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A NOTE ON THE DISTANCE POSET OF POSETS

Judita Lihová

Abstract. Let F_ω^* be the system of all non–isomorphic finite orders of a countable set P, ordered in such a way that $R \leq S$ if $f(R) \subseteq S$ for a bijective map $f : P \to P$. There are investigated some properties of $(F_\omega^*, \leq)$.

In [3] a metric d on the system F_n of isomorphism classes of ordered sets of the same finite cardinality n has been introduced. In [4] there is shown that this metric coincides with the distance–metric on the covering graph of F_n. The system F_n can be partially ordered. By the help of the above mentioned metric the author proves in [4] that the ordered system F_n is graded, i.e. all maximal chains with the same endpoints have the same length. The ordered system F_n, as each finite partially ordered set, is a multilattice. A natural question arises. Is F_n a metric multilattice with respect to d, in the sense of [5]?

In this note there is proved that F_n is not a metric multilattice with respect to any metric by showing that F_n is not a modular multilattice. In the second part some properties of the ordered system of all finite orders of the same infinite set P are mentioned.

0. Basic notions

A partially ordered set $(M, \leq)$ is said to be a multilattice if, whenever $a, b \in M, u \in M, u \geq a, u \geq b$, there exists a minimal upper bound u' of $\{a, b\}$ with $u' \leq u$, and dually. If, moreover, $(M, \leq)$ is a directed set, then $(M, \leq)$ is called a directed multilattice.

Let $a \vee b$ $(a \wedge b)$ denote the set of all minimal upper bounds of $\{a, b\}$ (maximal lower bounds of $\{a, b\}$). A multilattice $(M, \leq)$ is distributive if

$$a, b, c \in M, (a \wedge b) \cap (a \wedge c) \neq \emptyset, (a \vee b) \cap (a \vee c) \neq \emptyset \Rightarrow b = c,$$

and modular if

$$a, b, c \in M, b \leq c, (a \wedge b) \cap (a \wedge c) \neq \emptyset, (a \vee b) \cap (a \vee c) \neq \emptyset \Rightarrow b = c.$$

(For the above definitions see [1].)

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By a metric multilattice is meant a multilattice with a metric d fulfilling the following conditions (cf. [5]):

M1. $a \le b \le c$ implies $d(a,b) + d(b,c) = d(a,c)$,

M2. if $u \in a \wedge b, v \in a \vee b$, then $d(a,b) = d(u,v)$.

In [5] there is proved:

0.1. Theorem. *A metric multilattice is modular.*

0.2. Theorem. *A directed modular multilattice of locally finite length is a metric multilattice.*

1. Properties of F_n

Let $F_n (n \in N)$ be the set of all (non–isomorphic) orders of a set P of cardinality n. Set $R \le S$ $(R, S \in F_n)$ if there exists a permutation f of P satisfying $f(R) \subseteq S$ (the symbol $f(R)$ denotes the set $\{[f(a), f(b)] : [a,b] \in R\}$). In other words, $R \le S$ means that there exists an isotone bijection of (P, R) onto (P, S). The poset $(F_n, \le)$ is called the distance poset (of orders of an n–element set) (cf. [4]).

The following theorem is proved in [4].

1.1. Theorem. *The distance poset $(F_n, \le)$ is a graded poset with the least element and the greatest element.*

The diagrams of $(F_3, \le)$ and $(F_4, \le)$ are depicted in Fig. 1 and Fig. 2, respectively. Evidently $(F_1, \le)$ is a one element set and $(F_2, \le)$ is a two element chain. The least element of $(F_n, \le)$ is the discrete order, i.e. the order in which only comparable elements are the couples of equal elements and the greatest element is the linear order. Let us remark that $(F_n, \le)$, as a finite bounded partially ordered set, is a directed multilattice.

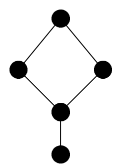

Fig. 1

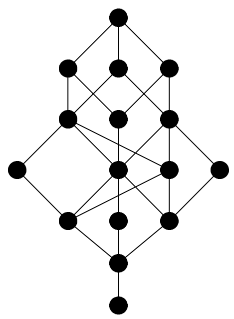

Fig. 2

If $R, S \in F_n, R \le S$ and S covers R, we will write $R \prec S$. The following lemma proved in [4] will be useful.

1.2. Lemma. *Let $R, S \in F_n, R \leq S$, f be a permutation of P satisfying $f(R) \subseteq S$. Then $R \prec S$ if and only if $S - f(R) = \{[a, b]\}$, where $a \prec_S b$.*

It is easy to see that $(F_1, \leq), (F_2, \leq)$ and $(F_3, \leq)$ are distributive lattices. In contrast with this, there holds:

1.3. Lemma. *If $n \geq 4$, then $(F_n, \leq)$ is not a lattice.*

Proof. Let $R, S \in F_n$ be as in Fig. 3 and Fig. 4, respectively. Using 1.2 it is easy to see that U shown in Fig. 5 and its dual U^δ are covered by R, S and V in Fig. 6 and its dual V^δ cover both R and S. Hence U and U^δ are maximal lower bounds of $\{R, S\}$ and V, V^δ are minimal upper bounds of $\{R, S\}$.

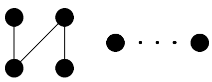

Fig. 3

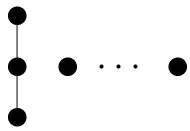

Fig. 4

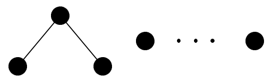

Fig. 5

As we have remarked, $(F_n, \leq)$ is a multilattice, hence for any $R, S \in F_n$ and any $U, V \in F_n$ satisfying $U \leq R$, $S \leq V$ there exists a maximal lower bound U' of $\{R, S\}$ and a minimal upper bound V' of $\{R, S\}$ with $U \leq U'$ and $V' \leq V$. In 1.4 and 1.6 there is described the set of all maximal lower bounds of $\{R, S\}$ and the set of all minimal upper bounds of $\{R, S\}$, respectively.

1.4. Lemma. *Let $R, S, U \in F_n, U \leq R, U \leq S$. Then $U \in R \wedge S$ if and only if for each couple of permutations f, g of P with $f(U) \subseteq R, g(U) \subseteq S$ there is $f^{-1}(R) \cap g^{-1}(S) = U$.*

Proof. Clearly for any permutation h of P and any order T of P, $h^{-1}(T)$ is an order of P and further the intersection of two orders of P is an order of P less than or equal to each of them. So if $U \leq R, U \leq S$, then for each couple of permutations f, g of P with $f(U) \subseteq R, g(U) \subseteq S$ there is $U \subseteq f^{-1}(R) \cap g^{-1}(S)$, $f^{-1}(R) \cap g^{-1}(S) \leq R$, $f^{-1}(R) \cap g^{-1}(S) \leq S$. Now if U is a maximal lower bound of $\{R, S\}$, then $U = f^{-1}(R) \cap g^{-1}(S)$. Conversely, if $U < U' \leq R, S$ and h, f_1, g_1 are permutations of P such that $h(U) \subset U', f_1(U') \subseteq R, g_1(U') \subseteq S$, then $f_1(h(U)) \subseteq R, g_1(h(U)) \subseteq S, h^{-1}(f_1^{-1}(R)) \cap h^{-1}(g_1^{-1}(S)) = h^{-1}(f_1^{-1}(R) \cap g_1^{-1}(S)) \supseteq h^{-1}(U') \supset U$.

Let us remark that it can happen that $f_1^{-1}(R) \cap g_1^{-1}(S) = U$ for some permutations f_1, g_1 of P and at the same time $f_2^{-1}(R) \cap g_2^{-1}(S) \supset U$ for some other permutations f_2, g_2 of P, as the following example shows.

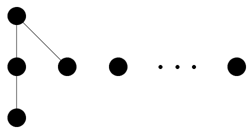

Fig. 6

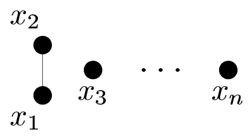

Fig. 7

1.5. Example. Let U, R, S be as in Fig. 7, Fig. 8 and Fig. 9, respectively. Define i to be the identity map on $P = \{x_1, \ldots, x_n\}, g = (x_3x_4)$. Then $i^{-1}(R) \cap i^{-1}(S) = U$ while $i^{-1}(R) \cap g^{-1}(S) \supset U$.

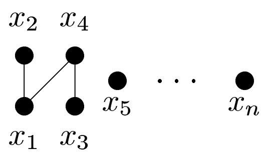

Fig. 8

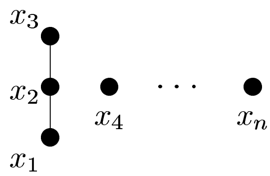

Fig. 9

Analogously can be proved:

1.6. Lemma. *Let $R, S, V \in F_n, R \leq V, S \leq V$. Then $V \in R \vee S$ if and only if for each couple of permutations f, g of P with $f(R) \subseteq V, g(S) \subseteq V$, V is the transitive cover of $f(R) \cup g(S)$.*

Considering the same R, S as in 1.5 and V as in Fig. 10, $V = i(R) \cup g_1(S)$, but V properly contains the transitive cover of $i(R) \cup g_2(S)$ for $g_1 = (x_1x_3x_4), g_2 = (x_3x_4)$.

Now we are going to investigate $(F_n, \leq)$ for $n \geq 4$ from the view of its distributivity and modularity. Obviously $(F_4, \leq)$ it is not distributive and a straightforward testing yields that $(F_4, \leq)$ is modular.

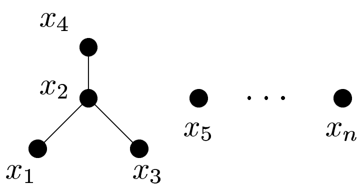

Fig. 10

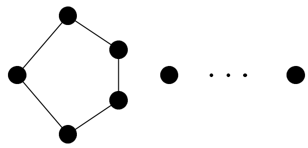

Fig. 11

1.7. Theorem. *If $n \geq 5$, then the multilattice $(F_n, \leq)$ is not modular.*

Proof. Let R, S, T, U, V be as in Fig. 11, 12, 13, 14 and 15, respectively. Then $U < S \prec V, U \prec R \prec T \prec V$, by 1.2, and $V \in S \vee R$, because $S \ngeq R$. Let us suppose that there exists $U' \in F_n$ satisfying $U' > U, U' \leq S, T$. Since S, T are incomparable orders, using 1.1 we obtain that U' must be covered by S and T. If we find all orders covered by S, using

1.2, we see that the order in Fig. 16 is the only one covered also by T, but it is not greater than U. We have a contradiction.

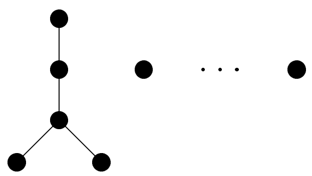

Fig. 12

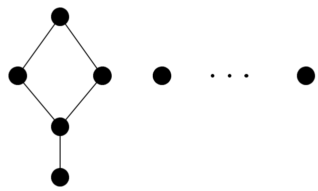

Fig. 13

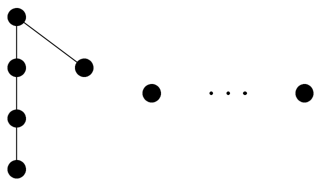

Fig. 14

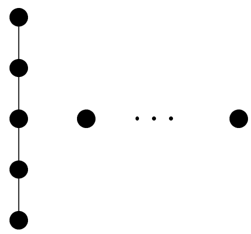

Fig. 15

Using 0.1 and 0.2 we obtain:

1.8. Corollary. *If $n \geq 5$, then the multilattice $(F_n, \leq)$ is not a metric multilattice. $(F_4, \leq)$ is a metric multilattice, $(F_1, \leq), (F_2, \leq)$ and $(F_3, \leq)$ are metric lattices.*

Note that if $n \leq 4$, the metric d introduced in [3] (for the definition see below) fulfils the conditions M1 and M2.

2. Distance poset F_ω^*

In this section P will be any countable set (in fact, it could be of any infinite cardinality). An order R of P will be said to be finite if R contains only finitely many couples of distinct elements. Let F_ω^* denote the system of all (non–isomorphic) finite orders of P. It can be partially ordered by

$$R \leq S \text{ if there exists a bijective map } f : P \to P \text{ with } f(R) \subseteq S.$$

Evidently $(F_\omega^*, \leq)$ has the least element (the discrete order of P), but it contains no maximal elements, so it is of infinite length.

Denote by F_n' the set of all orders $R \in F_\omega^*$ with the property that there exists an n–element subset P' of P satisfying

$$[a, b] \in R, a \neq b \Rightarrow \{a, b\} \subseteq P'.$$

2.1. Theorem. *For each $n \in N (F_n, \leq)$ is isomorphic to $(F'_n, \leq)$. F'_n is an interval of F^*_ω with the discrete order of P as the least element. Further $F'_1 \subset F'_2 \subset F'_3 \subset \ldots$ and $F^*_\omega = \cup_{n \in N} F'_n$.*

This statement is evident.

The preceding theorem yields immediately that $(F^*_\omega, \leq)$ is of locally finite length and graded. So $(F^*_\omega, \leq)$ is a directed multilattice. If $R, S \in F^*_\omega$, let us denote by $R \vee_\omega S (R \wedge_\omega S)$, $R \vee_n S (R \wedge_n S)$ the set of all minimal upper bounds (maximal lower bounds) of $\{R, S\}$ in $(F^*_\omega, \leq)$ and $(F'_n, \leq)$, respectively. It is easy to verify:

2.2. Theorem. *Let $R, S \in F^*_\omega$ and let n_0 be the least positive integer such that both R and S belong to F'_{n_0}. Then $R \wedge_\omega S = R \wedge_{n_0} S, R \vee_\omega S = \cup_{n \geq n_0} R \vee_n S$.*

One can see that for any $R, S \in F^*_\omega$ the set $R \wedge_\omega S$ is finite. Since some $R, S \in F^*_\omega$ can have minimal upper bounds in various F'_n (cf. the following example), it is not quite evident that the same holds for the set $R \vee_\omega S$.

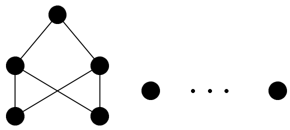

Fig. 16

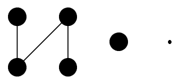

Fig. 17

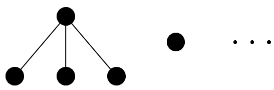

Fig. 18

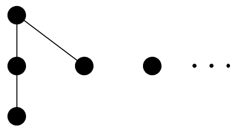

Fig. 19

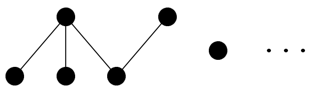

Fig. 20

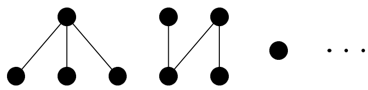

Fig. 21

2.3. Example. Let R, S be as in Fig. 17 and Fig. 18, respectively. Then each of Figures 19, 20, 21 represents a minimal upper bound of $\{R, S\}$.

2.4. Theorem. *For any $R, S \in F^*_\omega$ the set $R \vee_\omega S$ is finite.*

Proof. Let $R, S \in F^*_\omega$, $V \in R \vee_\omega S$. We are going to show that V contains at most card $R+$ card S couples of elements a, b with $a \prec_V b$. Suppose that this is not the case. Let f, g be bijective maps $P \to P$ satisfying $f(R) \subseteq V, g(S) \subseteq V$. Then there exist a, b with

$a \prec_V b$ such that $[a, b] \notin f(R) \cup g(S)$. Then in view of 1.2 $V - \{[a, b]\}$ is an order covered by V. Evidently $f(R) \subseteq V - \{[a, b]\}$, $g(S) \subseteq V - \{[a, b]\}$, so $V - \{[a, b]\}$ is an upper bound of $\{R, S\}$ less than V, a contradiction.

Using 1.7 and 2.2 we obtain:

2.5. Theorem. *The multilattice* $(F_\omega^*, \leq)$ *is not modular.*

In view of 0.1 we have:

2.6. Corollary. *The multilattice* $(F_\omega^*, \leq)$ *is not a metric multilattice.*

Nevertheless, there can be introduced a metric into F_ω^*, but not satisfying both M1 and M2. Namely, the metric d on the system F_n of all non–isomorphic orders of an n–element set P_n, defined in [3] by

$$d(R, S) = \text{ min } \{d_f(R, S) : f \text{ is a permutation of } P_n\},$$

where $d_f(R, S) = \text{ card } (f(R) - S) + \text{ card } (S - f(R))$, evidently yields a metric on F_ω^*, too.

In [4] there is proved that if $R, S \in F_n$, $d(R, S) = \delta(R, S)$, where $\delta(R, S)$ is the distance of vertices R, S of the covering graph of F_n (Th. 2.2). Further by 2.1 of [2] $\delta(R, S) = h(R) - h(S)$ (h denotes the height) provided that $S \leq R$, thanks to the fact that $(F_n, \leq)$ is a graded poset.

So we have:

2.7. Theorem. *An order* $R \in F_\omega^*$ *has the height* k *in the partially ordered set* $(F_\omega^*, \leq)$ *if and only if card* $\{[a, b] \in R : a \neq b\} = k$.

Proof. Evidently the height of R in $(F_\omega^*, \leq)$ is the same as in F_n', if $R \in F_n'$. Therefore $h(R) = k$ if and only if $d(R, D) = k$ with D being the discrete order. But obviously $d(R, D) = \text{ card } \{[a, b] \in R : a \neq b\}$.

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Department of Geometry and Algebra, University of P. J. Šafárik, Jesenná 5, 041 54 Košice, Slovakia

E-mail address: lihova@duro.upjs.sk

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