-1}g \in B(P)$ is an optimal map of the poset (P,R) onto the poset (P,S).

It remains to prove that the relation structure $(P, h^{-1}g(R) \cup S)$ is isomorphic to the poset (P,Q). For all $[x,y] \in h^{-1}g(R) \cup S$ we put $\psi([x,y]) = [h(x), h(y)]$.

a) If $[x,y] \in S$, then $\psi([x,y]) = [h(x), h(y)] \in Q$, since h is an isotone map of (P,S) onto (P,Q).

b) If $[x,y] \in h^{-1}g(R) \setminus S$, then there is $[a,b] \in R$ with $[x,y] = [h^{-1}g(a), h^{-1}g(b)]$ and thus $\psi([x,y]) = [h(x), h(y)] = [hh^{-1}g(a), hh^{-1}g(b)] = [g(a), g(b)] \in Q$, since g is an isotone map of (P,R) onto (P,Q).

By the above, ψ is a map of $h^{-1}g(R) \cup S$ to Q. It is obvious that the map ψ is injective. Since $|h^{-1}g(R) \cup S| = |Q|$, the map ψ is bijective. Thus ψ^{-1} is a bijective map of Q onto $h^{-1}g(R) \cup S$ and for $[u,v] \in Q$,

$$\psi^{-1}([u,v]) = [h^{-1}(u), h^{-1}(v)] \in h^{-1}g(R) \cup S.$$

From this it follows that the relation structure $(P, h^{-1}g(R) \cup S)$ is a poset isomorphic to the poset (P,Q). $\square$

Example 6. Let (P,R), (P,S), (P,T) be posets with $|R| = |S| = 16$, $|T| = 15$ given in Figure.

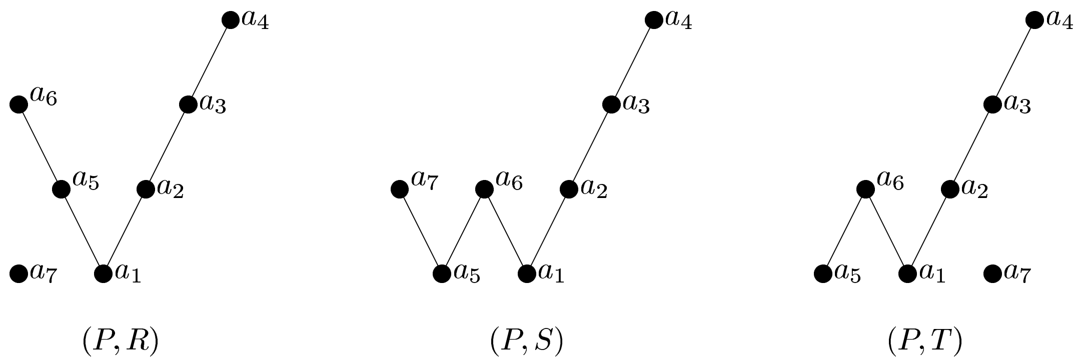

Fig.

The map $\mathrm{id}_P \in B(P)$ is the only optimal map of the poset (P,R) onto the poset (P,S).

The union $\mathrm{id}_P(R) \cup S$ is not a partial ordering, since $[a_1, a_5], [a_5, a_7] \in \mathrm{id}_P(R) \cup S$ but $[a_1, a_7] \notin \mathrm{id}_P(R) \cup S$. Let (P, Q) be an mCWO of the posets $(P, R), (P, S)$. By Proposition 5, from $|\,\mathrm{id}_P(R) \cup S| = 17$ it follows $|Q| \geq 18$. Thus $d^o(R, S) = 2|Q| - |R| - |S| \geq 4$. The poset $(P, \mathrm{id}_P(R) \cup S \cup \{[a_1, a_7]\})$ with $|\,\mathrm{id}_P(R) \cup S \cup \{[a_1, a_7]\}| = 18$ is a mCWO of the posets $(P, R), (P, S)$ and so $d^o(R, S) = 4$.

Since the poset (P, T) is a w-subposet of the posets $(P, R), (P, S)$, we have $d^o(R, T) = 1$, $d^o(T, S) = 1$. From this it follows that

$$2 = d^o(R, T) + d^o(T, S) < d^o(R, S) = 4.$$

Therefore d^o is not a metric.

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