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CONSISTENT ORTHOGONAL ATOMIC PARTITIONS

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ABSTRACT. The present paper defines consistent orthogonal atomic partitions of two elements of an orthollatice. Then this notion is studied in the realm of ortholattices associated with a vector space with a scalar product over an ordered field.

This paper deals with ortholattices in which every element is a union of finitely many pairwise orthogonal atoms. Let us recall the notion of an ortholattice - [1], p. 75.

Definition 1. Let L be a lattice with element 0 and 1 and a mapping $\perp: L \to L, a \mapsto$ $\mapsto a^{\perp}$ such that

$$
\begin{array}{ll}
0 \le a \le 1 & \text{for any } a \in L \\
a^{\perp\perp} = a & \text{for any } a \in L \\
(a \wedge b)^{\perp} = a^{\perp} \vee b^{\perp},\ (a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp} & \text{for any } a, b \in L \\
a \wedge a^{\perp} = 0,\ a \vee a^{\perp} = 1 & \text{for any } a \in L\ .
\end{array}
$$

Then L is said to be an ortholattice. We define $a \perp b$ if and only if $a \le b^{\perp}$, we say that a and b are orthogonal in this case.

Definition 2. Let L be an ortholattice. A sequence $a_1, \dots, a_k$ of pairwise orthogonal atoms of L is said to be an orthogonal atomic partition of an element $u \in L$, if $u = a_1 \vee \dots \vee a_k$.
If v is another element of L with an orthogonal atomic partition $b_1, \dots, b_m$, then these two orthogonal atomic partitions are said to be consistent if $a_i \perp b_j$ for $1 \le i \le k$, $1 \le j \le m$ and $i \ne j$. (Relation $a_i \perp b_i$ is not required.)

The definition of consistent orthogonal atomic partitions is motivated by Theorem 1.

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Example 1. Let F be an ordered field and X be a finite dimensional vector space over F with a scalar product $\cdot$, i.e. symmetric bilinear positively defined form with values in F. The case $F = \mathbb{R}$ is well known, see [2], pp. 432 – 465. Many results and basic notions may be easily extended to the general case. We need mainly Gram-Schmidt ortogonalization, i.e. the following procedure. Let $u_i, \ldots, u_m$ be linearly independent vectors of X. Define vectors $a_1, \ldots, a_m$ by induction

$$a_1 = u_1 \quad \text{and} \quad a_k = u_k - \sum_{i=1}^{k-1} \frac{(u_k \cdot a_i)}{(a_i \cdot a_i)} a_i \quad \text{for} \quad 1 < k \leq m \ .$$

Then we obtain an orthogonal system of vectors which generate the same subspace as $u_1, \ldots, u_m$. So, any subspace of X has an orthogonal basis. When any nonnegative element of the field F has a square root, then any orthogonal system may be orthonormalized.

Let $L(X)$ be the lattice of all subspaces of X and put $U^{\perp} = \{v \in X : \ u \cdot v = 0$ for all $u \in U\}$ for a subspace U of X.
Then we obtain an ortholattice in which any element has an orthogonal atomic partition, because any subspace of X has an orthogonal basis. Example 2 shows that consistent orthogonal atomic partitions of two subspaces need not exist. On the other hand Theorem 2 gives a sufficient condition under which consistent orthogonal partitions exist.

Definition 3. Let F be an ordered field and X be a finite dimensional vector space over F with a scalar product $\cdot$. A linear operator $A : X \to X$ is said to be selfadjoint if $(Ax) \cdot y = x \cdot Ay$ for all $x, y \in X$.
Let U be a subspace and $P : X \to X$ be a selfadjoint linear operator such that $P^2 = P$ and $P(X) = U$, then P is said to be the orthogonal projection onto U.

Proposition 1. *Let F be an ordered field, X be a finite dimensional vector space over F with a scalar product and $a_1, \ldots, a_k$ be an orthogonal basis of a subspace U of X. Then the formula $Px = \sum_{i=1}^{k_1} \frac{(x \cdot a_i)}{(a_i \cdot a_i)} a_i$ defines uniquely the orthogonal projection onto U.*

Theorem 1. *Let X be a finite dimensional real vector space with a scalar product and U and V be subspaces of X with $\dim(U) = k_1$ and $\dim(V) = k_2$. Then there are bases $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ of U and V respectively such that*

(1)	$\|\|a_i\|\| = 1 = \|\|b_j\|\|$	*for* $1 \leq i \leq k_1$ *and* $j \leq k_2$
(2)	$a_i \cdot a_j = 0$	*for* $1 \leq i < j \leq k_1$
(3)	$b_i \cdot b_j = 0$	*for* $1 \leq i < j \leq k_2$
(4)	$a_i \cdot b_j = 0$	*for* $i \neq j,\ 1 \leq i \leq k_1$ *and* $1 \leq j \leq k_2$
(5)	$a_i \cdot b_i \geq 0$	*for* $1 \leq i \leq \min(k_1, k_2)$

The following Lemma shows the properties which the bases $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ must have.

Lemma 1. *Let F be an ordered field, X be a finite dimensional vector space over F with a scalar product, U and V be subspaces of X with bases $a_1, \ldots, a_{k_1}$ and $b_1, \ldots b_{k_2}$, which satisfy relations (2) - (4). Then all a_i are eigenvectors for PQ and all b_j are eigenvectors for QP, where P and Q denote orthogonal projections onto U and V respectively.*

Proof. Since $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ form orthogonal bases of U and V then $Px = \sum_{i=1}^{k_1} \frac{(x \cdot a_i)}{(a_i \cdot a_i)} a_i$ and $Qx = \sum_{i=1}^{k_2} \frac{(x \cdot b_i)}{(b_i \cdot b_i)} b_i$ for all $x \in X$ by Proposition 1.
We may assume $k_1 \leq k_2$. For $1 \leq i \leq k_1$ relation (4) implies

$$Pb_i = \frac{(b_i \cdot a_i)}{(a_i \cdot a_i)} a_i \quad \text{and} \quad Qa_i = \frac{(a_i \cdot b_i)}{(b_i \cdot b_i)} b_i \ .$$

Therefore

$$PQa_i = \frac{(a_i \cdot b_i)^2}{(a_i \cdot a_i)(b_i \cdot b_i)} a_i \quad \text{and} \quad QPb_i = \frac{(a_i \cdot b_i)^2}{(a_i \cdot a_i)(b_i \cdot b_i)} b_i \ ,$$

which means that a_i and b_j are eigenvectors for PQ and QP. If $k_1 < i \leq k_2$ then $Pb_i = 0$ and $QPb_i = 0$ and b_i is an eigenvector for QP.

Proof of Theorem 1. Let P and Q be the orthogonal projections onto U and V respectively. Let $u_1 \in U$ and $u_2 \in U$. Then $(PQu_1) \cdot u_2 = (Qu_1) \cdot (Pu_2) = Qu_1 \cdot u_2 = u_1 \cdot Qu_2 = Pu_1 \cdot Qu_2 = u_1 \cdot (PQu_2)$, which means that the restriction of PQ onto U is a selfadjoint linear operator. There is an orthonormal basis $a_1, \ldots, a_k$ of U such that $PQa_i = \lambda_i a_i$ for some $\lambda_i \in \mathbb{R}$, see [2], p. 461. Now, consider the elements Qa_i. Let $i \neq j$ and $i, j \in \{1, \ldots, k_1\}$. Then $a_i \cdot Qa_j = Pa_i \cdot Qa_j = a_i \cdot PQa_j = a_i \cdot (\lambda_j a_j) = \lambda_j (a_i \cdot a_j) = 0$ and

$$(Qa_i \cdot Qa_j) = (a_i \cdot Q^2 a_j) = (a_i \cdot Qa_j) = 0 \ .$$

Put $k_3 = \dim(Q(U))$. Obviously $k_3 \leq \min(k_1, k_2)$. We may assume that $Qa_i \neq 0$ for $1 \leq i \leq k_3$ and $Qa_i = 0$ for $k_3 < i \leq k_1$. Put $b_j = \frac{Qa_j}{||Qa_j||}$ for $1 \leq j \leq k_3$. If $k_3 = k_2$, the proof is complete. If $k_3 < k_2$, take an orthonormal basis $c_1, \ldots, c_{k_2 - k_3}$ of $V \cap Q(U)^{\perp}$ and put $b_{k_3+j} = c_j$ for $1 \leq j \leq k_3 - k_2$. It is sufficient to verify $a_i \cdot c_j = 0$ for $1 \leq i \leq k_1$ and $1 \leq j \leq k_2 - k_3$. We have $a_i \cdot c_j = a_i \cdot (Qc_j) = (Qa_i) \cdot c_j = 0$.

Example 2. Let X be $\mathbb{Q}^4$ with the standard scalar product, $u_1 = (1,0,0,0)$, $u_2 = (0,1,0,0)$, $v_1 = (1,1,1,1)$, $v_2 = (1,-2,-2,3)$, U and V be linear spans of u_1, u_2 and v_1, v_2 respectively. Note that $u_1 \perp u_2$ and $v_1 \perp u_2$. Using Proposition 1, it is easy to see that $PQu_1 = \frac{1}{36}(11u_1 + 5u_2)$ and $PQu_2 = \frac{1}{36}(5u_1 + 17u_2)$. It means that the restriction of PQ onto U has the matrix $\frac{1}{36}\begin{pmatrix} 11 & 5 \\ 5 & 17 \end{pmatrix}$ with respect to the basis $\{u_1, u_2\}$. The eigenvalues of this matrix are $\frac{1}{36}(14 \pm \sqrt{34})$, which are irrational and PQ has no eigenvectors in U. By Lemma 1. consistent orthogonal atomic partitions of U and V do not exist.

Example 2 and Lemma 1 indicate that the existence of consistent atomic partitions for subspaces U and V is connected with the solvability of algebraic equations over the field F. In fact, in Theorem 1 is essential only the fact, that the field or real numbers is a maximal ordered field ([3], pp. 276 – 282), which means that every polynomial is a product of linear polynomials and quadratic polynomials with the negative discriminant. The following theorems shows that the condition of maximality of the ordered field F may be weakened according to $\dim(X)$.

Theorem 2. *Let X be a n-dimensional vector space with a scalar product over an ordered field F with the property, that any polynomial of the degree $\leq [\frac{n}{2}]$ is a product of linear polynomials and quadratic polynomials with the negative discriminant. Then for any vector subspaces U and V of X there are bases $a_1, \ldots, a_{k_1}$ and $b_1, \ldots, b_{k_2}$ satisfying relations (2) - (5). Relation (1) may be satisfied whenever every nonnegative element of F has a square root, which is automatically satisfied for $n \geq 4$.*

Proof. The case $n \leq 3$ may be easily studied. So, assume $n \geq 4$. In this case any nonnegative $\alpha \in F$ must have a square root, because $g(\lambda) = \lambda^2 - \alpha$ is a polynomial of degree $\leq [\frac{n}{2}]$ with the nonnegative discriminant and it must be reducible. We may assume $\dim(U) \leq \dim(V)$. We have

$$\begin{aligned} n &\geq \dim(U+V) = \dim(U) + \dim(V) - \dim(U \cap V) \geq [\dim(U) - \dim(U \cap V)] + \\ &+ [\dim(V) - \dim(U \cap V)] \geq 2[\dim(U) - \dim(U \cap V)] \ , \end{aligned}$$

which implies $[\dim(U) - \dim(U \cap V)] \leq [\frac{n}{2}]$.

Analogically to Theorem 1. the restriction of PQ onto U is a selfadjoint linear operator. Denote this restriction by A. Let $f(\lambda)$ be the characteristic polynomial of A. For $x \in U \cap V$ we have $Ax = PQx = x$. It means that $f(\lambda)$ is divisible by $(\lambda - 1)^m$, where $m = \dim(U \cap V)$. Therefore $f(\lambda) = (\lambda - 1)^m g(\lambda)$, where $\deg(g) = \deg(f) - m = = \dim(U) - \dim(U \cap V) \leq [\frac{n}{2}]$. Since $\deg(g) \leq [\frac{n}{2}]$, the polynomial g is a product of linear and quadratic polynomials with the negative discriminant. Let K be the complexification of F. Then $g(\lambda) = \prod_{i=1}^{\deg g} (\lambda - \lambda_i)$, where $\lambda_i \in K$. Every λ_i is an eigenvalue of the selfadjoint operator A and in fact $\lambda_i \in F$. (The proof of this fact is fully analogical to the case $F = \mathbb{R}$. For this case see [2], p. 460.) Now, analogically to the real case ([2], p. 461) using induction it may be constructed an orthonormal basis $a_1, \ldots, a_{k_1}$ of U consisting of eigenvectors of A. The proof may be finished in the same way as the proof of Theorem 1.

Let K be either the complexification of an ordered field F or the quaternion algebra over F. All results of this paper may be reformulated for the case, when F is replaced by K. The axioms for the scalar product are the following

$$\begin{aligned} &x \cdot y \in K \text{ for all } \ x, y \in X \\ &y \cdot x = (x \cdot y)^* \\ &(\alpha x) \cdot y = \alpha(x \cdot y) \\ &(x_1 + x_2) \cdot y = x_1 \cdot y + x_2 \cdot y \\ &x \cdot x > 0 \text{ for } x \neq 0 \ . \end{aligned}$$

(If $x{\cdot}y = \alpha+\beta i$, then $(x{\cdot}y)^* = \alpha-\beta i$. If $x{\cdot}y = \alpha+\beta i+\gamma j+\delta k$, then $(x{\cdot}y)^* = \alpha-\beta i-\gamma j-\delta k$.)

In the reformulation of Theorem 2 F has the same property and X is a vector space over K.

It would be interesting to characterize the class of ortholattices in which any two elements have consistent orthogonal atomic partitions. By the results of the present paper this problem is connected with eigenvalues when it is considered in the realm of ortholattices associated with a vector space with a scalar product over an ordered field. So, in a general case this problem may be difficult. Therefore any partial result will be interesting.

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