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SPECIAL ALMOST R-PARACONTACT CONNECTIONS

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Abstract. For an almost r-paracontact manifold M_n with a structure Σ and any linear connection Γ on M_n, all almost r-paracontact connections (making all structure tensors parallel) have been found. Also, D-connections for which some distributions on M_n are parallel have been considered. Finally, pairs of connections compatible with a structure Σ have been discussed.

1. Almost r-paracontact manifolds. An almost r-paracontact structure is the generalization of an almost product structure on a differentiable manifold. The study of almost r-paracontact structures (utilizing, in part, certain distributions of tangent bundles of manifolds generated by these structures) and almost r-paracontact connections on manifolds provides a foundation for the investigation of geometric and topological properties of these manifolds.

In this section we recall the definition of an almost r-paracontact manifold [1] and present some of their properties.

Definition 1.1. Let M_n be an n-dimensional differentiable manifold. If on M_n there exist: a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, \dots, \xi_r$ $(r < n)$, r 1-forms $\eta^1, \eta^2, \dots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \qquad \alpha, \beta \in (r) = 1, 2, ..., r \tag{1.1}$$

$$\phi^2 = Id - \eta^\alpha \otimes \xi_\alpha, \quad \text{where } a^\alpha b_\alpha \stackrel{\text{def}}{=} \sum_\alpha a^\alpha b_\alpha \tag{1.2}$$

$$\eta^\alpha \circ \phi = 0, \qquad \alpha \in (r) \tag{1.3}$$

then $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ is said to be an *almost r-paracontact structure* on M_n, and M_n is an *almost r-paracontact manifold.*

From (1.1), (1.2), and (1.3) we also have

$$\phi(\xi_\alpha) = 0, \qquad \alpha \in (r). \tag{1.4}$$

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There exists a positive Riemannian metric g on M_n [1] such that

$$\eta^\alpha(X) = g(X, \xi_\alpha), \qquad \alpha \in (r) \tag{1.5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y). \tag{1.6}$$

Then, $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is called a *metric almost r-paracontact structure* on M_n, and g is said to be compatible Riemannian metric.

From (1.1) through (1.6) we get

$$g(\phi X, Y) = g(X, \phi Y). \tag{1.7}$$

Remark 1.1. On an almost r-paracontact manifold M_n with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ the tensor ϕ has constant eigenvalues $1, -1$, and 0. Let p, q, and r be their multiplicities, respectively, with $p + q + r = n = \dim M_n$. M_n is said to be of type (p, q).

Proposition 1.1 [1]. *An almost r-paracontact manifold M_n admits the following complementary distributions*

$$D^+ = \{X;\quad \phi X = X\} \tag{1.8}$$

$$D^- = \{X;\quad \phi X = -X\} \tag{1.9}$$

$$D^0 = \{X;\quad \phi X = 0\} \tag{1.8}$$

with $\dim D^+ = p,\ \dim D^- = q,\ \dim D^0 = r,$ *and* $p + q + r = n.$

Theorem 1.1. *A necessary and sufficient condition for M_n to admit an almost r-paracontact structure is that there exist three complementary distributions D_1, D_2, and D_3 of dimensions p, q, and r, respectively, with $p + q + r = n$.*

Proof. The necessary condition follows immediately from Proposition 1.1. Now, let D_1, D_2, and D_3 be three complementary distributions of dimensions p, q, and r, respectively, with $p + q + r = n$. For any $p \in M_n$ we have $T_pM_n = D_{1p} \oplus D_{2p} \oplus D_{3p}$. Let $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}, e_{p+q+1} = \xi_1, \ldots, e_n = \xi_r\}$ be a basis for T_pM_n, and let $\{e^1, \ldots, e^p, e^{p+1}, \ldots, e^{p+q}, e^{p+q+1} = \eta^1, \ldots, e^n = \eta^r\}$ be the dual basis for the cotangent space $T_p^*M_n$, i.e., $e^i(e_j) = \delta^i_j,\ \ i, j \in (n)$. Hence, we get $e^a \otimes e_a + e^{p+\lambda} \otimes e_{p+\lambda} + \eta^\alpha \otimes \xi_\alpha = Id,\ \ a \in (p), \lambda \in (q), \alpha \in (r)$. Let $\phi = e^a \otimes e_a + \varepsilon\, e^{p+\lambda} \otimes e_{p+\lambda},\ \ \varepsilon = \pm 1$. Then, the structure $(\phi, \xi_\alpha, \eta^\alpha)$ is an almost r-paracontact structure on M_n. □

Definition 1.2. If M_n is an almost r-paracontact manifold with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$, then Σ is said to be normal if an almost product structure F defined on $M_n \times R^r$ by $F(X, f^\alpha \frac{d}{dt^\alpha}) = (\phi X + f^\alpha \xi_\alpha, \eta^\alpha(X)\frac{d}{dt^\alpha})$ is integrable, i.e., its Nijenhuis tensor field N_F vanishes.

Theorem 1.2 [1]. *An almost r-paracontact structure Σ on M_n is normal if and only if $N(X, Y) = N_\phi(X, Y) - 2d\eta^a(X, Y)\xi_\alpha = 0$, where N_ϕ is the Nijenhuis tensor for ϕ.*

2. Projection operators on almost r-paracontact manifolds. In this section various projection operators on an almost r-paracontact manifold are defined.

Definition 2.1. A multilinear operator Φ on an appropriate space is said to be a *projection operator* if $\Phi^2 = \Phi$.

Definition 2.2. A set of operators $\{\Phi_i\}$ is said to be the *set of complementary projection operators* if $\sum_i \Phi_i = Id, \quad \Phi_i^2 = \Phi_i, \quad \Phi_i\Phi_j = 0, \quad i \neq j$.

Proposition 2.1 [3]. *If Φ is a projection operator and $\Psi = Id - \Phi$, then Φ and Ψ are complementary projection operators and all solutions of the equation $\Phi x = y$ are of the form $x = y + \Psi w$, where w is an arbitrary element.*

Remark 2.1. If operators Φ and Ψ are tensor fields of type (2,2) and S, ϕ, X are tensor fields of type (1,2), (1,1), and (0,1) respectively, then the operations $\Phi\Psi, \Phi S, \Phi\phi, \Phi X$ are expressed locally as follows: $\Phi^{ij}_{kl}\Psi^{ml}_{in}, \Phi^{ij}_{kl}S^{l}_{mi}, \Phi^{ij}_{kl}\phi^{l}_{i}, \Phi^{ij}_{kl}X^k$.

Let $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ be an almost r-paracontact structure on M_n. We define the following operators on M_n.

$$\phi_1 = \tfrac{1}{2}(Id + \phi - \eta^\alpha \otimes \xi_\alpha) = \tfrac{1}{2}(\phi^2 + \phi) \tag{2.1}$$

$$\phi_2 = \tfrac{1}{2}(Id - \phi - \eta^\alpha \otimes \xi_\alpha) = \tfrac{1}{2}(\phi^2 - \phi) \tag{2.2}$$

$$\phi_3 = \eta^\alpha \otimes \xi_\alpha = Id - \phi^2 \tag{2.3}$$

$$\phi_4 = \phi^2 = Id - \phi_3 \tag{2.4}$$

with the properties

$$\begin{array}{c} \phi = \phi_1 - \phi_2, \quad \phi\phi_1 = \phi_1\phi = \phi_1, \quad \phi\phi_2 = \phi_2\phi = -\phi_2, \\ \phi\phi_3 = \phi_3\phi = 0, \quad \phi\phi_4 = \phi_4\phi = \phi \end{array}. \tag{2.5}$$

Proposition 2.2. *The operators ϕ_1, ϕ_2, ϕ_3 are complementary projection operators on an almost r-paracontact manifold M_n.*

Proposition 2.3. *The operators ϕ_3 and ϕ_4 are complementary projection operators on an almost r-paracontact manifold M_n.*

The distributions (1.8), (1.9), and (1.10) can be expressed as follows

$$D^+ = \{X; \quad \phi_1 X = X\} \tag{2.6}$$

$$D^- = \{X; \quad \phi_2 X = X\} \tag{2.7}$$

$$D^0 = \{X; \quad \phi_3 X = X\}. \tag{2.8}$$

Proposition 2.4. *The distributions D^+, D^-, and D^0 are generated by the projection operators ϕ_1, ϕ_2, and ϕ_3, respectively.*

Let A, B, and C be tensor fields of type (2,2) defined as

$$A = \tfrac{1}{2}(Id \otimes Id - \phi \otimes \phi) \tag{2.9}$$

$$B = \tfrac{1}{2}(Id \otimes Id + \phi \otimes \phi) \tag{2.10}$$

$$C = \tfrac{1}{2}(\phi_3 \otimes Id + Id \otimes \phi_3 - \phi_3 \otimes \phi_3). \tag{2.11}$$

The operators A, B, and C possess the following properties

$$\begin{array}{c} A + B = Id \otimes Id, \quad AA = A - \dfrac{1}{2}C, \quad BB = B - \dfrac{1}{2}C, \\ AB = BA = AC = CA = BC = CB = CC = \dfrac{1}{2}C \end{array}. \tag{2.12}$$

Define two operators

$$F = A + C \tag{2.13}$$

$$H = B - C. \tag{2.14}$$

Proposition 2.5. *The operators F and H are complementary projection operators on an almost r-paracontact manifold M_n.*

Remark 2.2. The operators F and H can be expressed in the following form

$$F = Id \otimes Id - \phi_1 \otimes \phi_1 - \phi_2 \otimes \phi_2 \tag{2.15}$$

$$H = \phi_1 \otimes \phi_1 + \phi_2 \otimes \phi_2. \tag{2.16}$$

Define two more operators

$$P = F - \phi_3 \otimes \phi_3 \tag{2.17}$$

$$Q = H + \phi_3 \otimes \phi_3 \tag{2.18}$$

or

$$P = Id \otimes Id - \phi_1 \otimes \phi_1 - \phi_2 \otimes \phi_2 - \phi_3 \otimes \phi_3 \tag{2.19}$$

$$Q = \phi_1 \otimes \phi_1 + \phi_2 \otimes \phi_2 + \phi_3 \otimes \phi_3. \tag{2.20}$$

Proposition 2.6. *The operators P and Q are complementary projection operators on an almost r-paracontact manifold M_n.*

3. Almost r-paracontact connections. In this section the definition of an almost r-paracontact connection on an almost r-paracontact manifold has been given and all such connections are found.

Definition 3.1 [2]. For an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ a linear connection Γ, given by its covariant derivative ∇, is said to be an *almost r-paracontact* if

$$\nabla_X \phi = 0 \tag{3.1}$$

$$\nabla_X \eta^\alpha = 0, \qquad \alpha \in (r) \tag{3.2}$$

for any vector field X.

From (3.1) and (3.2) it follows

$$\nabla_X \xi_\alpha = 0, \qquad \alpha \in (r). \tag{3.3}$$

If M_n is a metric almost r-paracontact manifold and an almost r-paracontact connection Γ satisfies

$$\nabla_X g = 0 \tag{3.4}$$

then it is called a *metric almost r-paracontact* on M_n.

Now, assume that Γ, given by its covariant derivative ∇, is a linear connection on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$. We are going to find an almost r-paracontact connection $\overline{\Gamma}$ given by $\overline{\nabla}$ in the form

$$\overline{\nabla}_X = \nabla_X + S_X \tag{3.5}$$

where S is a tensor field of type (1.2) with $S_X(Y) = S(X,Y)$.

For tensors fields f, Y, and ω of type (1,1), (0,1), and (1,0) respectively, we have

$$\overline{\nabla}_X f = \nabla_X f + S_X \circ f - f \circ S_X \tag{3.6}$$

$$\overline{\nabla}_X Y = \nabla_X Y + S_X(Y) \tag{3.7}$$

$$\overline{\nabla}_X \omega = \nabla_X \omega - \omega \circ S_X. \tag{3.8}$$

For the structure tensors ϕ, ξ_α, and η^α, we have

$$\overline{\nabla}_X \phi = \nabla_X \phi + S_X \circ \phi - \phi \circ S_X \tag{3.9}$$

$$\overline{\nabla}_X \xi_\alpha = \nabla_X \xi_\alpha + S_X(\xi_\alpha), \qquad \alpha \in (r) \tag{3.10}$$

$$\overline{\nabla}_X \eta^\alpha = \nabla_X \eta^\alpha - \eta^\alpha \circ S_X, \qquad \alpha \in (r). \tag{3.11}$$

Since $\overline{\Gamma}$ is an almost r-paracontact connection, so S_X satisfies

$$\nabla_X \phi = \phi \circ S_X - S_X \circ \phi \tag{3.12}$$

$$\nabla_X \xi_\alpha = -S_X(\xi_\alpha), \qquad \alpha \in (r) \tag{3.13}$$

$$\nabla_X \eta^\alpha = \eta^\alpha \circ S_X, \qquad \alpha \in (r). \tag{3.14}$$

Applying ϕ on the right to (3.12) and making use of (3.13) and (1.2) we obtain

$$S_X - \phi S_X \phi = \phi \nabla_X \phi + \nabla_X \eta^\alpha \otimes \xi_\alpha \tag{3.15}$$

or using the projection operator A from (2.9)

$$AS_X = \tfrac{1}{2}(\phi \nabla_X \phi + \nabla_X \eta^\alpha \otimes \xi_\alpha) \tag{3.16}$$

or making use of (2.5), (2.1)–(2.4), and Proposition 2.2

$$AS_X = \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \tfrac{1}{2}\phi_3 \nabla_X \phi_3 + \tfrac{1}{2}\eta^\alpha \otimes \nabla_X \xi_\alpha - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha. \tag{3.17}$$

Acting with the operator C from (2.11) on (3.16) and using (2.12) we get

$$\begin{aligned} CS_X = \tfrac{1}{2}(&\phi \nabla_X \phi \phi_3 + \nabla_X \eta^\alpha \circ \phi_3 \otimes \xi_\alpha + \phi_3 \phi \nabla_X \phi + \nabla_X \eta^\alpha \otimes \phi_3(\xi_\alpha) \\ &- \phi_3 \phi \nabla_X \phi \phi_3 - \nabla_X \eta^\alpha \circ \phi_3 \otimes \phi_3(\xi_\alpha)). \end{aligned}$$

Making use of (2.5), (2.1)–(2.4), and Proposition 2.2 we get

$$CS_X = \tfrac{1}{2}(\phi_3 \nabla_X \phi_3 - \eta^\alpha \otimes \nabla_X \xi_\alpha) \tag{3.18}$$

Adding (3.17) and (3.18) up, and using (2.13) we get

$$FS_X = \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha. \tag{3.19}$$

Making use of Propositions 2.5 and 2.1 we obtain from (3.19)

$$S_X = \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha + HW_X \tag{3.20}$$

where W is an arbitrary tensor field of type (1,2) with $W_X(Y) = W(X,Y)$, or

$$S_X = \tfrac{1}{2}\phi \nabla_X \phi + \nabla_X \eta^\alpha \otimes \xi_\alpha - \tfrac{1}{2}\eta^\alpha \otimes \nabla_X \xi_\alpha + \tfrac{1}{2}\eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha + HW_X. \tag{3.21}$$

Theorem 3.1 [2]. *The general family of the almost r-paracontact connections $\overline{\Gamma}$ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ is given by*

$$\overline{\nabla}_X = \nabla_X + \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha + HW_X \tag{3.22}$$

where H is defined by (2.16), *W is an arbitrary tensor field of type* (1,2) *with $W_X(Y) = W(X,Y)$, and ∇ is the covariant derivative of an arbitrary linear connection Γ on M_n.*

Corollary 3.1. *If the initial connection Γ is an a.r-p.-connection on M_n, then the general family of a.r-p.-connections $\overline{\Gamma}$ on M_n is given by*

$$\overline{\nabla}_X = \nabla_X + HW_X \tag{3.23}$$

where W is an arbitrary tensor field.

Now, suppose that M_n is an almost r-paracontact Riemannian manifold with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$. Let Γ, given by its covariant derivative ∇, be the Riemannian connection generated by the Riemannian metric g on M_n. Of all almost r-paracontact connections $\overline{\Gamma}$ given by (3.22) we will find metric connections, i.e., those satisfying $\overline{\nabla} g = 0$. For the Riemannian metric g and $\overline{\Gamma}$ given by (3.5) we have $(\overline{\nabla}_Z g)(X,Y) = -g(S_Z X, Y) - g(X, S_Z Y)$. $\overline{\Gamma}$ is a metric almost r-paracontact connection if and only if

$$g(S_Z X, Y) + g(X, S_Z Y) = 0. \tag{3.24}$$

From (1.5) and (1.1) we have

$$\eta^\alpha(\nabla_Z \xi_\beta) + \eta^\beta(\nabla_Z \xi_\alpha) = 0 \tag{3.25}$$

$$(\nabla_Z \eta^\alpha)(X) = g(X, \nabla_Z \xi_\alpha). \tag{3.26}$$

From (1.6), using (1.5) and (1.2), we get

$$\begin{aligned} &g(\nabla_Z(\phi X), \phi Y) + g(\phi X, \nabla_Z(\phi Y)) - g(\phi^2(\nabla_Z X), Y) \\ &-g(X, \phi^2(\nabla_Z Y)) + \sum_i (\nabla_Z \eta^\alpha)(X)\eta^\alpha(Y) + \sum_i (\nabla_Z \eta^\alpha)(Y)\eta^\alpha(X) = 0. \end{aligned} \tag{3.27}$$

From (3.24), using (3.21), (1.7), (3.26), and then (3.27) and (3.25), we get

$$\begin{aligned} &\tfrac{1}{2} g(\phi\nabla_Z \phi X, Y) - \tfrac{1}{2}\eta^\alpha(X) g(\nabla_Z \xi_\alpha, Y) + \nabla_Z \eta^\alpha(X) g(\xi_\alpha, Y) + \tfrac{1}{2} g(X, \phi\nabla_Z \phi Y) \\ &-\tfrac{1}{2}\eta^\alpha(Y) g(X, \nabla_Z \xi_\alpha) + \nabla_Z \eta^\alpha(Y) g(X, \xi_\alpha) + \eta^\alpha(\nabla_Z \xi_\beta)\eta^\beta(X) g(\xi_\alpha, Y) \\ &+\eta^\alpha(\nabla_Z \xi_\beta)\eta^\beta(Y) g(X, \xi_\alpha) + g(H_X W_Z, Y) + g(X, H_Y W_Z) = \\ &= \tfrac{1}{2} g(\nabla_Z(\phi X), \phi Y) - \tfrac{1}{2} g(\phi^2(\nabla_Z X), Y) + \tfrac{1}{2} g(\phi X, \nabla_Z(\phi Y)) - \tfrac{1}{2} g(X, \phi^2(\nabla_Z Y)) \\ &+\tfrac{1}{2}\sum_\alpha (\nabla_Z \eta^\alpha)(X)\eta^\alpha(Y) + \tfrac{1}{2}\sum_\alpha (\nabla_Z \eta^\alpha)(Y)\eta^\alpha(X) \\ &+\tfrac{1}{2}\sum_{\alpha,\beta} \eta^\alpha(X)\eta^\beta(Y)[\eta^\alpha(\nabla_Z \xi_\beta) + \eta^\beta(\nabla_Z \xi_\alpha)] + g(H_X W_Z, Y) + g(X, H_Y W_Z) = \\ &= g(H_X W_Z, Y) + g(X, H_Y W_Z) = 0. \end{aligned}$$

Hence,

Theorem 3.2. *Let M_n be an almost r-paracontact Riemannian manifold with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$. An almost r-paracontact connection $\overline{\Gamma}$ given by* (3.22) *on M_n is metric if and only if there exists a tensor field W of type* (1,2) *with $W(X,Y) = W_X(Y)$ satisfying*

$$g(H_X W_Z, Y) + g(X, H_Y W_Z) = 0. \tag{3.28}$$

From Theorem 3.1 we get

Theorem 3.3. *The linear connection $\overline{\Gamma}$ given by*

$$\overline{\nabla}_X = \nabla_X + \phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3 - \eta^\alpha(\nabla_X\xi_\beta)\eta^\beta \otimes \xi_\alpha \tag{3.29}$$

is a metric almost r-paracontact connection on an almost r-paracontact Riemannian manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, where ∇ is the Riemannian connection, i.e., $\overline{\nabla}\phi = 0$, $\overline{\nabla}\eta^\alpha = 0$, $\overline{\nabla}\xi_\alpha = 0$, $\overline{\nabla} g = 0$.

4. D-connections. In this section the definition of a D-connection on an almost r-paracontact manifold is given and all such connections are determined.

Definition 4.1. Let D be a distribution on a manifold M_n. A linear connection Γ given by its covariant derivative ∇ on M_n is said to be a *D-connection*, or D is said to be *parallel with respect to* Γ, if for any vector field Y and a vector field X from D the vector field $\nabla_Y X$ belongs to D.

Theorem 4.1. *The distribution D^+ given by* (1.8) *or* (2.6) *is parallel with respect to a linear connection Γ given by its covariant derivative ∇ on an almost r-paracontact manifold M_n, or Γ is a D^+-connection, if and only if*

$$\nabla\phi_1 \circ \phi_1 = 0. \tag{4.1}$$

Proof. For a vector field X from D^+ one obtains

$$\nabla_Y X = \nabla_Y(\phi_1 X) = (\nabla_Y \phi_1)X + \phi_1(\nabla_Y X). \tag{4.2}$$

If D^+ is parallel, then from (4.2) and (2.6) $(\nabla_Y\phi_1)X = 0$ for any vector field X in D^+, or $(\nabla_Y\phi_1)(\phi_1 Z) = 0$ for any vector field Z. Hence, (4.1) is obtained. Conversely, if (4.1) is satisfied, then for any $X \in D^+$, $0 = (\nabla\phi_1 \circ \phi_1)X = (\nabla\phi_1)X$, so by (4.2) $\nabla_Y X = \phi_1(\nabla_Y X)$, and $\nabla_Y X$ is in D^+. □

In similar way we obtain

Theorem 4.2. *The distribution D^- given by* (1.9) *or* (2.7) *is parallel with respect to a linear connection Γ given by its covariant derivative ∇ on an almost r-paracontact manifold M_n, or Γ is a D^--connection, if and only if*

$$\nabla\phi_2 \circ \phi_2 = 0. \tag{4.3}$$

Theorem 4.3. *The distribution D^0 given by* (1.10) *or* (2.8) *is parallel with respect to a linear connection Γ given by its covariant derivative ∇ on an almost r-paracontact manifold M_n, or Γ is a D^0-connection, if and only if*

$$\nabla\phi_3 \circ \phi_3 = 0. \tag{4.4}$$

If Γ is an almost r-paracontact connection, then from (2.1), (2.2), and (2.3) we get $\nabla\phi_1 = \nabla\phi_2 = \nabla\phi_3 = 0$, and on account of Theorems 4.1, 4.2, and 4.3 we obtain

Theorem 4.4. *If a linear connection Γ on an almost r-paracontact manifold M_n is an almost r-paracontact connection, then the distributions D^+, D^-, and D^0 given by* (1.8), (1.9) *and* (1.10) *are parallel with respect to this connection, or Γ is a D^+-connection, and a D^--connection, and a D^0-connection.*

Definition 4.2. A linear connection Γ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ is said to be a *D_Σ-connection* if it is a D^+-connection, and a D^--connection, and a D^0-connection.

Theorem 4.5. *Every almost r-paracontact connection Γ on an almost r-paracontact manifold M_n with a structure Σ is a D_Σ-connection.*

If a linear connection Γ is a D_Σ-connection, then from Proposition 2.2 and Theorems 4.1, 4.2, and 4.3 we have

$$\nabla\phi_i = \phi_i\nabla\phi_i, \qquad i = 1, 2, 3 \tag{4.5}$$

and since $\phi_1 + \phi_2 + \phi_3 = Id$, we get

$$\sum_i \phi_i\nabla\phi_i = 0. \tag{4.6}$$

Acting with ϕ_j on the left to (4.6) we obtain

$$\phi_j\nabla\phi_j = 0, \qquad j = 1, 2, 3 \tag{4.7}$$

then, from (4.5) we get

Theorem 4.6. *A linear connection Γ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ is a D_Σ-connection if and only if*

$$\nabla\phi_i = 0, \qquad i = i, 2, 3. \tag{4.8}$$

Now, we shall find all D_Σ-connections on an almost r-paracontact manifold M_n which are of the form

$$\hat\nabla_X = \nabla_X + J_X \tag{4.9}$$

where ∇ is the covariant differentiation operator of arbitrary linear connection Γ on M_n, and J is a tensor field of type (1,2) with $J_X(Y) = J(X, Y)$. From (3.6) we have for ϕ_i, $i = 1, 2, 3$

$$\hat\nabla_X\phi_i = \nabla_X\phi_i + J_X \circ \phi_i - \phi_i \circ J_X \qquad i = 1, 2, 3. \tag{4.10}$$

Since $\hat\Gamma$ is a D_Σ-connection, then from (4.10) and Theorem 4.6 we have

$$\nabla_X\phi_i + J_X \circ \phi_i - \phi_i \circ J_X = 0 \qquad i = 1, 2, 3. \tag{4.11}$$

Applying ϕ_i on the left to (4.11) and using Proposition 2.2 we get

$$\phi_i\nabla_X\phi_i + \phi_iJ_X\phi_i - \phi_iJ_X = 0, \qquad i = 1, 2, 3. \tag{4.12}$$

Using Proposition 2.2 we get from (4.12)

$$\phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3 + \phi_1J_X\phi_1 + \phi_2J_X\phi_2 + \phi_3J_X\phi_3 - J_X = 0. \tag{4.13}$$

Using (2.19) we obtain

$$PJ_X = \phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3. \tag{4.14}$$

Hence, in virtue of Propositions 2.6 and 2.1 we obtain

$$J_X = \phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3 + QV_X \tag{4.15}$$

where V is an arbitrary tensor field of type (1,2) with $V_X(Y) = V(X, Y)$.

Hence,

Theorem 4.7. *The general family of the D_Σ-connections $\hat{\Gamma}$ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ is given by*

$$\hat{\nabla}_X = \nabla_X + \phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3 + QV_X \tag{4.16}$$

where Q is defined by (2.20)*, V is an arbitrary tensor field of type* (1,2) *with $V_X(Y) = V(X,Y)$, and ∇ is the covariant derivative of an arbitrary linear connection Γ on M_n.*

Corollary 4.1. *If the initial connection Γ is a D_Σ-connection on M_n, then the general family of D_Σ-connections $\hat{\Gamma}$ on M_n is given by*

$$\hat{\nabla}_X = \nabla_X + QV_X \tag{4.17}$$

where V is an arbitrary tensor field.

5. Pairs of connections compatible with a structure. In this section a definition of a pair of connections compatible with an almost r-paracontact structure on an almost r-paracontact manifold is given and all such pairs are found.

Let $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$ be two linear connections, given by their covariant derivatives $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, on an almost r-paracontact manifold M_n.

For a function f, a vector field Y, a 1-form ω, and a tensor field ψ of type (1,1), (0,2), or (2,0), we define the following *mixed covariant derivatives*

$$\overset{ij}{\nabla}_Z f = \overset{ji}{\nabla}_Z f = Zf \tag{5.1}$$

$$\overset{ij}{\nabla}_Z Y = \overset{i}{\nabla}_Z Y \tag{5.2}$$

$$\overset{ij}{\nabla}_Z \omega = \overset{i}{\nabla}_Z \omega \tag{5.3}$$

$$(\overset{ij}{\nabla}_Z \psi)(A,B) = Z\psi(A,B) - \psi(\overset{i}{\nabla}_Z A, B) - \psi(A, \overset{j}{\nabla}_Z B) \tag{5.4}$$

where $i, j = 1, 2;\ i \neq j$.

Proposition 5.1. $\overset{m}{\nabla}_Z = \frac{1}{2}(\overset{1}{\nabla}_Z + \overset{2}{\nabla}_Z)$ *is a covariant differentiation operator of a certain connection $\overset{m}{\Gamma}$ on M_n.*

Definition 5.1. The connection $\overset{m}{\Gamma}$ on M_n is called a *mean connection* of $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$ if its *mean covariant derivative* is

$$\overset{m}{\nabla}_Z = \frac{1}{2}(\overset{1}{\nabla}_Z + \overset{2}{\nabla}_Z). \tag{5.5}$$

Proposition 5.2. $d_Z = \overset{2}{\nabla}_Z - \overset{1}{\nabla}_Z$ *is a tensor field of type* (1,1) *on M_n* for any vector field Z.

Definition 5.2. The tensor field d of type (1,2) defined by

$$d_Z = \overset{2}{\nabla}_Z - \overset{1}{\nabla}_Z \tag{5.6}$$

with $d_Z(X) = d(Z, X)$ on M_n is called a *deformation tensor field* of connections $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$.

For the structure tensors ϕ and η^α we obtain

$$d_Z(\eta^\alpha) = -\eta^\alpha \circ d_Z \tag{5.7}$$

$$d_Z(\phi) = d_Z \circ \phi - \phi \circ d_Z \tag{5.8}$$

$$\overset{12}{\nabla}_Z \phi = \overset{2}{\nabla}_Z \phi + \phi \circ d_Z \tag{5.9}$$

$$\overset{21}{\nabla}_Z \phi = \overset{1}{\nabla}_Z \phi - \phi \circ d_Z. \tag{5.10}$$

Making use of these formulas we get for the structure tensors ϕ, η^α, and ξ_α

$$\tfrac{1}{2}(\overset{12}{\nabla}_Z \eta^\alpha + \overset{21}{\nabla}_Z \eta^\alpha) = \overset{m}{\nabla}_Z \eta^\alpha \tag{5.11}$$

$$\overset{12}{\nabla}_Z \eta^\alpha - \overset{21}{\nabla}_Z \eta^\alpha = \eta^\alpha \circ d_Z \tag{5.12}$$

$$\tfrac{1}{2}(\overset{12}{\nabla}_Z \phi + \overset{21}{\nabla}_Z \phi) = \overset{m}{\nabla}_Z \phi \tag{5.13}$$

$$\overset{12}{\nabla}_Z \phi - \overset{21}{\nabla}_Z \phi = d_Z \circ \phi + \phi \circ d_Z \tag{5.14}$$

$$\tfrac{1}{2}(\overset{12}{\nabla}_Z \xi_\alpha + \overset{21}{\nabla}_Z \xi_\alpha) = \overset{m}{\nabla}_Z \xi_\alpha \tag{5.15}$$

$$\overset{12}{\nabla}_Z \xi_\alpha - \overset{21}{\nabla}_Z \xi_\alpha = -d_Z \xi_\alpha. \tag{5.16}$$

For (1,1) tensor fields ψ and χ, using (5.8), (5.14) and (5.9), (5.10) we get

$$\overset{1}{\nabla}_Z(\psi\chi) = (\overset{21}{\nabla}_Z \psi)\chi + \psi \overset{12}{\nabla}_Z \chi \tag{5.17}$$

$$\overset{2}{\nabla}_Z(\psi\chi) = (\overset{12}{\nabla}_Z \psi)\chi + \psi \overset{21}{\nabla}_Z \chi. \tag{5.18}$$

Definition 5.3. A pair of linear connections $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ is said to be *compatible with* Σ if

$$\overset{12}{\nabla} \phi = 0, \qquad \overset{1}{\nabla} \eta^\alpha = 0, \qquad \overset{2}{\nabla} \xi_\alpha = 0, \qquad \alpha \in (r). \tag{5.19}$$

Theorem 5.1. *A pair of linear connections* $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ *on an almost r-paracontact manifold* M_n *with a structure* $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ *is compatible with* Σ *if and only if all structure tensor fields are parallel with respect to both mixed covariant derivatives* $\overset{12}{\nabla}$ *and* $\overset{21}{\nabla}$.

Proof. We have to show that the following conditions are satisfied

(5.20) $$\overset{21}{\nabla}\phi = 0, \qquad \overset{2}{\nabla}\eta^\alpha = 0, \qquad \overset{1}{\nabla}\xi_\alpha = 0, \qquad \alpha \in (r).$$

From (1.1) we get

(5.21) $$(\overset{i}{\nabla}\eta^\alpha)(\xi_\beta) + \eta^\alpha(\overset{i}{\nabla}\xi_\beta) = 0, \qquad i = 1, 2$$

From (1.3), using (5.9) and $(5.19)_1$, we get $\overset{2}{\nabla}\eta^\alpha \circ \phi = 0$. Hence $\overset{2}{\nabla}\eta^\alpha - (\overset{2}{\nabla}\eta^\alpha)(\xi_\beta)\eta^\beta = 0$, and using (5.21) we obtain $(5.20)_2$. From (1.4), using $(5.19)_1$ and (5.9), we get $\phi\overset{1}{\nabla}\xi_\alpha = 0$. Hence $\overset{1}{\nabla}\xi_\alpha - \eta^\beta(\overset{1}{\nabla}\xi_\alpha)\xi_\beta = 0$. Again, using (5.21) we obtain $(5.20)_3$. Now, making use of (5.17) and (5.18) with $\psi = \chi = \phi$ we obtain from (1.2), after using $(5.19)_2$, $(5.19)_3$, $(5.20)_2$, $(5.20)_3$, and $(5.19)_1$, $(\overset{21}{\nabla}\phi)\phi = 0$, $\phi\overset{21}{\nabla}\phi = 0$. Hence $\overset{21}{\nabla}\phi = \overset{21}{\nabla}\phi^3 = \phi\overset{21}{\nabla}\phi\phi + \overset{21}{\nabla}\phi\phi^2 + \phi^2\overset{21}{\nabla}\phi = 0$. □

Remark 5.1. From Theorem 5.1 we obtain the symmetry of compatibility, i.e., a pair $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ is compatible with Σ if and only if $(\overset{2}{\Gamma}, \overset{1}{\Gamma})$ is compatible with Σ.

Now, we obtain the following

Theorem 5.2. *A pair of linear connections $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ is compatible with Σ if and only if*

(i) *the mean connection $\overset{m}{\Gamma}$ given by* (5.5) *is an almost r-paracontact connection on M_n*
(ii) *$Bd_Z = 0$, where B is given by* (2.10)

Proof. Using (5.11) through (5.16) the conditions (5.19) and (5.20) are equivalent to

(5.22) $$\overset{m}{\nabla}\phi = 0, \qquad \overset{m}{\nabla}\eta^\alpha = 0, \qquad \overset{m}{\nabla}\xi_\alpha = 0, \qquad \alpha \in (r)$$

(5.23) $$d_Z\phi + \phi d_Z = 0, \qquad \eta^\alpha d_Z = 0, \qquad d_Z(\xi_\alpha) = 0, \qquad \alpha \in (r).$$

The conditions (5.22) are equivalent to (i) and (5.23) to (ii). □

Remark 5.2. The condition (ii) of Theorem 5.2 implies

(5.24) $$Hd_Z = 0$$

where H is given by (2.14).

Hence we get

Theorem 5.3. *If a pair of linear connections $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ is compatible with an almost r-paracontact structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ on a manifold M_n, then*

(5.25) $$\overset{1}{\nabla}_Z = \nabla_Z + S_Z - \tfrac{1}{2}FV_Z$$

(5.26) $$\overset{2}{\nabla}_Z = \nabla_Z + S_Z + \tfrac{1}{2}FV_Z$$

where ∇ is an arbitrary linear connection on M_n, S_Z is given by (3.21), *F is defined by* (2.15), *and V is an arbitrary tensor field of type* (1,2) *with $V_Z(X) = V(Z, X)$.*

Proof. From (5.24), using Proposition 2.1, we obtain

$$d_Z = FV_Z. \tag{5.27}$$

From Theorem 5.2(i), using Theorem 3.1, we get

$$\overset{m}{\nabla}_Z = \nabla_Z + S_Z. \tag{5.28}$$

From (5.27), (5.28), (5.5), and (5.6) we obtain (5.25) and (5.26). □

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ON NON-VERTICAL LINEAR (1,1)-TENSOR FIELDS AND CONNECTIONS ON TANGENT BUNDLES

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Abstract. Special (1,1)-tensor fields α on tangent bundles TM which do not preserve the vertical subbundles VTM and satisfy some linearity conditions determine linear connections Γ_α on TM. The relationships between α and Γ_α are studied in this paper.

INTRODUCTION

In the papers [1], [2] we have dealt with connections on TM which are canonicaly determined by (1,1)-tensor fields on tangent bundles TM, especially by almost complex structures on TM. This contribution completes our previous considerations by some (1,1)-tensor fields which do not preserve the subbundle VTM of all vertical tangent vectors on TM and satisfy some linearity conditions. We suppose that all manifolds and maps are infinitely differentiable.

Let (x^i, x_1^i) be the local chart on the tangent bundle $\pi : TM \to M$ induced by a local chart (x^i) on a manifold M. Then the coordinate form of an arbitrary (1,1)-tensor α is as follows

$$\alpha = (a_j^i dx^j + b_j^i dx_1^j) \otimes \partial/\partial x^i + (c_j^i dx^j + h_j^i dx_1^j) \otimes \partial/\partial x_1^i \ ,$$

where $a_j^i, b_j^i, c_j^i, h_j^i$ are functions of the variables x^k, x_1^k.

A connection Γ on TM can be considered as a special (1,1)-tensor field $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma_j^i(x, x_1) dx^j \otimes \partial/\partial x_1^i$ such that $h_\Gamma(VTM) = 0$ and $T\pi \cdot h_\Gamma = T\pi$ where Tf denotes the tangent prolongation of a map f. Then $v_\Gamma = Id_{TM} - h_\Gamma$ is the vertical form of Γ. Denote by $H\Gamma := h_\Gamma(TM) \subset TM$ the vector subbundle of the Γ-horizontal vectors on TM, i.e. such vectors $(x^i, x_1^i, dx^i, dx_1^i)$ on TM which satisfy the equation $dx_1^i = \Gamma_j^i dx^j$. The functions $\Gamma_j^i(x, x_1)$ will be called local functions of Γ. Recall that a connection Γ is linear iff $\Gamma_j^i = \Gamma_{jk}^i(x) x_1^k$.

Let us introduce a short survey of the main results published in [1] we will need.

Let $Y = \eta^i \partial/\partial x_1^i$ be an arbitrary vertical vector field on TM and $X = \xi^i \partial/\partial x^i + \Gamma_j^i \xi^j \partial/\partial x_1^i$ be a horizontal vector field of a given connection Γ on TM. Then $\alpha(Y) =$

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$b^i_j\eta^j\partial/\partial x^i + h^i_j\eta^j\partial/\partial x^i_1$ or $\alpha(X) = (a^i_j + b^i_t\Gamma^t_j)\xi^j\partial/\partial x^i + (c^i_j + h^i_t\Gamma^t_j)\xi^j\partial/\partial x^i_1$, is Γ-horizontal for any vertical vector field Y or vertical for any Γ-horizontal vector field X iff

$$(1) \qquad \Gamma^i_k b^k_j = h^i_j \quad \text{or} \quad a^i_j + b^i_t\Gamma^t_j = 0 \ ,$$

respectively.

Let $J = dx^i \otimes \partial/\partial x^i_1$ be the canonical morphism (almost tangent structure) determined by Id_{TM} and the canonical identification $VTM = TM \times_M TM$. Then the (1, 1)-tensor field $B := J\alpha J = b^i_j dx^j \otimes \partial/\partial x^i_1$ can be interpreted as the vector bundle morphism $T\pi \cdot \alpha|_{VTM} = b^i_j dx^j_1 \otimes \partial/\partial x^i : VTM \to TM$ over $\pi : TM \to M$ or as a vector bundle morphism $VTM \to VTM$, $B = b^i_j dx^j_1 \otimes \partial/\partial x^i_1$.

If $B = 0$ or $B \neq 0$ i.e. if $\alpha(VTM) \subset VTM$ or $\alpha(VTM) \not\subset VTM$ we say that α is vertical or non vertical, respectively.

It is evident from (1) that in the case when α is non vertical and B is an isomorphism then there is a unique connection Γ^1_2 and a unique connection Γ^2_α on TM such that $\alpha(H\Gamma^1_\alpha) = VTM$ and $\alpha(VTM) = H\Gamma^2_\alpha$. Then $\Gamma^i_j = -\tilde{b}^i_k a^k_j$ or $\Gamma^i_j = h^i_k \tilde{b}^k_j$, $\tilde{b}^i_k b^k_j = \delta^i_j$, are local functions of Γ^1_α or Γ^2_α, respectively.

The following coordinate conditions

$$(2) \qquad a^i_s a^s_j + b^i_s c^s_j = -\delta^i_j, \quad a^i_s b^s_j + b^i_s h^s_j = 0, \quad c^i_s a^s_j + h^i_s c^s_j = 0, \quad c^i_s b^s_j - h^i_s h^s_j = -\delta^i_j$$

under which a (1,1)-tensor field α is an almost complex structure on TM and the equalities (1) immediately give

Lemma 1. *Let Γ be a connection on TM. Let $B = b^i_j dx^j_1 \otimes \partial/\partial x^i : VTM \to TM$ be a vector bundle isomorphism over $\pi : TM \to M$. Then there exists a unique almost complex structure $\alpha(\Gamma, B)$ on TM such that $T\pi \cdot \alpha|_{VTM} = B$ and $\Gamma^1_\alpha = \Gamma^2_\alpha = \Gamma$.*

Remark 1. It is easy to see that in the case when B is regular then the third and fourth equations of (2) are the consequence of the first and the second ones. The second equality of (2) is the coordinate condition for α^2 to be vertical and for the equality $\Gamma^1_\alpha = \Gamma^2_\alpha$.

NON VERTICAL LINEAR (1,1)-TENSOR FIELD ON TM

Definition 1. A non vertical (1,1)-tensor field α on TM is said to be v-projectable if the (1,1)-tensor field $B = J\alpha J$ is the v-lift of a (1,1)-tensor field $\overline{B} = b^i_j(x) dx^j \otimes \partial/\partial x^i$ on M.

Definition 2. A v-projectable (1,1)-tensor field α is called l-linear if $T\pi \cdot \alpha X : TM \to TM$ is a vector bundle morphism over Id_M for any projectable and linear vector field $X : TM \to TTM$. Analogously a v-projectable (1,1)-tensor field α is called r-linear if $\alpha(Y^v)$ is a projectable and linear vector field on TM for any vertical lift Y^v of any vector field Y on M.

In coordinates, let $X = \xi^i(x)\partial/\partial x^i + \eta^i_j(x) x^j_1 \partial/\partial x^i_1$ be a projectable and linear vector field on TM and α be v-projectable. Then the map $T\pi \cdot \alpha X$ is given by the equations

$\overline{x}^i = x^i$, $\overline{x}^i_1 = a^i_j(x,x_1)\xi^j + b^i_k(x)\eta^k_j x^j_1$. So it is a vector bundle morphism, i.e. α is l-linear iff $a^i_j(x,x_1) = a^i_{jk}(x)x^k_1$.

Analogously for a field $Y = \xi^i(x)\partial/\partial x^i$ on M its v-lift is $Y^v = \xi^i(x)\partial/\partial x^i_1$ and so $\alpha(Y^v) = b^i_j(x)\xi^j(x)\partial/\partial x^i + h^i_j(x,x_1)\xi^j(x)\partial/\partial x^i_1$ is projectable and linear, i.e. α is r-linear iff $h^i_j(x,x_1) = h^i_{jk}(x)x^k_1$.

Definition 3. A non vertical (1,1)-tensor field α is called linear if it is r- and l-linear.

The following lemma is evident from the equalities (1) and (2).

Lemma 2. *Let α be such a (1,1)-tensor field on TM that B is regular. Then the connection Γ^1_α or Γ^2_α is linear if α is l-linear or r-linear, respectively. Both connections Γ^1_α and Γ^2_α are linear if α is linear. If moreover α^2 is vertical then if α is l-linear or r-linear then α is linear. If connection Γ is linear and B is v-lift of a regular (1,1)-tensor field $\overline{B}$ on M then the almost complex structure $\alpha(\Gamma, B)$ is linear.*

Recall that a semispray S on TM is such a vector field on TM that $J(S) = V$ where $V = x^i_1\partial/\partial x^i_1$ is the canonical (Liouville) vector field the flows of which are the homotheties on individual fibres T_xM. S is a spray if $[S, V] = S$. In local coordinates $S = x^i_1\partial/\partial x^i + \eta^i(x,x_1)\partial/\partial x^i_1$ and it is a spray if $\eta = \eta^i_k(x)x^k_1$.

Every connection Γ on TM with local functions Γ^i_j determines a unique semispray $S_\Gamma = x^i_1\partial/\partial x^i + \Gamma^i_j x^j\partial/\partial x^i_1$ (called Γ-horizontal).

Lemma 3. *Let α be such a (1,1)-tensor field on TM that B is regular. Then $\alpha(V)$ is Γ^2_α-horizontal and $\alpha(B^{-1}(V))$ is the Γ^2_α-horizontal semispray.*

Proof in coordinates. $\alpha(V) = b^i_j x^j_1\partial/\partial x^i + h^i_j x^j_1\partial/\partial x^i_1$ and considering B as a morphism on VTM we have $B^{-1}(V) = \tilde{b}^i_j x^j_1\partial/\partial x^i$, $\alpha(B^{-1}(V)) = x^i_1\partial/\partial x^i + h^i_k\tilde{b}^k_j x^j_1\partial/\partial x^i_1$. The proof is finished.

In concordance with many authors (for example Modugno [4], Yano [5]) using the Nijenhuis-Frölicher bracket $[\alpha, \beta]$ of two vector valued tensor fields we introduced the following notions

Definition 4. We will say that a (1,1)-tensor field α is quasi-symmetric or semi-symmetric or symmetric if $[\alpha, J]$ is a vertical valued or a semi-basic vertical valued (1,2)-form on TM or vanishes, respectively. We will say that a connection Γ is symmetric, (equivalently without torsion), when its torsion $T_\Gamma = [h_\Gamma, J]$ vanishes.

Throughout this paper we will use the denotations $f_i := \frac{\partial f}{\partial x^i}$, $f_{i_1} := \frac{\partial f}{\partial x^i_1}$.

By direct computations we get in the case of a v-projectable (1,1)-tensor field:

$$[\alpha, J] = a^i_{sj_1}dx^j \wedge dx^s \otimes \partial/\partial x^i + [(c^i_{sj_1} + a^i_{js})dx^j \wedge dx^s + (b^i_{js} - h^i_{js_1} - a^i_{sj_1})dx^j_1 \wedge dx^s] \otimes \otimes \partial/\partial x^i_1$$

$$[h_\Gamma, J] = \Gamma^i_{sj_1}dx^j \wedge dx^s \otimes \partial/\partial x^i_1 \ , \tag{3}$$

where Γ^i_j are the local functions of a connection Γ. So

$$a^i_{sj_1} = a^i_{js_1} \quad \text{or} \quad a^i_{sj_1} = a^i_{js_1},\ b^i_{js} - h^i_{js_1} - a^i_{sj_1} = 0 \tag{4}$$

are the coordinate conditions of α to be quasi- or semi-symmetric.

Proposition 1. *Let α be a v-projectable (1,1)-tensor field on TM such that $\overline{B}$ is regular. Then the connection Γ^1_α is symmetric if and only if α is quasi-symmetric.*

Proof follows from the relationships (3), (4) and from the local functions $-b^i_k(x)a^k_j$ of the connection Γ^1_α.

Remark 2. If α is l-linear and $\overline{B}$ is regular then $a^i_j = a^i_{jk}(x)x^k_1$ and so α is quasi-symmetric iff $a^i_{jk} = a^i_{kj}$.

Let $i_2 : TTM \to TTM, (x^i, x^i_1, dx^i, dx^i_1) \to (x^i, dx^i, x^i_1, dx^i_1)$ be the canonical involution on TTM. Let $T\overline{B} : (x^i, x^i_1, dx^i, dx^i_1) \to (x^i, b^i_j x^j_1, dx^i, b^i_{kj}x^k_1 dx^j + b^i_j dx^j_1)$ denote the tangent prolongation of the map $\overline{B} : TM \to TM, (x^i, x^i_1) \to (x^i, b^i_j x^j_1)$ given by a (1,1)-tensor field $\overline{B}$ on M. Consider the following maps:

$$\begin{aligned} i_2 \cdot T\overline{B} \cdot i_2 :&(x^i, x^i_1, dx^i, dx^i_1) \to (x^i, x^i_1, b^i_j dx^j, b^i_{kj} dx^k x^j_1 + b^i_j dx^j_1) \\ J\alpha :&(x^i, x^i_1, dx^i, dx^i_1) \to (x^i, x^i_1, 0, a^i_j dx^j + b^i_j dx^j_1) \\ \alpha J :&(x^i, x^i_1, dx^i, dx^i_1) \to (x^i, x^i_1, b^i_j dx^j, h^i_j dx^j) \ . \end{aligned}$$

Then the map

$$\begin{aligned} i_2 \cdot T\overline{B} \cdot i_2 - J\alpha - \alpha J :&(x^i, x^i_1, dx^i, dx^j_1) \to (x^i, x^i_1, 0, b^i_{kj} dx^k x^j_1 - (a^i_j + h^i_j)dx^j) = \\ &= (x^i, b^i_{kj} dx^k x^j_1 - (a^i_j + h^i_j)dx^j) \end{aligned}$$

determines in the case of a linear (1,1)-tensor field α, (i.e. $a^i_j = a^i_{jk}x^k_1,\ h^i_j = h^i_{jk}x^k_1$), a (1,2)-tensor α_B on M,

$$\alpha_B = (b^i_{jk} - a^i_{jk} - h^i_{jk}) dx^j \otimes dx^k \otimes \partial/\partial x^i \ . \tag{5}$$

Proposition 2. *Let α be a quasi-symmetric and linear (1,1)-tensor field on TM. Then α is semi-symmetric if and only if $\alpha_B = 0$.*

Proof. Using $a^i_{jk} = a^i_{kj}$ and comparing (4) with (5) we complete our proof.

By the map $T\overline{B}$ we can construct another connection $\tilde{\Gamma}$ on TM as follows: We have

$$i_2 \cdot T\overline{B} \cdot i_2 - \alpha J = [(b^i_{jk} x^k_1 - h^i_j) dx^j + b^i_j dx^j_1] \otimes \partial/\partial x^i_1 \ .$$

Considering B as a vector bundle isomorphism $VTM \to VTM$ over Id_{TM} we see that the following (1,1)-tensor field

$$B^{-1}[i_2 \cdot T\overline{B} \cdot i_2 - \alpha J] = [\tilde{b}^i_t (b^t_{jk} x^k_1 - h^t_j) dx^j + dx^j_1] \otimes \partial/\partial x^i_1$$

is the vertical form of the connection $\tilde{\Gamma}$ the local functions of which are $\Gamma^i_j = -\tilde{b}^i_t(b^t_{jk}x^k_1 - h^t_j)$.

Proposition 3. *Let α be a linear quasi-symmetric (1,1)-tensor field on TM the morphism B of which is regular. Then α is semi-symmetric if and only if the equality $\tilde{\Gamma} = \Gamma^1_\alpha$ is satisfied.*

Proof. Under our suppositions the equality $\tilde{\Gamma} = \Gamma^1_\alpha$ has the following coordinate form

$$-\tilde{b}^i_t(b^t_{jk} - h^t_{jk})x^k_1 = -\tilde{b}^i_t a^t_{jk} x^k_1, \quad \text{i.e.} \quad b^i_{jk} - h^i_{jk} = a^i_{jk} \ .$$

Comparing it with (4) we finish our proof.

Recall that if Γ is a linear connection on TM with local functions $\Gamma^i_j = \Gamma^i_{jk}x^k_1$, $\overline{B} = b^i_j dx^j \otimes \partial/\partial x^i$ is a (1,1)-tensor field on M and $X = \xi^i \partial/\partial x^i$ is a vector field on M then the covariant derivative $\nabla^\Gamma_X \overline{B}$ is the (1,1)-tensor field on M the coordinate form of which is as follows

$$\nabla^\Gamma_X \overline{B} = (b^i_{kj} + b^i_s \Gamma^s_{jk} - \Gamma^i_{js} b^s_k)\xi^j dx^k \otimes \partial/\partial x^i \ .$$

Then $\nabla^\Gamma \overline{B} = (b^i_{kj} + b^i_s \Gamma^s_{jk} - \Gamma^i_{js} b^s_k) dx^j \otimes dx^k \otimes \partial/\partial x^i$ is a (1,2)-tensor field on M and the equality

$$b^i_{kj} + b^i_s \Gamma^s_{jk} - \Gamma^i_{js} b^s_k = 0 \tag{6}$$

is the coordinate condition for $\overline{B}$ to be constant under the covariant derivative with respect to the connection Γ.

Remark 3. The covariant derivative $\nabla^\Gamma \overline{B}$ can be interpreted as follows. If $\overline{B} = b^i_j dx^j \otimes \partial/\partial x^i$ and if $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma^i_{jk} x^k_1 dx^j \otimes \partial/\partial x^i_1$ then the map $TB \cdot h_\Gamma - h_\Gamma \cdot TB \cdot h_\Gamma$: $\overline{x}^i = x^i$, $\overline{x}^i_1 = b^i_j x^j_1$, $d\overline{x}^i = 0$, $d\overline{x}^i_1 = (b^i_{kj} + b^i_s \Gamma^s_{jk} - \Gamma^i_{js} b^s_k) x^k_1 dx^j$ determines the tensor field $\nabla^\Gamma \overline{B}$ on M.

Lemma 4. *Let $\overline{B} = b^i_j dx^j \otimes \partial/\partial x^i$ be a (1,1)-tensor field on M and $B = b^i_j dx^j \otimes \partial/\partial x^i_1$ be its vertical lift on TM. Let Γ be a symmetric linear connection on TM. Then the Nijenhuis-Frölicher bracket $[h_\Gamma, B]$ is the skew-symmetrization of the v-lift of the tensor field $\nabla^\Gamma \overline{B}$.*

Proof follows immediately from the coordinate form $[h_\Gamma, B] = (b^i_{kj} + \Gamma^i_{ku} b^u_j) dx^j \wedge dx^k \otimes \partial/\partial x^i_1$.

Let $S = x^i_1 \partial/\partial x^i + \eta(x, x_1)\partial/\partial x^i_1$ be a semi-spray on TM and $B = b^i_j(x) dx^j \otimes \partial/\partial x^i_1$ be the v-lift of a regular (1,1)-tensor field $\overline{B}$ on M. Then the Lie derivative

$$L_S B = -b^i_j dx^j \otimes \partial/\partial x^i + [(b^i_{jk} x^k_1 - \eta^i_{k_1} b^k_j) dx^j + b^i_j dx^j_1] \otimes \partial/\partial x^i_1$$

is the so-called skew 2-projectable (1,1)-tensor field on TM, see [2]. Such a (1,1)-tensor field β determines a unique connection Γ_{BS} the horizontal subbundle $H\Gamma_{BS}$ of which spannes the vectors Y on TM for which $L_{\overline{S}} L_S B(Y) \in VTM$, where $\overline{S}$ is an arbitrary semi-spray on TM on which the connection Γ_{BS} is independent. It is easy to calculate that

$$\Gamma^i_j = -\frac{1}{2}\tilde{b}^i_s (2b^s_{jk} x^k_1 - \eta^s_{k_1} b^k_j) \tag{7}$$

are its local functions.

Denote by $A : \alpha \to \Gamma^1_\alpha$ and by $H : \alpha \to \Gamma^2_\alpha$ the operators from the space α_v of all v-projectable (1,1)-tensor fields α on TM with regular $B = J\alpha J$ into the space Conn TM of all connections on TM. Evidently A, H are zero- order natural operators in the category $\mathcal{M}_m$ of all smooth manifolds and smooth local diffeomorphisms.

Let $DB : V(1,1) \times S \to$ Conn TM denote the operator from the product space of all v-lifts of all regular (1,1)-tensor fields on M and of all semisprays on TM into the space Conn TM given by the above described rule $DB : (S, B) \to \Gamma_{BS}$. It is easy to see that DB is a natural operator of first order in the category $\mathcal{M}_n$. Reader is kindly refered to [3] for the theory of natural operators.

Proposition 4. *Let B be the v-lift of a regular (1,1)-tensor field $\overline{B}$ on M. Let Γ be a linear symmetric connection on TM. Then the following conditions are equivalent*

a) $\nabla^\Gamma \overline{B} = 0$

b) $DB(S_\Gamma, B) = \Gamma$

where S_Γ is the Γ-horizontal spray on TM.

Proof. Let $S_\Gamma = x^i_1 \partial/\partial x^i + \eta^i \partial/\partial x^i_1,\ \eta^i = \Gamma^i_{jk} x^j_1 x^k_1$, be Γ-horizontal spray. Then by (7) $\Gamma^i_j = -\tilde{b}^i_s (b^s_{jk} - \Gamma^s_{su} b^u_j) x^k_1$ are the local functions of the connection $DB(S_\Gamma, B)$. So locally $DB(S_\Gamma, B) = \Gamma$ if and only if

$$b^i_{kj} - \Gamma^i_{js} b^s_k + b^i_s \Gamma^s_{kj} = 0$$

which coincides with (6), i.e. with the condition $\nabla^\Gamma \overline{B} = 0$. Proof is finished.

It is evident from the definitions of operators A and DB that

Proposition 5. $A(L_S L_S B) = DB(S, B)$.

Remember that if α^2 is vertical then $\Gamma^1_\alpha = \Gamma^2_\alpha$, i.e., $H(\alpha) = A(\alpha)$.

Proposition 6. *Let α be a l-linear and quasi-symmetric (1,1)-tensor field on TM such that α^2 is vertical and B is regular. Then the following conditions are equivalent*

i) $\nabla^{A(\alpha)} B = 0$,

ii) *α is semi-symmetric.*

Proof. The equality (6) for the connection $\Gamma^1_\alpha = A(\alpha)$ yields $b^i_{kj} - b^i_s \tilde{b}^s_t a^t_{jk} + \tilde{b}^i_t a^t_{js} b^s_k = 0$. Then using the second relationship of (2) we have $b^i_{kj} - a^i_{jk} - h^i_{kj} = 0$. So the equalities (4) are satisfied, i.e. α is semi-symmetric iff $\nabla^{A(\alpha)} B = 0$.

Corollary. *Under the conditions of Proposition 6 α is semi-symmetric if and only if $DB(S_{\Gamma^1_\alpha}, B) = \Gamma^1_\alpha$.*

Let Γ be a given connection on TM. Recall that the Γ-lift of a vector $X \in T_x M$ at $u \in TM$ is the Γ-horizontal vector $\Gamma(X) \in T_u TM$ such that $T\pi(\Gamma(X)) = X$. Let $\overline{B}$ be a (1,1)-tensor field on M. Then $\overline{B}$ acts on the Γ-horizontal vectors on TM by the rule: If $Y \in T_u TM$ then $\overline{B} Y = \Gamma(\overline{B}(T\pi Y)) \in T_u TM$, i.e. $\overline{B} Y$ is Γ-lift of the vector $\overline{B}(T\pi Y)$ at u. In coordinates, $\overline{B}(\xi^i \partial/\partial x^i + \Gamma^i_j \xi^j \partial/\partial x^i_1) = b^i_j \xi^j \partial/\partial x^i + \Gamma^i_t b^t_j \xi^j \partial/\partial x^i_1$.

Let α be a r-linear (1,1)-tensor field on TM. Recall that $\overline{B}$ is a (1,1)-tensor on M determined by $B = J\alpha J$. Let Γ be a linear connection on TM then the map $\overline{B}h_\Gamma - \alpha J : (x^i, x_1^i, dx^i, dx_1^i) \to (x^i, x_1^i, 0, (\Gamma^i_{sk} b^s_j - h^i_{jk}) x_1^k dx^j)$ determines the (1,2)-tensor field $\overline{B}h_\Gamma - \alpha J = (\Gamma^i_{sk} b^s_j - h^i_{jk}) dx^j \otimes dx^k \otimes \partial/\partial x^i$ on M.

Proposition 7. *Let α be a quasi-symmetric and l-linear (1,1)-tensor field on TM such that $\overline{B}$ is regular, α^2 is vertical and α_B is symmetric (1,2)-tensor. Then just the connection $\Gamma^2_\alpha = \Gamma^1_\alpha = \Gamma_\alpha$ is such a linear symmetric connection on TM that the tensor $\nabla^\Gamma B$ is symmetric and the tensor $\overline{B}h_{\Gamma_\alpha} - \alpha J$ is skew-symmetric.*

Proof. By our assumptions α is linear and (see (2), (5)),

$$a^i_{jk} = a^i_{kj}, \quad a^i_{sk} b^s_j = -b^i_s h^s_{jk}, \quad b^i_{jk} - h^i_{jk} = b^i_{kj} - h^i_{kj} \ . \tag{8}$$

Then the tensor $\nabla^\Gamma B$ is symmetric and $\overline{B}h_{\Gamma_\alpha} - \alpha J$ is skew-symmetric iff

$$b^i_{jk} - b^i_{kj} - \Gamma^i_{ku} b^u_j + \Gamma^i_{ju} b^u_k = 0 \quad \text{i.e.,} \quad h^i_{jk} - h^i_{kj} - \Gamma^i_{uk} b^u_j + \Gamma^i_{uj} b^u_k = 0 \ , \tag{9}$$

$$\Gamma^i_{uk} b^u_j - h^i_{jk} + \Gamma^i_{uj} b^u_k - h^i_{kj} = 0 \ , \tag{10}$$

where Γ is a linear symmetric connection. If the relationships (9), (10) are satisfied then

$$\Gamma^i_{uj} b^u_k = h^i_{kj}, \quad \text{i.e.} \quad \Gamma^i_{jk} = h^i_{tk} \tilde{b}^t_j, \quad \text{i.e.} \quad \Gamma = \Gamma^2_\alpha = \Gamma_\alpha \ .$$

Conversely, it easy to show that under the conditions (8) the connection Γ_α is symmetric and satisfies the equalities (9) and (10). The proof is finished.

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VALUATIONS AND METRICS ON A POSET

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Abstract. The aim of the paper is to characterize metrics (pseudometrics) induced on connected posets by positive (isotone) valuations. Further, as an application it is shown that there exists a positive valuation on some posets of locally finite length.

Introduction

Several authors investigated valuations and metrics on posets (see, e.g. [1] - [3]). M. Kolibiar and J. Lihová [5] and J. Lihová [6] gave characterizations of metrics induced by positive (isotone) valuations on directed multilattices. In this paper we will present similar results for richier families of posets.

Recall some basic definitions. Let $\mathbf{F}_n = (\{a_1, \ldots, a_n\}, \leq)$ be a poset. Let n be an odd integer, $n \geq 3$. The poset $\mathbf{F}_n$ is called *a fence* if

$$a_1 < a_2 > a_3 < \cdots < a_{n-1} > a_n$$

or

$$a_1 > a_2 < a_3 > \cdots > a_{n-1} < a_n$$

are the only comparability relations of the poset $\mathbf{F}_n$ (see Figures 1a, 1b). Let n be an even integer, $n \geq 4$. The poset $\mathbf{F}_n$ is called *a fence* if

$$a_1 < a_2 > a_3 < \cdots > a_{n-1} < a_n$$

are the only comparability relations of the poset $\mathbf{F}_n$ (see Fig. 1c).

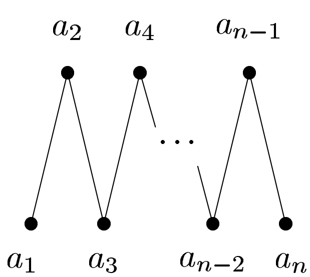

Fig. 1a

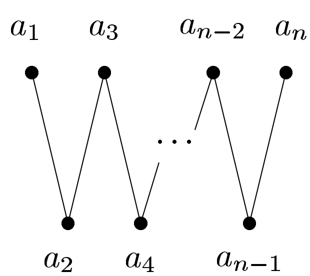

Fig. 1b

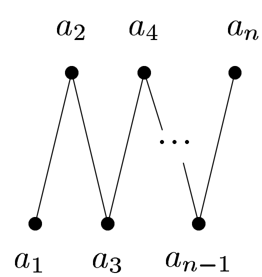

Fig. 1c

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A poset $(P,\leq)$ is called *a connected poset* if for every elements $a,b\in P$ there is a fence $\mathbf{F}=(\{a_1,a_2,\ldots,a_n\},\leq)$ such that $a=a_1$, $b=a_n$. In this case we say that $\mathbf{F}$ is *a fence from a to b.*

Let $\mathbf{C}=(\{a_0,a_1,\ldots,a_n\},\leq)$ be a chain. The number n is called *the length* of the chain $\mathbf{C}$.

A poset $(P,\leq)$ is called *a poset of finite length* if all chains of the poset $(P,\leq)$ are finite and if there exists the maximum of their lengths. We write $l(P)=l_0$ if the maximum of all chains of the poset $(P,\leq)$ is l_0 and the number l_0 is called the *length of the poset* $(P,\leq)$. A poset $(P,\leq)$ is called *a poset of locally finite lenght* if all bounded chains in $(P,\leq)$ are finite. Note, it is possible that a poset of locally finite length has neither maximal nor minimal elements.

Let $(P,\leq)$ be a poset. A real valued function v defined on P is called

a) *a positive valuation* on a poset $(P,\leq)$ if

$$a<b \quad\Longrightarrow\quad v(a)<v(b),$$

b) *an isotone valuation* on a poset $(P,\leq)$ if

$$a\leq b \quad\Longrightarrow\quad v(a)\leq v(b).$$

Throughout the paper we will denote by Z and R the set of all integers and the set of all reals, respectively. We will denote by $|x|$ the absolute value of x.

1. Valuations and metrics on posets

The aim of this part is to characterize metrics induced by isotone (positive) valuations on posets. First we will give some definitions needed for our purposes.

Definition 1.1. Let $(P,\leq)$ be a poset. A finite sequence $(x_i)_{i=0}^{n}$ is said to be a way from a to b if

(a) $x_0=a$, $x_n=b$,
(b) $x_i\leq x_{i+1}$ or $x_i\geq x_{i+1}$ for each $i=0,1,\ldots,n-1$.

Definition 1.2. Let $(P,\leq)$ be a poset and $\mathbf{C}_n=(a_1,b_1,\ldots,a_n,b_n)$ be a 2n-element subposet of $(P,\leq)$. The subposet $\mathbf{C}_n$ is said to be a cycle-fence of $(P,\leq)$ if

(1.0) $$a_1<b_1>a_2<b_2>\cdots>a_n<b_n>a_1.$$

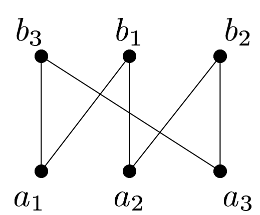

Fig. 2a

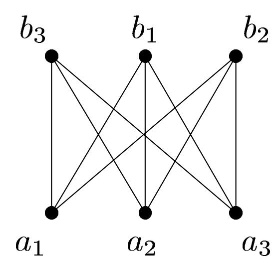

Fig. 2b

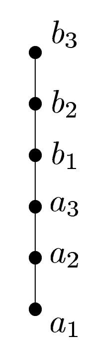

Fig. 2c

Note that the mentioned poset $\mathbf{C}_n$ is a crown if $n > 1$ and (1.0) are the only comparability relations on $\{a_1, b_1, \ldots, a_n, b_n\}$ (see Fig. 2a). The cycle-fences (by the above definition) are also the posets in Figures 2b and 2c.

Definition 1.3. Let $(P, \leq)$ be a poset, v an isotone valuation on $(P, \leq)$ and $(x_i)_{i=0}^n$ a way from a to b, $a, b \in P$. We define the length $l(x_0, x_1, \ldots, x_n)$ of the way $(x_i)_{i=0}^n$ by

$$l(x_0, x_1, \ldots, x_n) =: \sum_{i=0}^{n-1} |v(x_i) - v(x_{i+1})| \tag{1.1}$$

A way $(x_0, x_1, \ldots, x_n)$ from a to b is said to be minimal if

$$l(x_0, x_1, \ldots, x_n) \leq l(y_0, y_1, \ldots, y_m)$$

for every way $(y_0, y_1, \ldots, y_m)$ from a to b.

Evidently, it is possible that there is no way from a to b or there exists a way from a to b but there is no minimal way from a to b.

Definition 1.4. An isotone valuation v on a connected poset $(P, \leq)$ is said to be a distance-valuation if there exists a minimal way from a to b for all $a, b \in P$.

Definition 1.5. Let v be a distance-valuation on a connected poset $(P, \leq)$. We define a non-negative real function $d_v : P \times P \longrightarrow R$ by

$$d_v(a, b) =: l(x_0, x_1, \ldots, x_n) \tag{1.2}$$

where $(x_i)_{i=0}^n$ is a minimal way from a to b. The function d_v will be called the distance function induced by the distance-valuation v on the poset $(P, \leq)$.

Lemma 1.6. *Let d_v be a distance function induced by a distance-valuation v on a connected poset $(P, \leq)$. Then for all $a, b \in P$*

$$a < b \quad \Longrightarrow \quad d_v(a, b) = v(b) - v(a). \tag{1.3}$$

Proof. It is sufficient to prove that $(x_0, x_1) = (a, b)$ is a minimal way from a to b. Let $(x_i)_{i=0}^n$ be an arbitrary way from a to b. Then
$l(a, x_1, \ldots, x_{n-1}, b) = |v(a) - v(x_1)| + |v(x_1) - v(x_2)| + \cdots + |v(x_{n-1}) - v(b)| \geq$
$\geq |v(a) - v(x_1) + v(x_1) - v(x_2) + \cdots + v(x_{n-1}) - v(b)| = |v(a) - v(b)| = l(a, b)$, i.e. (a, b) is a minimal way from a to b. $\square$

Corollary 1.7. *Let $d_v : P \times P \longrightarrow R$ be a distance function induced by a distance-valuation v on a connected poset $(P, \leq)$. Let $(m_0, m_1, \ldots, m_{k-1}, m_k)$ be a minimal way from a to b. Then*

$$d_v(a, b) = d_v(m_0, m_1) + d_v(m_1, m_2) + \cdots + d_v(m_{k-1}, m_k). \tag{1.4}$$

Lemma 1.8. *Let d_v be a distance function induced by a distance-valuation v on a connected poset $(P, \leq)$. Then*

$$\forall a, b, c \in P \quad a < b < c \quad \Longrightarrow \quad d_v(a,c) = d_v(a,b) + d_v(b,c) \tag{1.5}$$

and

$$\begin{aligned} &d_v(a_1,b_1) + d_v(a_2,b_2) + \cdots + d_v(a_n,b_n) = \\ &= d_v(b_1,a_2) + d_v(b_2,a_3) + \cdots + d_v(b_n,a_1) \end{aligned} \tag{1.6}$$

holds for every cycle-fence $(a_1, b_1, \ldots, a_n, b_n)$ of the poset $(P, \leq)$.

Proof. It is easy to verify that (1.5) and (1.6) follow from (1.3). □

Theorem 1.9. *Let d_v be a distance function induced by a distance-valuation v on a connected poset $(P, \leq)$. Then*
d_v is a metric on the poset $(P, \leq)$ if v is a positive valuation,
d_v is a pseudometric on the poset $(P, \leq)$ if v is a isotone valuation.

Proof. Let $a, b, c \in P$. Obviously, $d_v(a,a) = l(a,a) = 0$. If $a \neq b$ and v is a positive valuation then $d_v(a,b) > 0$. If $(x_0, x_1, \ldots, x_n)$ is a way from a to b then $(y_0, y_1, \ldots, y_n) = (x_n, x_{n-1}, \ldots, x_0)$ is the way from b to a, hence $d_v(a,b) = d_v(b,a)$. Let $(x_0, x_1, \ldots, x_n)$, $(y_0, y_1, \ldots, y_m)$ be minimal ways from a to b and from b to c, respectively. Then $(x_0, x_1, \ldots, x_n, y_1, \ldots, y_m)$ is the way from a to c and consequently, $d_v(a,c) \leq d_v(a,b) + d_v(b,c)$. □

Theorem 1.10. *Let d be a metric (pseudometric) on a poset $(P, \leq)$ satisfying the following three conditions*

(i) *for all $a, b \in P$ there exists a (minimal) way $(m_0, m_1, \ldots, m_k)$ from a to b for which*
$$d(a,b) = d(m_0,m_1) + d(m_1,m_2) + \cdots + d(m_{k-1},m_k),$$

(ii) $a < b < c \quad \Longrightarrow \quad d(a,c) = d(a,b) + d(b,c)$ *for all $a, b, c \in P$,*

(iii)
$$\begin{aligned} &d(a_1,b_1) + d(a_2,b_2) + \cdots + d(a_n,b_n) = \\ &= d(b_1,a_2) + d(b_2,a_3) + \cdots + d(b_n,a_1) \end{aligned}$$
for every cycle-fence $(a_1, b_1, \ldots, a_n, b_n)$ of the poset $(P, \leq)$.

Then there exists a positive (isotone) distance-valuation v_d on $(P, \leq)$ and $d_{v_d} = d$.

Proof. By (ii) we can consider for noncomparable elements a, b only such ways $(x_0, x_1, \ldots, x_n)$ from a to b for which

$$x_i < x_{i+1} > x_{i+2} \quad \text{or} \quad x_i > x_{i+1} < x_{i+2} \tag{1.7}$$

for each $i = 0, 1, \dots, n-2$, i.e. so-called fence ways.

Fix an element $c \in P$. Let $a \in P$. We define the valuation v_d on P by

$$(1.8) \qquad v_d(a) =: \begin{cases} v_d(c) + d(c, x_1) - d(x_1, x_2) + \cdots + (-1)^{n+1} d(x_{n-1}, a), \\ \text{if } c < x_1 \\ \\ v_d(c) - d(c, x_1) + d(x_1, x_2) - \cdots + (-1)^{n} d(x_{n-1}, a), \\ \text{if } c > x_1 \end{cases}$$

where $v_d(c)$ is any but fixed value from R and $(x_0, x_1, \dots, x_{n-1}, x_n)$ is any fence way from c to a.

Now we will show that the function v_d is well-defined, i.e. $v_d(a)$ does not depend on a choice of the way from c to a and that v_d is the positive (isotone) valuation.

Let $(x_0, x_1, \dots, x_{n-1}, x_n)$, $(y_0, y_1, \dots, y_{m-1}, y_m)$ be two arbitrary fence ways from c to a (i.e. $c = x_0 = y_0$, $a = x_n = y_m$). Then one of the following subposet is a cycle-fence

1) $(c, x_1, \dots, x_{n-1}, a, y_{m-1}, \dots, y_1)$,
2) $(c, x_1, \dots, x_{n-1}, y_{m-1}, \dots, y_1)$,
3) $(x_1, \dots, x_{n-1}, a, y_{m-1}, \dots, y_1)$,
4) $(x_1, \dots, x_{n-1}, y_{m-1}, \dots, y_1)$.

In the first case we distinguish

1a) $c < x_1$ and $c < y_1$ and $a < x_{n-1}$ and $a < y_{m-1}$ (see Fig 3),
1b) $c < x_1$ and $c < y_1$ and $a > x_{n-1}$ and $a > y_{m-1}$,
1c) $c > x_1$ and $c > y_1$ and $a < x_{n-1}$ and $a < y_{m-1}$,
1d) $c > x_1$ and $c > y_1$ and $a > x_{n-1}$ and $a > y_{m-1}$.

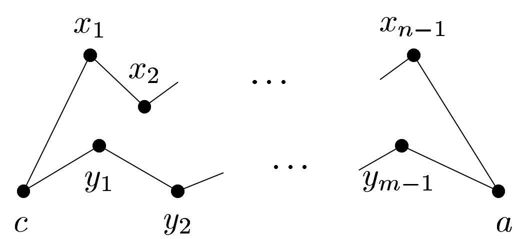

Fig. 3

(We illustrate in Fig. 3 only comparable relations among the elements of the fence ways $(x_0, x_1, \dots, x_n)$ and $(y_0, y_1, \dots, y_m)$, respectively.)

Let 1a) hold. By (iii) we get

$d(c, x_1) + d(x_2, x_3) + \cdots + d(x_{n-2}, x_{n-1}) + d(y_{m-1}, a) + d(y_{m-2}, y_{m-3}) +$
$+ \cdots + d(y_1, y_2) =$
$= d(x_1, x_2) + d(x_3, x_4) + \cdots + d(x_{n-1}, a) + d(y_{m-1}, y_{m-2}) + \cdots + d(y_1, c).$

It implies

$$v_d(a) = v_d(c) + d(c,x_1) - d(x_1,x_2) + d(x_2,x_3) - \cdots + d(x_{n-2},x_{n-1}) - d(x_{n-1},a) =$$
$$= v_d(c) + d(c,y_1) - d(y_1,y_2) + d(y_2,y_3) - \cdots + d(y_{m-2},y_{m-1}) - d(y_{m-1},a),$$

i.e. $v_d(a)$ does not depend on the choice of a way from c to a.
The other cases 1b), 1c), 1d) can be handled in the same way. Analogously to the case 1), we can distinguish four subcases for each of the cases 2), 3) and 4).
For example, let $(x_1, x_2, \dots, x_{n-1}, y_{m-1}, \dots, y_1)$ be a cycle-fence (the case 4)) and let $x_1 > c > y_1$ and $y_{m-1} > a > x_{n-1}$ (see Fig. 4). By (iii) we obtain

$$d(x_1,x_2) + d(x_3,x_4) + \cdots + d(x_{n-2},x_{n-1}) + d(y_{m-1},y_{m-2}) + \cdots + d(y_2,y_1) =$$
$$= d(y_1,x_1) + d(x_2,x_3) + \cdots + d(x_{n-1},y_{m-1}) + d(y_{m-2},y_{m-3}) + \cdots + d(y_3,y_2),$$

where $d(y_1,x_1) = d(y_1,c) + d(c,x_1)$ and $d(x_{n-1},y_{m-1}) =$
$= d(x_{n-1},a) + d(y_{m-1},a)$ by (ii).

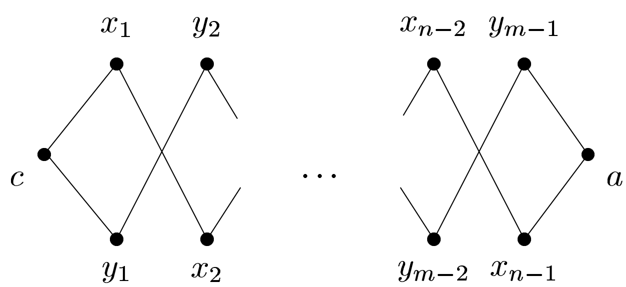

Fig. 4

Therefore,

$$v_d(a) = v_d(c) + d(c,x_1) - d(x_1,x_2) + d(x_2,x_3) - \cdots + d(x_{n-1},a) =$$
$$= v_d(c) - d(c,y_1) + d(y_1,y_2) - d(y_2,y_3) + \cdots - d(y_{m-1},a).$$

Now we are going to prove that v_d is a positive (isotone) valuation.
Let $b < a$ and let $(x_0, x_1 \dots, x_n)$, $(y_0, y_1, \dots, y_m)$ be fence ways from c to a and from c to b, respectively. Similarly as above we can distinguish some cases.

1) $(c, x_1, \dots, x_{n-1}, a, y_{m-1}, \dots, y_1)$ is a cycle-fence and $c < x_1$, $c < y_1$, $x_{n-1} < a$, $y_{m-1} < b$ (Fig. 5).

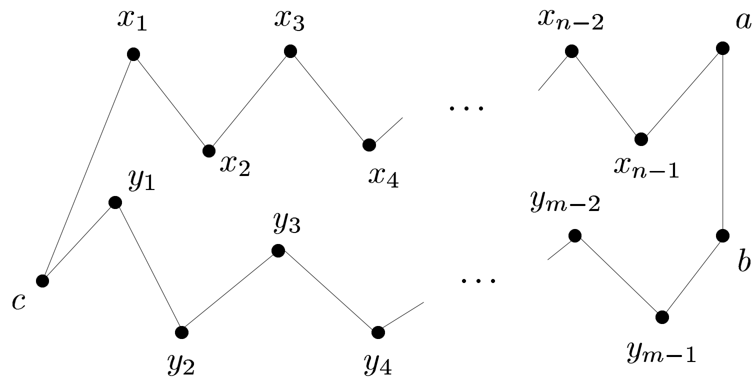

Fig. 5

Then we get by (iii)
$d(c,x_1)+d(x_2,x_3)+\cdots+d(x_{n-1},a)+d(y_{m-1},y_{m-2})+\cdots+d(y_3,y_4)+d(y_1,y_2)-$
$-d(x_1,x_2)-d(x_3,x_4)-\cdots-d(x_{n-2},x_{n-1})-d(a,y_{m-1})-d(y_{m-3},y_{m-2})-\cdots-d(y_2,y_3)-$
$d(y_1,c)=0$

and
$d(y_{m-1},a)=d(y_{m-1},b)+d(a,b)$ by (ii).

It implies
$v_d(a)-v_d(b)=v_d(c)+d(c,x_1)-d(x_1,x_2)+d(x_2,x_3)-\cdots-d(x_{n-2},x_{n-1})+d(x_{n-1},a)-$
$v_d(c)-d(c,y_1)+d(y_1,y_2)-d(y_2,y_3)+\cdots+d(y_{m-2},y_{m-1})-d(y_{m-1},b)=d(a,b)$ and $d(a,b)>0$ if d is a metric and $d(a,b)\geq 0$ if d is a pseudometric. Thus $v_d(a)>v_d(b)$, $v_d(a)\geq v_d(b)$, respectively.

The other subcases of the case 1) can be handled in the same way.

2) Let $(x_1,\ldots,x_{n-1},y_{m-1},y_{m-2},\ldots,y_1)$ be a cycle-fence and $y_1<c<x_1$, $y_{m-1}<b<a<x_{n-1}$ (Fig. 6).

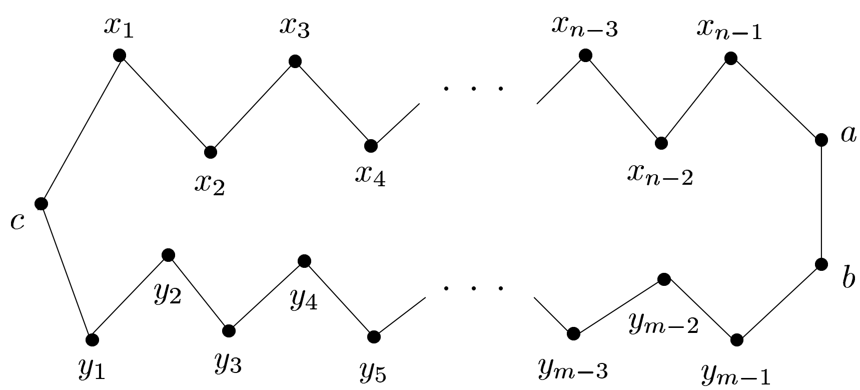

Fig. 6

By (iii) holds
$d(y_1,x_1)+d(x_2,x_3)+\cdots+d(x_{n-2},x_{n-1})+d(y_{m-1},y_{m-2})+d(y_{m-3},y_{m-4})+\cdots+$
$+d(y_2,y_3)-d(x_1,x_2)-d(x_3,x_4)-\cdots-d(x_{n-3},x_{n-2})-d(x_{n-1},y_{m-1})-d(y_{m-2},y_{m-3})-$

$\cdots - d(y_3, y_4) - d(y_1, y_2) = 0$

and by (ii)

$d(y_1, x_1) = d(y_1, c) + d(c, x_1)$ and $d(x_{n-1}, y_{m-1}) = d(x_{n-1}, a) + d(a, b) + d(b, y_{m-1})$.

Hence

$v_d(a) - v_d(b) = v_d(c) + d(c, x_1) - d(x_1, x_2) + d(x_2, x_3) - \cdots - d(x_{n-3}, x_{n-2}) + d(x_{n-2}, x_{n-1}) - d(x_{n-1}, a) - v_d(c) + d(c, y_1) - d(y_1, y_2) + d(y_2, y_3) - \cdots - d(y_{m-3}, y_{m-2}) + d(y_{m-2}, y_{m-1}) - d(y_{m-1}, b) = d(a, b)$,

i.e. $v_d(a) > v_d(b)$ if d is a metric and $v_d(a) \geq v_d(b)$ if d is a pseudometric.

The proofs in all other cases can be done analogously.

We are going to show that $d_{v_d} = d$.

Let $a, b \in P$ and let $(m_0, m_1, \ldots, m_k)$ be a way from a to b for which (i) holds. Let, for example, $a < m_1$ and $m_{k-1} < b$. Then

$$\begin{aligned} d_{v_d}(a,b) =& (v_d(m_1) - v_d(a)) + (v_d(m_1) - v_d(m_2)) + \cdots + (v_d(b) - v_d(m_{k-1})) = \\ =& (v_d(a) + d(a, m_1) - v_d(a)) + (v_d(a) + d(a, m_1) - v_d(a) - d(a, m_1) + \\ &+ d(m_1, m_2)) + \cdots + (v_d(a) + d(a, m_1) - d(m_1, m_2) + \cdots - \\ &- d(m_{k-2}, m_{k-1}) + d(m_{k-1}, b) - v_d(a) - d(a, m_1) + \\ &+ d(m_1, m_2) - \cdots + d(m_{k-2}, m_{k-1})) = \\ =& d(a, m_1) + d(m_1, m_2) + \cdots + d(m_{k-1}, b) = d(a, b) \end{aligned}$$

The proof is complete. □

Theorem 1.11. *Let d_v be a metric (pseudometric) induced by a positive (isotone) distance-valuation v on a poset $(P, \leq)$ and $c \in P$ be an arbitrary but fixed element. Let $a \in P$. We define (in the same way as in (1.8)) the valuation v_{d_v} on P by*

$v_{d_v}(c) =: v(c)$

$$v_{d_v}(a) =: \begin{cases} v(c) + d_v(c, x_1) - d_v(x_1, x_2) + d_v(x_2, x_3) - \cdots + (-1)^{n+1} d_v(x_{n-1}, a), \\ \text{if } c < x_1 \\ \\ v(c) - d_v(c, x_1) + d_v(x_1, x_2) - d_v(x_2, x_3) + \cdots + (-1)^{n} d_v(x_{n-1}, a), \\ \text{if } c > x_1 \end{cases}$$

where $(x_0, x_1, \ldots, x_{n-1}, x_n)$ is any fence way from c to a.
Then v_{d_v} is the positive (isotone) valuation on the poset $(P, \leq)$ and $v_{d_v} = v$.

Proof. We can show that v_{d_v} is correctly defined and v_{d_v} is a positive (isotone) valuation by the same way as in the proof of Theorem 1.10. It is immediate that $v_{d_v} = v$. □

2. Positive valuations on connected posets of locally finite length

In this section we apply the above results in order to show that there exist positive valuations on some connected graded posets of locally finite length. Particularly, we will deal with modular posets (multilattices).

Let $(P, \leq)$ be a poset. A graph $C(P) = (P, E)$ is called *the covering graph associated with the poset* $(P, \leq)$, if the edge set E consists of the pairs ab for which a covers b in $(P, \leq)$.

Let a, b be vertices of a covering graph $C(P)$ of a poset $(P, \leq)$. Let $W = (a_0, a_1, \ldots, a_n)$ be a finite sequence mutually different vertices of the graph $C(P)$. We call that W is *the way from a to b* (in $C(P)$) if

(j) $a = a_0$, $b = a_n$ and

(jj) a_i, a_{i+1} are adjacent vertices of the graph $C(P)$, for each $i = 0, 1, \ldots, n-1$ (i.e. a_i covers a_{i+1} or a_{i+1} covers a_i in the poset $(P, \leq)$).

The number n is called *the length of the way W*. *The distance* of vertices a and b in a covering graph $C(P)$ we mean the length of the shortes way from a to b (if it exists). We write $d(a, b) = d_0$ if the distance of the vertices a, b is d_0.

Let $(P, \leq)$ be a poset. In this section we will denote by d the distance function $d : P \times P \longrightarrow Z$ defined above (i.e. $d(a, b)$ is the distance of the vertices a, b in the covering graph $C(P)$ of the poset $(P, \leq)$).

Let $(P, \leq)$ be a connected poset of locally finite length. Since the set of all non-negative integers is well ordered, for every $a, b \in P$ there exists a fence way from a to b of shortest length. Thus, the function d is the metric on P and moreover the metric d satisfies the condition (i) (Theorem 1.10). On the other hand the metric d do not need satisfy the conditions (ii) and (iii) (see Fig. 7).

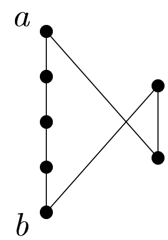

Fig. 7

Lemma 2.1. *Let $(P, \leq)$ be a connected poset of locally finite length. If the metric d satisfies the condition (ii) (Theorem 1.10) then the poset $(P, \leq)$ is graded.*

Proof. Let $a, b \in P$, $a < b$. The interval $[a, b]$ is the subposet of $(P, \leq)$ with the least element a. If $x, y \in [a, b]$ and y covers x we have

$$d(a, y) = d(a, x) + d(x, y) = d(a, x) + 1$$

by (ii). It implies that all maximal chains of the interval $[a, b]$ are of the same length. $\square$

Remark. The poset depicted in Fig. 7 is graded but this poset does not satisfy the condition (ii).

Theorem 2.2. *Let $(P, \leq)$ be a connected poset of locally finite length. If the metric d the conditions (ii) and (iii) satisfies then there exists a positive distance-valuation v on the poset $(P, \leq)$.*

Proof. We can define the positive distance-valuation v on the poset $(P, \leq)$ by Theorem 1.10. □

Let $(P, \leq)$ be a graded poset of locally finite length and $a < b,\ a, b \in P$. In this section we will denote by $l(a, b)$ the length of a maximal chain (i.e. of all maximal chains) of the interval $[a, b]$ of the graded poset $(P, \leq)$. Note it is possible that $l(a, b) \neq d(a, b)$ (for a, b in Fig. 7 we have $l(a, b) = 4 > d(a, b) = 3$).

Theorem 2.3. *Let $(P, \leq)$ be a directed graded poset of locally finite length. There exits a positive distance-valuation v on the poset $(P, \leq)$ for which*

$$a < b \implies v(b) = v(a) + l(a, b) = v(a) + d(a, b). \tag{2.1}$$

Proof. Let $(a_1, b_1, \ldots, a_n, b_n)$ be a cycle-fence of the poset $(P, \leq)$. Let w and u be a lower bound and an upper bound of the poset $\{a_1, b_1, \ldots, a_n, b_n\}$, respectively. Since the poset $(P, \leq)$ is graded we have

$$\begin{aligned} &l(a_1, b_1) + \cdots + l(a_n, b_n) = \\ =&l(w, u) - l(w, a_1) - l(b_1, u) + \cdots + l(w, u) - l(w, a_n) - l(b_n, u) = \\ =&l(a_2, b_1) + l(a_3, b_2) + \cdots + l(a_1, b_n). \end{aligned}$$

Let $a < b$, $a, b \in P$. Let $(a, x_1, \ldots, x_{n-1}, b)$ be a minimal fence way from a to b for which

$$d(a, b) = l(a, x_1) + l(x_1, x_2) + \cdots + l(x_{n-1}, b).$$

The result of the first part of the proof implies (e.g. $x_1 < a$)

$$l(a, b) = -l(a, x_1) + l(x_1, x_2) - l(x_2, x_3) + \cdots + (-1)^n l(x_{n-1}, b)$$

(see Fig. 8).

Thus, $l(a, b) \leq d(a, b)$, i.e. $l(a, b) = d(a, b)$. Now it is immediate that (ii) and (iii) hold for the metric d.

We may now define the distance-valuation v by (1.8). Obviously, the valuation v is positive and (2.1) holds. □

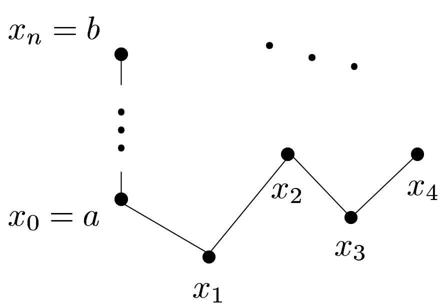

Fig. 8

Let $(M, \leq)$ be a poset and $a, b \in M$. Let $a \vee b$ be the set of all minimal elements of the set of all upper bounds of the set $\{a, b\}$ and $a \wedge b$ be the set of all maximal elements of the set of all lower bounds of the set $\{a, b\}$. We call a poset $(M, \leq)$ a multilattice if for every upper bound h the set $\{t;\, t \in a \vee b \ \& \ t \leq h\}$ is non-empty and for every lower bound k the set $\{t;\, t \in a \wedge b \ \& \ t \geq k\}$ is also non-empty for all $a, b \in M$. It is easy to verify that every poset of locally finite length is a multilattice.

A multilattice $(M, \leq)$ is said to be modular if whenever $a, b, c \in M$

$$(a \vee b) \cap (a \vee c) \neq \emptyset \ \& \ (a \wedge b) \cap (a \wedge c) \neq \emptyset \ \& \ b \leq c \quad \Longrightarrow \quad b = c.$$

Every modular multilattice of locally finite length is a graded poset.

Lemma 2.4. *Let* **M** *be a modular directed multilattice of locally finite length. If* $a, b \in M$*,* $u \in a \vee b$*,* $w \in a \wedge b$ *then* (a, u, b) *and* (a, w, b) *are minimal ways from* a *to* b *(i.e.* $d(a, b) = l(a, u) + l(b, u) = l(w, a) + l(w, b)$*).*

Proof. By Theorem 2.3 $d(a, b) = l(a, b)$ for any comparable elmements $a, b \in M$. Let a, b be two noncomparable elements and let $(a, x_1, \ldots, x_{n-1}, b)$ be a way from a to b. Let, for instance, $a > x_1 < x_2$ and let $x_1 \notin a \wedge x_2$ (Fig. 9).

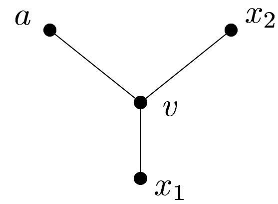

Fig. 9

Then there exists an element $v \in a \wedge x_2$, $v > x_1$ and the way $(a, v, x_2, \ldots, x_{n-1}, b)$ is shorter than the way $(a, x_1, \ldots, x_{n-1}, b)$. In the next part of the proof we will consider only fence ways $(a, x_1, \ldots, x_{n-1}, b)$ from a to b for which

$$(2.2) \qquad x_i \in x_{i-1} \wedge x_{i+1} \quad \text{or} \quad x_i \in x_{i-1} \vee x_{i+1}$$

holds for each $i = 1, \ldots, n-1$. We will do the proof by induction.

Let $n = 2$. We distinguish two cases. Let $a > x_1 < b$. Then $x_1 \in a \wedge b$ and $l(a, x_1) = l(u, b)$ and $l(x_1, b) = l(a, u)$, because the multilattice is modular (see [5]). It implies that $d(a, u, b) = d(a, x_1, b)$. Let $a < x_1 > b$. Then $x_1 \in a \vee b$ and $u \in a \vee b$, too. The multilattice is modular, therefore $l(a, x_1) = l(w, b) = l(a, u)$ and $l(b, x_1) = l(w, a) = l(b, u)$, hence $l(a, x_1) + l(b, x_1) = l(a, u) + l(b, u)$, again.

Assume that for the lengths of all n-element fence ways from a to b the statement holds. We prove it for n+1-element fence ways.

Let $(a, x_1, \ldots, x_{n-1}, b)$ be an $(n+1)$-element fence way. For instance, let $a > x_1 < x_2 > \cdots < x_{n-1} > b$ (Fig. 10). If $y \in a \vee x_2$ then by induction hypothesis $l(a, y) + l(y, x_3) + \cdots + l(x_{n-1}, b) \geq d(a, u) + d(b, u)$. Because $l(a, x_1) = l(x_2, y)$ and $l(x_1, x_2) = l(a, y)$, we have

$$\begin{aligned} & l(a, x_1) + l(x_1, x_2) + l(x_2, x_3) + \cdots + l(x_{n-1}, b) = \\ =& l(a, y) + l(y, x_2) + l(x_2, x_3) + \cdots + l(x_{n-1}, b) = \\ =& l(a, y) + l(x_3, y) + + \cdots + l(x_{n-1}, b) \geq d(a, u) + d(b, u). \end{aligned}$$

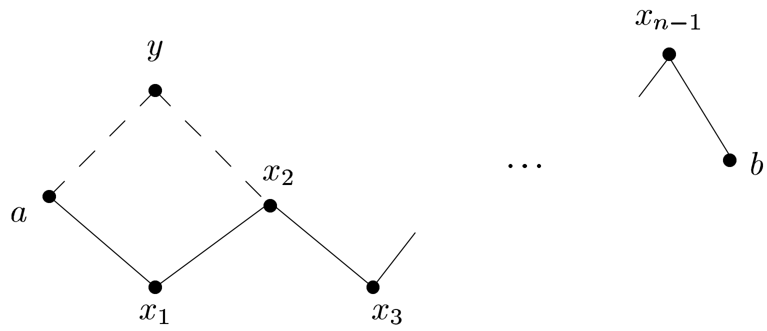

Fig. 10

Analogously, we can show the inequality in the other cases. The statement holds for a lower bound w by duality. □

Corollary 2.5. *Let* $\mathbf{M}$ *be a modular directed multilattice of locally finite length. There exists a positive valuation* v *on the multilattice* $\mathbf{M}$ *for which*

$$v(a) + v(b) = v(u) + v(w) \tag{2.3}$$

for all $a, b \in M$, $u \in a \vee b$, $w \in a \wedge b$.

Proof. We can define a positive distance-valuation v on the poset $\mathbf{M}$ by Theorem 2.3. The valuation v satifies (2.1). So, for every $a, b \in M$, $u \in a \vee b$, $w \in a \wedge b$ we have

$$\begin{aligned} v(a) + v(b) &= v(w) + l(w, a) + v(w) + l(w, b) = \\ &= v(w) + v(w) + l(w, a) + l(a, u) = \\ &= v(w) + v(u). \end{aligned}$$

□

In [2], by a valuation on a lattice L is meant a function $v : L \longrightarrow R$ for which (2.3) holds. This definition was accepted by authors of [5] and [6], too. If L is a modular multilattice of locally finite length, we have for the induced metric d_v

$$\begin{aligned} d_v(a, b) &= l(a, u) + l(b, u) = l(a, u) + l(w, a) = \\ &= v(u) - v(a) + v(a) - v(w) = v(u) - v(w) = \\ &= d_v(w, u) \end{aligned}$$

whenever $u \in a \vee b$, $w \in a \wedge b$.

Definition 2.6. Let $(P, \leq)$ be a poset. A positive valuation v on the poset $(P, \leq)$ is said to be a modular valuation if for every $a, b \in P$

$$\begin{aligned} & a \vee b \neq \emptyset \ \& \ a \wedge b \neq \emptyset \ \& \ u \in a \vee b \ \& \ w \in a \wedge b \\ & \implies \quad v(a) + v(b) = v(u) + v(w). \end{aligned} \tag{2.4}$$

Two intervals $[a, b]$, $[c, d]$ (of a poset $(P, \leq)$) are said to be *transposed* if either $a \in b \wedge c$, $d \in b \vee c$ or $c \in a \wedge d$, $b \in a \vee d$. The intervals I, J are *projective* if there exists a finite sequence of intervals $I = I_0, I_1, \ldots, I_n = J$ such that all adjoining intervals I_i, I_{i+1} are transposed.

Definition 2.7. Let $(P, \leq)$ be a connected poset of locally finite length. The poset $(P, \leq)$ is said to be a modular poset if the next two conditions are satisfied

(k) the lengths of every two projective intervals are equal

(kk) the metric d satisfies (ii) and (iii).

Theorem 2.8. *A connected graded poset $(P, \leq)$ of locally finite length is modular if and only if there exists a modular valuation v on $(P, \leq)$ satisfying (2.1).*

Proof. a) Let P be a modular connected poset of locally finite length. The metric d satisfies (i), (ii) and (iii), hence we can define a positive distance-valuation v on the poset $(P, \leq)$ by (1.8). If $u \in a \vee b$, $w \in a \wedge b$ then according to (1.8) we get

$$v(w) + v(u) = v(w) + v(w) + d(w, u) = 2v(w) + d(w, a) + d(a, u).$$

The intervals $[a, u]$, $[w, b]$ are projective, for this reason $d(a, u) = d(w, b)$ and we get

$$v(w) + v(u) = 2v(w) + d(w, a) + d(w, b) = v(a) + v(b)$$

Obviously, the valuation v satisfies (2.1).

b) Let there exist a modular valuation v satisfying (2.1) on a poset P. Let $[a, b]$, $[c, d]$ be two transposed intervals and let $a \in b \wedge c$, $d \in b \vee c$. From $v(b) + v(c) = v(a) + v(d)$ we get $v(b) - v(a) = v(d) - v(c)$, i.e. $d(a, b) = d(c, d)$ by (2.1). The transitivity of equality implies (k).

Let $a < b < c$. Using (2.1) we get

$$\begin{aligned} d(a, c) = v(c) - v(a) &= v(c) - v(b) + v(b) - v(a) = \\ &= d(a, b) + d(b, c). \end{aligned}$$

Let $(a_1, b_1, \ldots, a_n, b_n)$ be cycle fence of the poset $(P, \leq)$. By (2.1)

$$\begin{aligned} d(a_1, b_1) + \cdots + d(a_n, b_n) &= v(b_1) - v(a_1) + \cdots + v(b_n) - v(a_n) = \\ &= d(b_1, a_2) + \cdots + d(b_n, a_1). \end{aligned}$$

□

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AFFINE COMPLETE ALGEBRAS ABSTRACTING DOUBLE STONE AND KLEENE ALGEBRAS

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Abstract. In this paper we generalize R. Beazer's characterization of affine complete double Stone algebras with a non-empty bounded core [1] to the class of double K_2-algebras with a non-empty bounded core. These algebras have appeared in the literature as a common generalization of double Stone and Kleene algebras. We show that Post algebras of order 3 are the only locally affine complete (in a stronger sense of [12]) double K_2-algebras with a non-empty bounded core and the only finite affine complete double K_2-algebras. Then introducing some extension properties for congruence-preserving functions we characterize (infinite) affine complete double K_2-algebras with a non-empty bounded core. We finally derive the Beazer result for double Stone algebras.

1. Introduction. The problem of characterizing affine complete algebras was posed by G. Grätzer in [6] (Problem 6). Recall that an n-ary function f on an algebra A is *compatible* if for any congruence θ on A, $a_i \equiv b_i$ (θ) $(a_i, b_i \in A)$, $i = 1, \dots, n$ yields $f(a_1, \dots, a_n) \equiv f(b_1, \dots, b_n)$ (θ). A *polynomial* function of A is a function that can be obtained by composition of the basic operations of A, the projections and the constant functions. Clearly, all polynomial functions of A are compatible. An algebra A is called *affine complete* if the polynomial functions of A are the only compatible functions. Hence in general, affine complete algebras have 'many' congruences.

In [4] G. Grätzer proved that every Boolean algebra is affine complete. In [5] he showed that affine complete bounded distributive lattices are those which do not have proper Boolean subintervals. A list of particular varieties in which all affine complete members were characterized can be found in [3] and its up-to-date version in [10].

Two 'local' versions of affine completeness have been studied in the literature. A weaker notion of local affine completeness can be found e.g. in [16]. According to the stronger meaning of this concept, which we adopt here, an algebra A is *locally affine complete* if any finite partial function in $A^n \to A$ (i.e. function whose domain is a finite subset of A^n) which is compatible (where defined) can be interpolated by a polynomial of A (see e.g. [12]).

In [1] R. Beazer characterized affine complete algebras in the class of double Stone algebras with a non-empty bounded core. A generalization of this result, to the class of

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so-called quasi-modular double p-algebras, was presented in [9] where also locally affine complete Stone algebras (in the stronger sense of [12]) were characterized. Recently, affine complete Kleene algebras were successfully described in [11]. This allows us to investigate affine completeness in the class of double K_2-algebras which are known as a common generalization of double Stone and Kleene algebras [2]. This investigation, which uses techniques similar to those in [8] and [9], is the object of this paper.

First we show that for any double K_2-algebra L with a non-empty bounded core $K(L)$, (locally) affine completeness of L yields (locally) affine completeness of $K(L)$ as a bounded distributive lattice (Theorem 3.3). Consequently, we get that Post algebras of order 3 are the only locally affine complete double K_2-algebras with a non-empty bounded core and the only affine complete algebras among the double K_2-algebras with a finite skeleton and a finite non-empty core (Corollaries 3.6 and 3.9). Then we introduce some extension properties for compatible functions and with their use we reduce the question of characterizing (infinite) affine complete double K_2-algebras with non-empty bounded core to those questions for Kleene algebras and distributive lattices (Theorem 3.17) where the answers are already known. We finally derive from our characterization the Beazer result for double Stone algebras.

2. Preliminaries.

MS-algebras were introduced by T.S. Blyth and J.C. Varlet in the beginning of eighttees as a nice generalization of de Morgan and Stone algebras and have shown a fruitful development during the previous decade (cf. [2]).

Let us recall that an *MS-algebra* is an algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ of type (2,2,1,0,0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $^\circ$ is a unary operation such that for all $x, y \in L$

(1) x $\leq x^{\circ\circ}$,

(2) $(x \wedge y)^\circ = x^\circ \vee y^\circ$,

(3) $1^\circ = 0$.

One can show that the following rules of computation hold further in L:

$(x \vee y)^\circ = x^\circ \wedge y^\circ$,

$x^{\circ\circ\circ} = x^\circ$,

$0^\circ = 1$.

The class of all MS-algebras is equational. The subvariety K_2 of MS-algebras is defined by two additional identities

(4) $x \wedge x^\circ = x^{\circ\circ} \wedge x^\circ$ and

(5) $(x \wedge x^\circ) \vee y \vee y^\circ = y \vee y^\circ$,

The subvariety M of de Morgan algebras is defined by the identity (6) $x = x^{\circ\circ}$. Another important subvarieties of MS-algebras are the subvarieties B, S, K of Boolean, Stone and Kleene algebras, respectively which are characterized by the identities $B: x \vee x^\circ = 1$, $S: x \wedge x^\circ = 0$ and $K: (5), (6)$, respectively.

Let L be an MS-algebra from the subvariety K_2. Then

(i) $L^{\circ\circ} = \{x \in L;\ x = x^{\circ\circ}\}$ is a Kleene subalgebra of L ;

(ii) $L^\wedge = \{x \wedge x^\circ;\ x \in L\}$ is an ideal of L;

(iii) $L^\vee = \{x \vee x^\circ;\ x \in L\}$ is a filter of L.

Now we recall basic facts about *double* MS-algebras. A double MS-algebra is an algebra $(L; \vee, \wedge, ^\circ, ^+, 0, 1)$ of type (2,2,1,1,0,0) such that $(L; \vee, \wedge, ^\circ, 0, 1)$ is an MS-algebra, $(L; \vee, \wedge, ^+, 0, 1)$ is a dual MS-algebra, and the unary operations are linked by the identities

$x^{\circ +} = x^{\circ\circ}$ and $x^{+\circ} = x^{++}$.

Obviously, every de Morgan algebra $(L; \vee, \wedge, ^-, 0, 1)$ can be made into a double MS-algebra if one defines $x^\circ = x^+ = \overline{x}$. Conditions under which an MS-algebra can be made into a double MS-algebra are known (cf. [2]). It is proved that the subvarieties B, K, M of MS-algebras are *dense*, i.e. all algebras in these subvarieties can be made into double MS-algebras. Further, *bistable* subvarieties of MS-algebras are defined as those V that for every double MS-algebra $(L; \vee, \wedge, ^\circ, ^+, 0, 1)$, whenever $(L; \vee, \wedge, ^\circ, 0, 1) \in V$, then $(L; \vee, \wedge, ^+, 0, 1) \in V$ too. It is known which subvarieties of MS-algebras are bistable and which fail (cf. [2]). Among first, the subvarieties B, S, K, S∨K and M are included, among non-bistable one can find the subvariety K_2. It is true that the identity (5) implies the dual one, (5^d) $(x \vee x^+) \wedge y \wedge y^+ = y \wedge y^+$, however for the identity (4), which defines the subvariety $K_2 \vee M$, this is not the case. Therefore the variety of double K_2-algebras is defined by the identities (4), (5) and (4^d) $x \vee x^+ = x^{++} \vee x^+$.

It is known that there are precisely 22 non-isomorphic subdirectly irreducible double MS-algebras. The lattice of subvarieties of double MS-algebras has cardinality 381 (cf. [2]).

Some subsets of double MS-algebras play a significant role in investigations. By the *skeleton* $S(L)$ of a double MS-algebra L is meant a de Morgan algebra $L^{\circ\circ} = \{x \in L;\ x^{\circ\circ} = x\} = L^{++} = \{x \in L;\ x^{++} = x\}$. If L is a double K_2-algebra then $S(L)$ is a Kleene algebra. Further, in a double MS-algebra L, $x^\circ \leq x^+$ and consequently $x^{++} \leq x \leq x^{\circ\circ}$ hold for any element x. Therefore, the notion of the *core* $K(L)$, known for double Stone algebras, can be generalized for double K_2-algebras as follows:

$K(L) = \{x \vee x^\circ;\ x \in L\} \cap \{x \wedge x^+;\ x \in L\}$.

Double Stone algebras L which have a one-element core, $|K(L)| = 1$, are named *Post algebras of order 3*. They form a subclass (not a subvariety) of the variety of so-called three-valued Lukasiewicz algebras which are double Stone algebras defined by the identity $(x \wedge x^+) \vee (y \vee y^\circ) = y \vee y^\circ$ (cf. [1]). The variety of three-valued Lukasiewicz algebras is known to be arithmetical (i.e. congruence-distributive and congruence-permutable).

An important role is played by so-called *determination congruence* which is defined as follows:

$x \equiv y\ (\Phi^\circ_+)$ iff $x^\circ = y^\circ$ and $x^+ = y^+$.

For other properties of MS-algebras and double MS-algebras we refer the reader to [2].

We finally mention few facts concerning the affine completeness. We start with basic Grätzer's results.

2.1 Theorem ([4]). *Any Boolean algebra is affine complete.*

Let us recall that a function $f : L^n \to L$ on a lattice L is *order-preserving* if $x_i \leq y_i$ $(x_i, y_i \in L,\ i = 1, \dots, n)$ implies $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$ where $\leq$ is the lattice order. It is well-known that every polynomial function on a lattice is order-preserving.

2.2 Theorem ([5; Corollaries 1,3]). *Let L be a bounded distributive lattice. The following conditions are equivalent.*

(1) L is affine complete;

(2) every compatible function on L is order-preserving;

(3) L contains no proper Boolean interval.

Now we present the Beazer result for double Stone algebras and its immediate consequences.

2.3 Proposition ([1; Theorem 5]). *Let L be a double Stone algebra with a non-empty bounded core $K(L)$. The following conditions are equivalent.*

(1) L is affine complete;

(2) $K(L)$ is an affine complete distributive lattice;

(3) No proper interval of $K(L)$ is Boolean.

2.4 Corollary ([1; Corollary 6]). *Any Post algebra of order 3 is an affine complete double Stone algebra.*

2.5 Corollary ([1; Corollary 7]). *A finite double Stone algebra having a non-empty core is affine complete if and only if it is a Post algebra of order 3.*

In [8] it was shown that if a K_2-algebra L is affine complete then the filter $L^\vee$ is (as a lattice) affine complete, too. Since by 2.2 a finite distributive lattice $L^\vee$ is affine complete if and only if $|L^\vee| = 1$, we immediately get

2.6 Proposition ([8; Corollary 4]). *Let L be a K_2-algebra such that $L^\vee$ is finite. Then L is affine complete if and only if L is a Boolean algebra.*

In a Kleene algebra L, the filter $L^\vee$ is isomorphic to the ideal $L^\wedge$. Hence we have

2.7 Corollary. *Let L be a Kleene algebra such that $L^\wedge$ is finite. Then L is affine complete if and only if L is a Boolean algebra.*

The following few facts are considered to be a part of 'folklore':

2.8 Proposition. *If a lattice L contains a Boolean interval $[a,b]$ ($a < b$), then L is not affine complete.*

Proof. Define a function $f : L \to [a,b]$ by $f(x) = ((x \vee a) \wedge b)'$, where $'$ denotes the complement in the Boolean interval [a,b]. For any non-trivial congruence $\theta \in \mathrm{Con}(L)$ and $x \equiv y\ (\theta)\ \ (x,\ y \in L)$ we have $((x \vee a) \wedge b)' \equiv ((y \vee a) \wedge b)'\ (\theta)$, i.e. f is a compatible function of L. But f is not order-preserving because $f(a) = b,\ f(b) = a$, therefore f cannot be represented by a lattice polynomial. Hence L is not affine complete. □

2.2 implies that a finite distributive lattice L is affine complete if and only if $|L| = 1$. Now 2.8 yields that the assumption about distributivity of L can even be dropped.

2.9 Corollary. *A finite lattice L is affine complete if and only if $|L| = 1$.* □

2.10 Proposition. *For any lattice L the following are equivalent:*

(1) *L is locally affine complete;*
(2) *every finite partial compatible function of L is order-preserving;*
(3) $|L| = 1$.

Proof. (1) $\Longrightarrow$ (3): Let L be locally affine complete and let $a, b \in L,\ a < b$. The function $f = \{(a,b),(b,a)\}$ is a finite partial compatible function on L, thus by hypothesis it can be interpolated on $\{a,b\}$ by a polynomial of L, which is an order-preserving function. But we have $f(a) = b,\ f(b) = a$, a contradiction.

(2) $\Longrightarrow$ (3): If $|L| \neq 1$ then we can define the same partial function f as above which contradicts (2).

The rest of the proof is trivial. $\square$

The following result (see [13] or [14]) characterizes those varieties of which all members are locally affine complete as arithmetical.

2.11 Theorem. *A variety V is arithmetical if and only if for each algebra $A \in V$, a finite partial function f on A can be interpolated by a polynomial function of A just in the case f is Con(A)-compatible.*

Since the class of the Post algebras of order 3 is contained in the arithmetical variety of the three-valued Lukasiewicz algebras, we immediately get

2.12 Corollary. *Every Post algebra of order 3 is locally affine complete.*

We conclude with a technical lemma which will be applied in Section 3 (for D being a Boolean algebra and a bounded distributive lattice, respectively; its proof, which can be found in [7] or [9], will be repeated here as it is not long and we want this paper to be self-contained.)

2.13 Lemma. *Let D be any algebra such that its reduct is a bounded distributive lattice $(D, \vee, \wedge, 0, 1)$ and the algebra D is a subdirect product of 2-element algebras. Let $f', g' : D^n \to D$ be (partial) compatible functions with domains F and G ($F, G \subseteq D^n$), respectively, let $S := F \cap G$ and let $S \cap \{0,1\}^n \neq \varnothing$. For any $(0,1)$-homomorphism $h : D \to \{0,1\}$ between the algebra D and a 2-element algebra $\{0,1\}$, denote $h(S) := \{(h(x_1), \ldots, h(x_n)) \in \{0,1\}^n;\ (x_1, \ldots, x_n) \in S\}$ and let $h(S) = h(S \cap \{0,1\}^n)$ hold for every such h. Then $f' \equiv g'$ identically on S if and only if $f' \equiv g'$ identically on $S \cap \{0,1\}^n$.*

Proof. Let $f' \equiv g'$ identically on $S \cap \{0,1\}^n$. Suppose on the contrary that there exists an n-tuple $(d_1, \ldots, d_n) \in S$ such that $f'(d_1, \ldots, d_n) = a \neq b = g'(d_1, \ldots, d_n)$. Since $a \neq b$ in D which is a subdirect product of 2-element algebras, there exists a 'projection map' $h : D \to \{0,1\}$, which is a $(0,1)$-homomorphism between the algebra D and some algebra $\{0,1\}$, such that $h(a) \neq h(b)$. Define functions $f'_2,\ g'_2 : h(S) \to \{0,1\}$ by the following rules:

$f'_2(h(x_1), \ldots, h(x_n)) = h(f'(x_1, \ldots, x_n))$,
$g'_2(h(x_1), \ldots, h(x_n)) = h(g'(x_1, \ldots, x_n))$ where $(x_1, \ldots, x_n) \in S$.

Obviously, $f'_2,\ g'_2$ are well-defined, since f', g' preserve the kernel congruence of the homomorphism h. Obviously, $f'_2 \equiv g'_2$ identically on $h(S)$, because $h(S) = h(S \cap \{0,1\}^n)$, $h(0) = 0,\ h(1) = 1$ and $f' \equiv g'$ identically on $S \cap \{0,1\}^n$. Therefore

$h(a) = h(f'(d_1,\dots,d_n)) = f_2'(h(d_1),\dots,h(d_n)) = g_2'(h(d_1),\dots,h(d_n)) = h(g'(d_1,\dots,d_n)) = h(b)$, a contradiction. Hence $f' \equiv g'$ identically on S. The proof is complete. □

We finally mention that in order to abbreviate some expressions, we shall often use the notation $\tilde{x}$ for an n-tuple $(x_1,\dots,x_n)$, and $f(\tilde{x})$ for $f(x_1,\dots,x_n)$ in the next section. Further, $\tilde{x}^\circ$ and $\tilde{x}^+$ will denote $(x_1^\circ,\dots,x_n^\circ)$ and $(x_1^+,\dots,x_n^+)$, respectively, $(\tilde{x}\vee k)\wedge l$ will abbreviate $((x_1\vee k)\wedge l,\dots,(x_n\vee k)\wedge l)$, etc.

3. Affine completeness.

We start with a canonical form of any polynomial function on a double MS-algebra.

3.1 Lemma. *Any polynomial function $f(x_1,\dots,x_n)$ on a double MS-algebra L can be represented in the form*

$$f(x_1,\dots,x_n) = \bigvee_{\tilde{i},\tilde{j}\in\{-2,-1,0,1,2,3\}^n,\ \tilde{i}<\tilde{j}} (\alpha(i_1,j_1,\dots,i_n,j_n)\wedge x_1^{i_1}\wedge x_1^{j_1}\wedge\cdots\wedge x_n^{i_n}\wedge x_n^{j_n})$$

and dually, in the form

$$f(x_1,\dots,x_n) = \bigwedge_{\tilde{i},\tilde{j}\in\{-2,-1,0,1,2,3\}^n,\ \tilde{i}<\tilde{j}} (\beta(i_1,j_1,\dots,i_n,j_n)\vee x_1^{i_1}\vee x_1^{j_1}\vee\cdots\vee x_n^{i_n}\vee x_n^{j_n})$$

where $\bigvee$ and $\bigwedge$ are taken over all vectors $\tilde{i} = (i_1,\dots,i_n)$, $\tilde{j} = (j_1,\dots,j_n) \in \{-2,-1,0,1,2,3\}^n$, the coefficients $\alpha(i_1,j_1,\dots,i_n,j_n), \beta(i_1,j_1,\dots,i_n,j_n) \in L$ and $x^{-2}, x^{-1}, x^0, x^1, x^2$ and x^3 denote $x^{\circ\circ}, x^\circ, x, x^+, x^{++}$, and 1, respectively.

Proof. It follows from the following facts:

(i) for every $x \in L$ $x^{\circ +} = x^{\circ\circ}$, $x^{+\circ} = x^{++}$, $x^{\circ\circ\circ} = x^\circ$, $x^{+++} = x^+$ and $x^\circ \le x^+$, $x^{++} \le x \le x^{\circ\circ}$;

(ii) for every $x, y \in L$ $(x\vee y)^\circ = x^\circ \wedge y^\circ$, $(x\wedge y)^\circ = x^\circ \vee y^\circ$ and $(x\wedge y)^+ = x^+ \vee y^+$, $(x\vee y)^+ = x^+\wedge y^+$;

(iii) the lattice L is distributive. □

3.2 Lemma. *Let L be a double K_2-algebra with a non-empty core $K(L)$. Then for any $x,\ y \in K(L)$ $x^\circ \le y$ and $x^+ \ge y$.*

Proof. It follows from the facts that any element of $K(L)$ can be represented in the form $a\vee a^\circ$ as well as $b\wedge b^+$ for some $a,\ b \in L$, and that the identities (4),(5), (5^d) hold in L. □

3.3 Theorem. *Let L be a double K_2-algebra with a non-empty bounded core $K(L)$. If L is (locally) affine complete then $K(L)$ is a (locally) affine complete distributive lattice.*

Proof. Let L be (locally) affine complete . Let f' be an n-ary (finite partial) compatible function of the lattice $K(L) = [k,l]$. Define a (finite partial) function $f : L^n \to L$ by

$$f(x_1,\dots,x_n) = f'((x_1\vee k)\wedge l,\dots,(x_n\vee k)\wedge l).$$

Obviously, $f' = f \restriction K(L)^n$ and f is a (finite partial) compatible function of the algebra L (in local version we can always take $f \equiv f'$ to assure a finite domain of f). Indeed, if θ is a congruence of L and $x_i \equiv y_i\ (\theta)$, $i = 1,\dots,n$, then $(x_i\vee k)\wedge l \equiv (y_i\vee k)\wedge l\ (\theta)$, thus we have (where f' is defined) $f'((x_1\vee k)\wedge l,\dots,(x_n\vee k)\wedge l) \equiv f'((y_1\vee k)\wedge l,\dots,(y_n\vee k)\wedge l)\ (\theta\restriction K(L))$

as f' is compatible on $K(L)$ (where defined), whence $f(x_1,\dots,x_n) \equiv f(y_1,\dots,y_n)$ (θ). Therefore by 3.1 we can write for $x_1,\dots,x_n \in K(L)$ (where f' is defined)

$$(a)\ f'(\tilde{x}) = f(\tilde{x}) = \bigvee_{\tilde{\imath},\tilde{\jmath}\in\{-2,-1,0,1,2,3\}^n,\ \tilde{\imath}<\tilde{\jmath}} (\alpha(i_1,j_1,\dots,i_n,j_n) \wedge x_1^{i_1} \wedge x_1^{j_1} \wedge \dots \wedge x_n^{i_n} \wedge x_n^{j_n}).$$

By 3.2 and the fact that $f'(\tilde{x}) \in K(L)$, the terms x_i^+ can be omitted in (a). Further, the polynomial obtained can be, with use of the distributive laws and the relations $x^\circ \le x^+$ and $x^{++} \le x \le x^{\circ\circ}$, rewritten in the form

$$(b)\ \bigwedge_{\tilde{\imath},\tilde{\jmath}\in\{-2,-1,0,2,3\}^n,\ \tilde{\imath}<\tilde{\jmath}} (\beta(i_1,j_1,\dots,i_n,j_n) \vee x_1^{i_1} \vee x_1^{j_1} \vee \dots \vee x_n^{i_n} \vee x_n^{j_n}).$$

Now again from 3.2 and $f'(\tilde{x}) \in K(L)$ it follows that the terms x_i° can be omitted in (b). So we get

$$(c)\ \ f'(\tilde{x}) = \bigwedge_{\tilde{\imath},\tilde{\jmath}\in\{-2,0,2,3\}^n,\ \tilde{\imath}<\tilde{\jmath}} (\beta(i_1,j_1,\dots,i_n,j_n) \vee x_1^{i_1} \vee x_1^{j_1} \vee \dots \vee x_n^{i_n} \vee x_n^{j_n}).$$

Now we see that f' is order-preserving (where defined). The assertion follows from 2.2 (in local version from 2.10). □

3.4 Proposition. *If a double K_2-algebra L is (locally) affine complete then the Kleene algebra $S(L) = L^{\circ\circ}$ is (locally) affine complete, too.*

Proof. Let L be a (locally) affine complete double K_2-algebra. Let f' be an n-ary (finite partial) compatible function on $S(L)$. Define an n-ary (finite partial) function f on L by $f(x_1,\dots,x_n) = f'(x_1^{\circ\circ},\dots,x_n^{\circ\circ})$. Obviously, f is compatible since f' is compatible (where defined), so by hypothesis f can be represented (where defined) by a polynomial $p(x_1,\dots,x_n)$ of L. Hence for all $\tilde{x} = (x_1,\dots,x_n) \in (L^{\circ\circ})^n$ (where f' is defined) we have $f'(\tilde{x}) = f(\tilde{x}) = p(\tilde{x}) = p(\tilde{x})^{\circ\circ}$ as $f'(\tilde{x}) \in L^{\circ\circ}$. Clearly, in $p(x_1,\dots,x_n)^{\circ\circ}$ all constants are elements of $L^{\circ\circ}$, thus f' can be represented (where defined) by a polynomial of $S(L)$. □

3.5 Lemma. *Local affine completeness of the Kleene algebra $S(L)$ yields local affine completeness of the lattices $S(L)^\wedge$ and $S(L)^\vee$.*

Proof. We know that in a Kleene algebra $S(L)$, the ideal $S(L)^\wedge$ and the filter $S(L)^\vee$ are isomorphic. Let $f : F \subseteq (S(L)^\wedge)^n \to S(L)^\wedge$ be a finite partial compatible function of the lattice $S(L)^\wedge$. We claim that f also preserves the congruences of the Kleene algebra $S(L)$ where defined. Indeed, if θ is a congruence of $S(L)$ and $(x_1,\dots,x_n),(y_1,\dots,y_n) \in F$, $x_i \equiv y_i$ (θ), $i = 1,\dots,n$, then $f(x_1,\dots,x_n) \equiv f(y_1,\dots,y_n)$ $(\theta \restriction S(L)^\wedge)$ as f preserves the lattice congruence $\theta \restriction S(L)$. Now local affine completeness of $S(L)$ yields that for all $\tilde{x} = (x_1,\dots,x_n) \in F$, $f(\tilde{x})$ can be written as in (a) of the proof of 3.3. However, now in (a) we have only terms x_j and x_i^+ because $x^{\circ\circ} = x^{++} = x$ and $x^\circ = x^+$ hold in $S(L)$. Since $x_i \in S(L)^\wedge$ are of the form $x_i = a^+ \wedge a^{++}$, we have $x_i^+ = a^+ \vee a^{++} = a \vee a^+ \ge a^+ \wedge a^{++} = x_i$ by $(4^d), (5^d)$, consequently the terms x_i^+ can be omitted. Hence f is order-preserving (where defined). By 2.10, $S(L)^\wedge$ is locally affine complete. □

3.6 Corollary. *A double K_2-algebra with a non-empty bounded core $K(L)$ is locally affine complete if and only if it is a Post algebra of order 3.*

Proof. If $K(L)$ is locally affine complete then by 3.3 and 2.10, $|K(L)| = 1$. Further, by 3.4, 3.5 and 2.10 again, $|S(L)^\wedge| = |S(L)^\vee| = 1$. Hence for any $x \in L$, $0 = x^\circ \wedge x^{\circ\circ} = x \wedge x^\circ$ and

$1 = x^+ \vee x^{++} = x \vee x^+$. Hence L is a double Stone algebra ($S(L)$ is a Boolean algebra) and consequently L is a Post algebra of order 3. The converse follows from 2.12. $\square$

From now we shall deal with the affine completeness only.

3.7 Corollary. *Let L be a double K_2-algebra such that the ideal $L^\wedge = \{x \wedge x^\circ;\ x \in L\} \subseteq S(L)$ is finite. Then L is affine complete if and only if L is an affine complete double Stone algebra.*

Proof. If L is an affine complete double K_2-algebra then by 3.4 the Kleene algebra $S(L)$ is affine complete, too. Since $S(L)^\wedge \subseteq L^\wedge$ and $L^\wedge$ is finite, $S(L)$ is a Boolean algebra by 2.7. Therefore for any $x \in L$, $x^\circ \vee x^{\circ\circ} = 1$ and $x^+ \wedge x^{++} = 0$, hence (as in 3.6) $x \wedge x^\circ = x^\circ \wedge x^{\circ\circ} = 0$ and $x \vee x^+ = x^{++} \vee x^+ = 1$, i.e. L is a double Stone algebra. $\square$

3.8 Corollary. *A double K_2-algebra with a finite skeleton is affine complete if and only if it is an affine complete double Stone algebra .*

3.9 Corollary. *A double K_2-algebra L with a finite skeleton and a finite non-empty core (in particular, a finite double K_2-algebra) is affine complete if and only if L is a (finite) Post algebra of order 3.*

Proof. Let L be affine complete . By 3.8, L is affine complete double Stone algebra and by 3.3 and 2.9, $|K(L)| = 1$. Hence, L is a Post algebra of order 3. $\square$

Next, by L we always mean an (infinite) double K_2-algebra with a non-empty bounded core $K(L) = [k, l]$. Obviously, a mapping $\varphi : L \to K(L)$, $\varphi(x) = (x \vee k) \wedge l$ is a lattice homomorphism. We abbreviate by $\varphi(\tilde{x})$ the n-tuple $(\tilde{x} \vee k) \wedge l = ((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l)$.

3.10 Lemma. *Every element of L can be decomposed in the form*

(*d*) $\quad x = x^{++} \vee (x^{\circ\circ} \wedge (x \vee k) \wedge l)$.

Proof. We shall show that for any $a \in L$, $a = a^{\circ\circ} \wedge (a \vee k)$. By the distributivity of L, $a^{\circ\circ} \wedge (a \vee k) = a \vee (a^{\circ\circ} \wedge k)$. It suffices to show that $a^{\circ\circ} \wedge k = a \wedge k$. Suppose on the contrary that $a \wedge k < a^{\circ\circ} \wedge k$. Then for $x = a^{\circ\circ} \wedge k,\ y = a^\circ \wedge k,\ z = a \wedge k$ we have

$x \wedge y = a^{\circ\circ} \wedge a^\circ \wedge k = a \wedge a^\circ \wedge k = a \wedge a^\circ = z \wedge y$

as $k = c \vee c^\circ$ for some $c \in L$ and (4), (5) hold,

$x \vee y = (a^{\circ\circ} \vee a^\circ) \wedge k = k = (a \vee a^\circ) \wedge k = z \vee y.$

Hence, $\{a \wedge a^\circ,\ z,\ y,\ x,\ k\}$ is a five-element non-modular sublattice of L (pentagon), which contradicts to the distributivity of L.

Hence in L we have

$$x = x^{\circ\circ} \wedge (x \vee k)$$

and dually,

$$x = x^{++} \vee (x \wedge l).$$

These two equations imply (d). $\square$

We recall that $(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})$ $(x \in L^n)$ in the next definition is an abbreviation for the $5n$-tuple $((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l, (x_1^\circ \vee k) \wedge l, \ldots, (x_n^\circ \vee k) \wedge l, (x_1^{\circ\circ} \vee k) \wedge l, \ldots, (x_n^{\circ\circ} \vee k) \wedge l, (x_1^+ \vee k) \wedge l, \ldots, (x_n^+ \vee k) \wedge l, (x_1^{++} \vee k) \wedge l, \ldots, (x_n^{++} \vee k) \wedge l)$.

3.11 Definition. *We shall say that L satisfies an 'extension property'*

(EC) *if for any compatible function $f : L^n \to L$, the partial function $f'_K : K(L)^{5n} \to K(L)$ defined on the core such that for all $\tilde{x} \in L^n$*
$$f'_K(\varphi(\tilde{x}), \varphi(\tilde{x}^{\circ}), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^{+}), \varphi(\tilde{x}^{++})) = \varphi(f(\tilde{x}))$$
and f'_K is undefined elsewhere can be extended to a total compatible function of the lattice $K(L)$.

3.12 Lemma. *The partial function f'_K in the preceding definition is a (well-defined) partial compatible function of the lattice $K(L)$.*

Proof. We associate to any congruence θ_K of the lattice $K(L)$ an equivalence relation θ_L on L defined by the rule

(e) $x \equiv y\ (\theta_L)$ iff $\varphi(x^i) \equiv \varphi(y^i)\ (\theta_K)$ for all $i \in \{-2, -1, 0, 1, 2\}$,

where $x^0 = x,\ x^1 = x^+,\ x^2 = x^{++},\ x^{-1} = x^{\circ},\ x^{-2} = x^{\circ\circ}$. One can easily verify that θ_L is a congruence on L. Let $\varphi(x_i^j) \equiv \varphi(y_i^j)\ (\theta_K)$ for some elements $x_i, y_i \in K(L),\ i = 1, \ldots, n$ and all $j \in \{-2, -1, 0, 1, 2\}$. Then $x_i \equiv y_i\ (\theta_L)$, thus $f(\tilde{x}) \equiv f(\tilde{y})\ (\theta_L)$ as f is compatible on L. Now by (e) again $\varphi(f(\tilde{x})) \equiv \varphi\ (f(\tilde{y}))\ (\theta_K)$, i.e. f'_K preserves the congruences of $K(L)$ where defined. To show that f'_K is well-defined, it suffices to use $\theta_K = \triangle_{K(L)}$, the smallest congruence of $K(L)$. □

3.13 Proposition. *Let one of the following conditions hold in L:*
(i) L is affine complete;
(ii) $K(L)$ is simple (i.e. has only trivial congruences).
Then (EC) is fulfilled in L.

Proof. (i) For the function f'_K associated to a compatible function $f : L^n \to L$ we define a function $f_1 : L^n \to L$ by $f_1(\tilde{x}) = \varphi(f(\tilde{x}))$. This is compatible on L, hence it can be represented by a polynomial $p(x_1, \ldots, x_n)$ of L. Using the rules of computation for $^{\circ}$ and $^{+}$, $p(\tilde{x})$ can be rewritten as $l(\tilde{x}, \tilde{x}^{\circ}, \tilde{x}^{\circ\circ}, \tilde{x}^{+}, \tilde{x}^{++})$ for some lattice polynomial $l(x_1, \ldots, x_{5n})$, i.e. as a lattice polynomial in which terms $x_i, x_i^{\circ}, x_i^{\circ\circ}, x_i^{+}, x_i^{++}$ stand for variables. Further, using the homomorphism φ, one can show that for all $\tilde{x} \in L^n$
$$f'_K(\varphi(\tilde{x}), \ldots, \varphi(\tilde{x}^{++})) = f_1(\tilde{x}) = p(\tilde{x}) = \varphi(p(\tilde{x})) = l'(\varphi(\tilde{x}), \ldots, \varphi(\tilde{x}^{++})),$$
where all constants in l' are of the form $(a \vee k) \wedge l$, i.e. $l'(x_1, \ldots, x_{5n})$ is a polynomial of the lattice $K(L)$. Now, of course, l' can be chosen as the required total compatible extension of the partial function f'_K.

(ii) The statement is obvious as any total extension of f'_K is compatible. □

3.14 Lemma. *Let $f : L^n \to L$ be a compatible function on L. Let $f_s^{\circ},\ f_s^{+} : S(L)^{2n} \to S(L)$ be partial functions such that for all $(x_1, \ldots, x_n) \in L^n$*
$$f_s^{\circ}(x_1^{\circ}, \ldots, x_n^{\circ}, x_1^{+}, \ldots, x_n^{+}) = f(x_1, \ldots, x_n)^{\circ\circ},$$
$$f_s^{+}(x_1^{\circ}, \ldots, x_n^{\circ}, x_1^{+}, \ldots, x_n^{+}) = f(x_1, \ldots, x_n)^{++}$$
and $f_s^{\circ},\ f_s^{+}$ are undefined elsewhere. Then $f_s^{\circ},\ f_s^{+}$ are well-defined and preserve the congruences of $S(L)$ where defined.

Proof. Obviously, $f_s^{\circ},\ f_s^{+}$ are well-defined since $x_i^{\circ} = y_i^{\circ},\ x_i^{+} = y_i^{+},\ i = 1, \ldots, n$ yield $x_i \equiv y_i(\Phi_{+}^{\circ})$ (the determination congruence), which follows $f(\tilde{x}) \equiv f(\tilde{y})(\Phi_{+}^{\circ})$, thus $f(\tilde{x})^{\circ\circ} =$

$f(\tilde{y})^{\circ\circ}$, $f(\tilde{x})^{++} = f(\tilde{y})^{++}$. Further, for any congruence $\theta_{S(L)}$ of $S(L)$ we define an equivalence relation θ_L on L by $x \equiv y\ (\theta_L)$ iff $x^\circ \equiv y^\circ\ (\theta_{S(L)})$ and $x^+ \equiv y^+\ (\theta_{S(L)})$. Since $S(L)$ is a subalgebra of L, θ_L is obviously a congruence of L containing $\theta_{S(L)}$. Similarly as in Lemma 3.12 one can now easily show that f_s°, f_s^+ preserve the congruences of $S(L)$ where defined. □

3.15 Definition. *We shall say that L satisfies an 'extension property'*

(ES) *if for any compatible function $f : L^n \to L$, the partial functions f_s°, $f_s^+ : S(L)^{2n} \to S(L)$ defined as in 3.14 can be extended to total compatible functions of the Kleene algebra S(L).*

3.16 Proposition. *The extension property (ES) concerning the skeleton is fulfilled in L whenever one of the following conditions holds:*
(i) L is affine complete ;
(ii) $S(L)$ is a Boolean algebra;
(iii) $S(L)$ is simple.

Proof. (i) Let L be affine complete and $f : L^n \to L$ be a compatible function on L. We proceed similarly as in 3.13. Concerning the function f_s° associated to f, we define a function $f_1 : L^n \to S(L)$ by $f_1(\tilde{x}) = f(\tilde{x})^{\circ\circ}$. Clearly, f_1 is compatible on L, thus it can be represented by a polynomial $p(x_1, \dots, x_n)$ of the algebra L. Since for any $\tilde{x} \in L^n$, $f_1(\tilde{x}) \in L^{\circ\circ}$, we have $p(\tilde{x}) = p(\tilde{x})^{\circ\circ} = p(\tilde{x})^{++}$ and using the laws for $^\circ$ and $^+$, $p(\tilde{x})^{\circ\circ}$ can be rewritten as $k(x_1^\circ, \dots, x_n^\circ, x_1^+, \dots, x_n^+)$ for some polynomial $k(x_1, \dots, x_{2n})$ of the Kleene algebra S(L). Hence for all $\tilde{x} \in L^n$

$$f_s^\circ(\tilde{x}^\circ, \tilde{x}^+) = f(\tilde{x})^{\circ\circ} = p(\tilde{x})^{\circ\circ} = k(\tilde{x}^\circ, \tilde{x}^+)$$

showing that $k(x_1, \dots, x_{2n})$ can serve as the required total compatible extension of the partial function f_s°. The case of f_s^+ is analogical.

(ii) If $S(L) = L^{\circ\circ} = L^{++}$ is a Boolean algebra, then for any $x \in L$, $0 = x^{\circ\circ} \wedge x^\circ = x \wedge x^\circ$ by (4), and dually, $1 = x^{++} \vee x^+ = x \vee x^+$ by (4^d). Thus L is a double Stone algebra. Let S be the domain of f_s°, i.e.

$$S = \{(\tilde{x}^\circ, \tilde{x}^+);\ \tilde{x} \in L^n\} \subset S(L)^{2n}.$$

One can easily verify that the function f_s° can be interpolated on the set $S \cap \{0,1\}^{2n}$ by a Boolean polynomial function $b : S(L)^{2n} \to S(L)$ defined as follows:

$$b(x_1, \dots, x_{2n}) = \bigvee_{(\tilde{a}, \tilde{b}) \in S \cap \{0,1\}^{2n}} (f_s^\circ(\tilde{a}, \tilde{b}) \wedge x_1^{a_1} \wedge \dots \wedge x_n^{a_n} \wedge x_{n+1}^{b_1} \wedge \dots \wedge x_{2n}^{b_n})$$

where $x_i^0 = x_i$, $x_i^1 = x_i^+ = x_i^\circ = x_i'$.

We shall verify that the assumptions of Lemma 2.13 are satisfied for the Boolean algebra $S(L) = (S(L), \vee, \wedge, ', 0, 1)$ and the functions f_s° and b. It is easy to see that for $x_i = 0$, $(x_i^\circ, x_i^+) = (1,1)$ and for $x_i = 1$, $(x_i^\circ, x_i^+) = (0,0)$. Hence $S \cap \{0,1\}^{2n} \neq \varnothing$ and we claim that for any Boolean homomorphism $h : S(L) \to \{0,1\}$, $h(S) = h(S \cap \{0,1\}^{2n})$.

If for $(\tilde{x}^\circ, \tilde{x}^+) \in S$ we have $h(x_i^\circ) = 1$, then also $h(x_i^+) = h(x_i^\circ \vee x_i^+) = h(x_i^\circ) \vee h(x_i^+) = 1$. In this case $(h(x_i^\circ), h(x_i^+)) = (1,1) = (h(1), h(1))$. If $(h(x_i^\circ), h(x_i^+)) = (0,0)$ then trivially $(h(1^\circ), h(1^+)) = (h(0), h(0)) = (0,0) = (h(x_i^\circ), h(x_i^+))$. The remaining case is $(h(x_i^\circ), h(x_i^+)) = (0,1)$. Since we assume that L has a non-empty core $K(L) = [k, l]$, we have $(h(k^\circ), h(k^+)) = (h(0), h(1)) = (0,1) = (h(x_i^\circ), h(x_i^+))$. We have showed that $h(S) = h(S \cap \{0,1\}^{2n})$.

Applying Lemma 2.13 we get that $f_s^\circ \equiv b$ identically on the whole set S, hence the polynomial function b is the required compatible extension of f_s° on the skeleton $S(L)$. For f_s^+ one can proceed in the same way.

(iii) If $S(L)$ is simple, then both total extensions of f_s°, f_s^+ are compatible. □

3.17 Theorem. *Let L be an (infinite) double K_2-algebra with a non-empty bounded core $K(L)$. The following conditions are equivalent.*
(1) L is affine complete ;
(2) (i) $K(L)$ is an affine complete distributive lattice and
(ii) $S(L)$ is an affine complete Kleene algebra (cf. [11]) and
(iii) (EC) and
(iv) (ES).

Proof. The necessity follows from Theorem 3.3 and Propositions 3.4, 3.13 and 3.16. To prove the converse, let $f : L^n \to L$ be a compatible function on L. By Lemma 3.10 we can write

(f) $\quad f(\tilde{x}) = f(\tilde{x})^{++} \vee (f(\tilde{x})^{\circ\circ} \wedge (f(\tilde{x}) \vee k) \wedge l)$ for all $\tilde{x} \in L^n$.

To replace $(f(\tilde{x}) \vee k) \wedge l$, $f(\tilde{x})^{\circ\circ}$ and $f(\tilde{x})^{++}$ in (f) by polynomials of L, we take the partial functions f'_K and f_s°, f_s^+ associated to f as in 3.11 and 3.14. By (EC) and (ES), these partial compatible functions can be extended to total compatible functions $f_1(x_1, \ldots, x_{5n})$ and $f_2(x_1, \ldots, x_{2n})$, $f_3(x_1, \ldots, x_{2n})$ of $K(L)$ and S(L), respectively, which by hypothesis can be represented by polynomials $p_1(x_1, \ldots, x_{5n})$ and $p_2(x_1, \ldots, x_{2n})$, $p_3(x_1, \ldots, x_{2n})$ of $K(L)$ and S(L), respectively. Therefore in (f),

$f(\tilde{x}) = p_3(\tilde{x}^\circ, \tilde{x}^+) \vee [p_2(\tilde{x}^\circ, \tilde{x}^+) \wedge p_1(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \ldots, \varphi(\tilde{x}^{++})]$

for all $\tilde{x} \in L^n$, thus f can be represented by a polynomial of the algebra L. □

Now we derive the Beazer characterization of affine complete double Stone algebras with a non-empty bounded core (Proposition 2.3). The equivalence of (2) and (3) in 2.3 is known from 2.2, thus we show the equivalence of (1) and (2).

(1) $\Longrightarrow$ (2) of 2.3 is immediate from 3.3. Now let in a double Stone algebra L the core $K(L) = [k, l]$ is affine complete, i.e. $K(L)$ does not contain a proper Boolean interval. Since the skeleton $S(L)$ is a Boolean algebra, by 3.16 and 3.17 it only remains to show (EC). We first prove the following two lemmas.

3.18 Lemma. *Let L be a double Stone algebra with a non-empty bounded core $K(L) = [k, l]$ and $x, y \in L$. Then*

$\varphi(x^\circ) = \varphi(y^\circ)$ *iff* $\varphi(x^{\circ\circ}) = \varphi(y^{\circ\circ})$ *and*
$\varphi(x^+) = \varphi(y^+)$ *iff* $\varphi(x^{++}) = \varphi(y^{++})$.

Proof. Let $\varphi(x^+) = \varphi(y^+)$. The identities $x^+ \wedge x^{++} = 0$ and $x^+ \vee x^{++} = 1$ imply $\varphi(x^+) \wedge \varphi(x^{++}) = k$, $\varphi(x^+) \vee \varphi(x^{++}) = l$ for any $x \in L$. Hence

$\varphi(x^{++}) = (\varphi(y^+) \wedge \varphi(y^{++})) \vee \varphi(x^{++}) = (\varphi(x^+) \vee \varphi(x^{++})) \wedge (\varphi(y^{++}) \vee \varphi(x^{++})) = \varphi(y^{++}) \vee \varphi(x^{++})$.

In the same way one can show $\varphi(y^{++}) = \varphi(y^{++}) \vee \varphi(x^{++})$. The converse statement as well as the proof of the first one are analogical. □

3.19 Lemma. *Let L be a double Stone algebra with a non-empty bounded core $K(L) = [k,l]$, $k < l$ and let $x \in L$ such that $\varphi(x^\circ), \varphi(x^{\circ\circ}), \varphi(x^+), \varphi(x^{++}) \in \{k,l\}$. Then $\varphi(x^\circ) = l$ implies $\varphi(x^{\circ\circ}) = k$ and analogously, $\varphi(x^+) = l$ implies $\varphi(x^{++}) = k$.*

Proof. Let $\varphi(x^\circ) = l$. It is obvious that $\varphi(x^{\circ\circ}) = l$ would yield $l = \varphi(x^\circ) \wedge \varphi(x^{\circ\circ}) = \varphi(0) = k$, a contradiction. Analogously, if $\varphi(x^+) = l = \varphi(x^{++})$, then $l = \varphi(x^+) \wedge \varphi(x^{++}) = \varphi(0) = k$, a contradiction, using the identity $x^+ \wedge x^{++} = 0$. □

Now we are ready to prove the final result.

3.20 Proposition. *Let L be a double Stone algebra with a non-empty bounded core $K(L) = [k,l]$ such that $K(L)$ contains no proper Boolean interval. Then (EC) is fulfilled in L.*

Proof. If $k = l$, then L is a Post algebra of order 3 and trivially, (EC) is fulfilled in L. So we can further assume that $k < l$.

Let $f' = f'_K : K(L)^{5n} \to K(L)$ be the partial compatible function associated to a compatible function $f : L^n \to L$ as in 3.11. Let $S = \{(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++}));\ \tilde{x} \in L^n\}$ be the domain of f', $S \subset K(L)^{5n}$. We shall show that f' can be interpolated on the set $S \cap \{k,l\}^{5n}$ by the following polynomial of the lattice $K(L)$:

$$(g) \quad q(x_1,\ldots,x_{5n}) = \bigvee_{\tilde{b} \in S \cap \{k,l\}^{5n}} (f'(b_1,\ldots,b_{5n}) \wedge y_1 \wedge \cdots \wedge y_{5n}),$$

where $y_i = \begin{cases} x_i, & \text{if } b_i = l \\ l, & \text{if } b_i = k. \end{cases}$

Let $\tilde{x}$ be any (fixed) vector from $S \cap \{k,l\}^{5n}$. If $\tilde{a} \neq \tilde{x}$ and $b_j \neq x_j$ for some j, $n < j \leq 5n$, then either $b_j = l$, $x_j = k$ and then $f'(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n} = k$ or $b_j = k$, $x_j = l$ and then by Lemmas 3.18, 3.19 there exists $s \in \{j-n, j+n\}$ such that $x_s = k, b_s = l$, thus again $f'(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n} = k$. Hence it suffices to take into account in (g) only conjunctions $f'(\tilde{b}) \wedge y_1 \wedge \cdots \wedge y_{5n}$ such that $b_i = x_i$ for all i, $n < i \leq 5n$ and moreover, $b_i \leq x_i$ for all i, $1 \leq i \leq n$. So

$$q(x_1,\ldots,x_{5n}) = \bigvee_{\tilde{b} \in S \cap \{k,l\}^{5n},\ \tilde{b} \leq \tilde{x}} (f'(b_1,\ldots,b_n,x_{n+1},\ldots,x_{5n}).$$

Next, we show that $f'(\tilde{b}) \leq f'(\tilde{x})$ for any $\tilde{b} \in S \cap \{k,l\}^{5n}$ such that $b_i = x_i$ for $i = n+1,\ldots,5n$ and $b_i \leq x_i$ for $i = 1,\ldots,n$. Denote $u_s = b_s$ if $b_s = x_s$, otherwise $u_s = u$, $1 \leq s \leq n$. We get a unary compatible function $g : K(L) \to K(L)$, $g(u) = f'(u_1,\ldots,u_n,x_{n+1},\ldots,x_{5n})$ and we have to show that $g(k) \leq g(l)$. Since $g(k) \equiv g(u)\ (\theta_{\text{lat}}(k,u))$ and $g(u) \equiv g(l)\ (\theta_{\text{lat}}(u,l))$ for any $u \in K(L)$ ($\theta_{\text{lat}}(k,u)$ and $\theta_{\text{lat}}(u,l)$ denote the principal lattice congruences generated by the pairs (k,u) and (u,l), respectively), we get

$g(u) \vee u = g(k) \vee u$ and
$g(u) \wedge u = g(l) \wedge u.$

This means that for any $u \in [g(l), g(k) \vee g(l)]$, $g(u)$ is the relative complement of u in this interval, which is therefore Boolean. By the assumption of Proposition 3.20 this implies $g(k) \leq g(l)$, what was to be proved. Hence

$$q(x_1,\ldots,x_{5n}) = f'(x_1,\ldots,x_{5n}) \quad \text{for any } \tilde{x} \in S \cap \{k,l\}^{5n}.$$

We shall show that the assumptions of Lemma 2.13 are satisfied for the lattice $K(L)$ and the functions f' and q. If in the $5n$-tuple $(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})) \in S$ we take $x_i = 0$ then $(\varphi(x_i), \varphi(x_i^\circ), \varphi(x_i^{\circ\circ}), \varphi(x_i^+), \varphi(x_i^{++})) = (k, l, k, l, k)$ and if $x_i = 1$ then $(\varphi(x_i), \varphi(x_i^\circ), \varphi(x_i^{\circ\circ}), \varphi(x_i^+), \varphi(x_i^{++})) = (l, k, l, k, l)$. Hence $S \cap \{k, l\}^{5n} \neq \varnothing$ and we claim that $h(S) = h(S \cap \{k, l\}^{5n})$ for any $(0, 1)$-lattice homomorphism $h : K(L) \to \{k, l\}^{5n}$.

Let $(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})) \in S$ and $i \in \{1, \dots, n\}$. Since h and φ are lattice homomorphisms and L is a double Stone algebra, we have

$$h(\varphi(x_i^\circ)) \vee h(\varphi(x_i^{\circ\circ})) = h(\varphi(x_i^\circ \vee x_i^{\circ\circ})) = h(\varphi(1)) = h(l) = l,$$
$$h(\varphi(x_i^\circ)) \wedge h(\varphi(x_i^{\circ\circ})) = h(\varphi(x_i^\circ \wedge x_i^{\circ\circ})) = h(\varphi(0)) = h(k) = k$$

and analogously,

$$h(\varphi(x_i^+)) \vee h(\varphi(x_i^{++})) = h(\varphi(x_i^+ \vee x_i^{++})) = h(\varphi(1)) = h(l) = l,$$
$$h(\varphi(x_i^+)) \wedge h(\varphi(x_i^{++})) = h(\varphi(x_i^+ \wedge x_i^{++})) = h(\varphi(0)) = h(k) = k.$$

Hence

$$h(\varphi(x_i^\circ)) = k \text{ if and only if } h(\varphi(x_i^{\circ\circ})) = l \text{ and}$$
$$h(\varphi(x_i^+)) = k \text{ if and only if } h(\varphi(x_i^{++})) = l.$$

This yields that each 5-tuple $(h(\varphi(x_i)), h(\varphi(x_i^\circ)), h(\varphi(x_i^{\circ\circ})), h(\varphi(x_i^+)), h(\varphi(x_i^{++})))$ can only be one of the 5-tuples (k, l, k, l, k), (k, k, l, k, l), (l, k, l, k, l) or (l, l, k, l, k). Since moreover,

$$h(\varphi(x_i)) \wedge h(\varphi(x_i^\circ)) = h(\varphi(x_i \wedge x_i^\circ)) = h(\varphi(0)) = h(k) = k,$$

the last 5-tuple (l, l, k, l, k) is impossible. Hence we get

$$\{(h(\varphi(x_i)), h(\varphi(x_i^\circ)), h(\varphi(x_i^{\circ\circ})), h(\varphi(x_i^+)), h(\varphi(x_i^{++}))); \ x_i \in L\} =$$
$$\{(k, l, k, l, k), (k, k, l, k, l), (l, k, l, k, l)\} =$$
$$\{(h(\varphi(x_i)), h(\varphi(x_i^\circ)), h(\varphi(x_i^{\circ\circ})), h(\varphi(x_i^+)), h(\varphi(x_i^{++}))); \ x_i \in \{0, k, 1\} \}.$$

Note that for $x_i \in \{0, k, 1\}$ we have $(\varphi(x_i), \varphi(x_i^\circ), \varphi(x_i^{\circ\circ}), \varphi(x_i^+), \varphi(x_i^{++})) \in \{k, l\}^5$. Consequently, $h(S) = h(S \cap \{k, l\}^{5n})$.

By Lemma 2.13, $f' \equiv q$ on S, thus $q(x_1, \dots, x_{5n})$ is the required total compatible extension of the partial function f'. $\square$

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DEFINABILITY OF PASCAL'S TRIANGLES MODULO 4 AND 6 AND SOME OTHER BINARY OPERATIONS FROM THEIR ASSOCIATED EQUIVALENCE RELATIONS

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ABSTRACT. Pascal's triangles modulo n can be considered as binary operations on the set $\mathbb{N}$ of nonnegative integers. To every binary operation f on a set M an equivalence relation R on M^2 can be associated in which (x, y) is equivalent with (u, v) if and only if $f(x, y) = f(u, v)$. The equivalence R can be considered as a 4-ary relation on M, and we can try to reconstruct f from R, more precisely, to define elementarily f in $\langle M; R\rangle$. (Some abstract information about f is used by this.) This problem will be solved for Pascal's triangles modulo n for $1 \leq n \leq 6$. The answer is positive for $n = 4$ and $n = 6$, negative for $n = 1$ and n prime (even greater than 6). As a corollary we obtain that the operations $+$, $\times$ are definable in the structure $\langle \mathbb{N}; \mathrm{EqB}_6 \rangle$, where $\mathrm{EqB}_6 = \{(x, y, u, v) \in \mathbb{N}^4 \mid \binom{x+y}{x} \equiv \binom{u+v}{u} \pmod 6\}$; the integer 6 cannot be replaced by any smaller positive integer. The above mentioned problem will be solved (positively, resp. negatively) also for addition and multiplication on the set $\mathbb{N}$, resp. $\mathbb{Z}$, and for some other operations.

1. INTRODUCTION

To every mapping $f : X \to Y$ an equivalence relation R on X can be associated by the formula $R(x, y) \iff f(x) = f(y)$. In particular, to every binary operation $*$ on a set M an equivalence relation R on M^2 can be associated. We can consider R as a 4-ary relation on the set M in the obvious way.

Definition. *Let $*$ be a binary operation on a set M. We shall say that R is the associated equivalence relation of the operation $*$ if*

$$R = \{(x, y, u, v) \in M^4 \mid x * y = u * v\}.$$

Notice once more that the associate equivalence relation is not an equivalence relation on M but may be considered as an equivalence relation on M^2. (Analogously we could associate a $2n$-ary relation to an n-ary operation also for $n \neq 2$.) We can also define R in the groupoid $\langle M; *\rangle$ by the first order formula

$$R(x, y, u, v) \iff x * y = u * v.$$

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The relation R was constructed from $*$. We can ask whether, conversely, $*$ can be reconstructed from R. Generally speaking, it is impossible because distinct operations can have equal associated equivalence relations. However, sometimes it can be done if additional information about the structure $\langle M; *\rangle$ is available. In what follows usually the structure $\langle M; *\rangle$ will be given up to isomorphism; this is the strongest possible *abstract* information about $\langle M; *\rangle$, but it also need not suffice. Further, the answer can depend on the chosen type of reconstructability. We shall deal with first order definability, and so our goal is to define (elementarily, i.e., by a first order formula) the operation $*$ in the structure $\langle M; R\rangle$.

We shall investigate Pascal's triangles modulo n from this point of view. Pascal's triangle modulo n will be denoted by B_n, and it is defined by the formula

$$B_n(x, y) = \binom{x+y}{x} \operatorname{MOD} n;$$

the symbol MOD denotes the rest by integer division. In the present paper moduli $n \leq 6$ will be considered; the greater moduli will be considered later. However, the answer to our problem seems to depend on the factorization of n. So examples for all three typical cases are presented here: n prime, n prime power (with the exponent $e > 1$) and n divisible by at least two distinct primes. Besides Pascal's triangles modulo n also several more classical examples of binary operations will be considered.

We shall use the classical first order predicate calculus with equality. We shall use five usual logical connectives with usual priority rules and other method to simplify or shorter the formulas. Classical mathematical symbols (like $+$, $\times$ etc.) will be used in their usual sense; it may depend on the considered base set. Predicate and functional symbols are formed rather freely (groups of several letters, subscripts, superscripts, ...) but, of course, from the formal point of view every such symbol is considered as indecomposable.

2. Illustrating examples

Let us describe the problem from previous section informally for the case M finite. Let in the Cayley table of $\langle M; *\rangle$ the inside elements are replaced by colours (distinct elements by distinct colours). We obtain the coloured table (without information about the used association of colours to the elements of M). Then coloured table shows us the equivalence relation associated to $*$ (and nothing more: the coloured table can be constructed, up to the choice of colours, from this relation). Moreover, we obtain some additional information about $\langle M; *\rangle$. For example, we can obtain the Cayley table of an isomorphic copy of the structure; this is the strongest possible *abstract* information about $\langle M; *\rangle$. Of course, the order of the elements in its headings need not correspond to that in the coloured table. Our goal is to reconstruct the coloured table, i.e. to replace the colours by the elements of M in the original way. In principle, we could simply check all binary operations on M. However, we shall use the considerations which will be useful also for the infinite cases considered later; for example, we shall look for "invariants" expressible by first order formulas.

Example 2.1. Let $\langle M; *\rangle$ be a groupoid , where $M = \{0, 1, 2, 3\}$ and $*$ is the operation which must be reconstructed. We are given the Cayley table of an isomorphic copy $\mathcal{A}$

of $\langle M; *\rangle$ and the coloured table constructed as described above. They are given in the left-hand and the central part of Figure 1; let the capitals correspond to colours: Red, Green, Blue, Yellow.

	a	b	c	d
a	a	d	b	c
b	c	d	c	b
c	a	a	a	d
d	b	c	d	a

	0	1	2	3
0	R	G	B	Y
1	B	R	Y	G
2	G	B	Y	B
3	Y	R	R	R

	0	1	2	3
0	1	2	3	0
1	3	1	0	2
2	2	3	0	3
3	0	1	1	1

Figure 1.

We have to associate elements to colours and (as a by-product) find an isomorphism between $\langle M; *\rangle$ and the structure $\mathcal{A}$ (with Cayley's table) displayed on the left. Since $\mathcal{A}$ has no nontrivial automorphism the isomorphism will be determined uniquely. The Red colour occurs in the table 5 times, and the only element which occurs 5 times in the left table (except headings) is a; therefore Red must be associated to a. Similarly Green must be associated to b. The Yellow occurs once at the main diagonal, hence it must be associated to d. It remains to associate Blue to c. There are only two distinct colours in the row of 3, therefore 3 must be associated to c, and hence to Blue. The only pair of commuting elements of $\langle M; *\rangle$ is $\{0, 3\}$ and the only such pair in $\mathcal{A}$ is $\{c, d\}$. Therefore 0 must be associated to d, and hence to Yellow. There remain the colours Red, Green and the elements 1, 2. We cannot associate Red to 2 because the obtained algebra would have no idempotent, and $\mathcal{A}$ has one. Therefore we have to associate Red to 1, and finally Green to 2. The completed Cayley table is on the right-hand side of Figure 1.

Remark. A faster (but less illustrative) method in this case would be to consider the invariants "number of distinct symbols in the row and in the column" for the elements of $\mathcal{A}$ and the elements of M. So we obtain immediately the isomorphism mentioned above. Then we can "forget" colours, and fill in the table of $\langle M; *\rangle$.

Example 2.2. Let us consider the tables on the left-hand part of Figure 2; their roles are similar to those in Figure 1. Now we cannot reconstruct the operation $*$ uniquely (and we cannot define it from its associated equivalence relations) because there are two distinct (although isomorphic) algebras which fulfil the given conditions.

	a	b
a	a	b
b	b	a

	0	1
0	R	G
1	G	R

	0	1
0	0	1
1	1	0

	0	1
0	1	0
1	0	1

Figure 2.

More formally, let $\langle\{0, 1\}, \oplus\rangle$ be the additive group modulo 2. Then the operation $\oplus$

is not definable in the structure $\langle\{0,1\};R\rangle$, where

$$R=\{(x,y,u,v)\in\{0,1\}^4 \mid x\oplus y=u\oplus v\}$$

is the the associated equivalence relation of $\oplus$.

Example 2.3. Let us consider the left-hand and the central table of Figure 3; their role is similar to those in Example 2.1.

	a	b	c
a	a	c	c
b	c	b	c
c	c	c	c

	0	1	2
0	R	G	G
1	G	B	G
2	G	G	G

	0	1	2
0	0	2	2
1	2	1	2
2	2	2	2

Figure 3.

The algebra on the left is idempotent, hence the reconstructed algebra is also idempotent. Hence the elements 0, 1, 2 must be associated to the colours Red, Blue, Green, respectively; we can see it immediately from the diagonal of the central table. However, the element and the colour associated to a are not uniquely determined. It can be either 0 and Red or 1 and Blue; both choices are possible, and determine the two isomorphisms between the left-hand and the right-hand table.

Example 2.4. Let us consider the left-hand and the central table of Figure 3, and let us replace any non-diagonal G in the central table by R. The only possible solution of our problem can be the operation on the right. However, it is not a solution indeed. So we can see that the table of an isomorphic image and the coloured table cannot be combined arbitrarily. They really must correspond to the same algebra.

Example 2.5. Let us consider two operations on the set $\{0,1\}$: logical conjuction (now denoted by $\bullet$) and Sheffer's function denoted by $*$; the tables are given in Figure 4.

$\bullet$	0	1
0	0	0
1	0	1

$*$	0	1
0	1	1
1	1	0

Figure 4.

The corresponding algebras are not isomorphic; the first one is idempotent while the other is not. Nevertheless, we can see that $\mathrm{Eq}_\bullet=\mathrm{Eq}_*$. Both elements 0, 1 are definable (as constants) from the relation $\mathrm{Eq}_\bullet$, and hence the operations $\bullet$ and $*$ (as well as all other binary operations on $\{0,1\}$) are definable, too. (Of course, the defining formulas must be distinct.)

Example 2.6. Let us consider a set M of cardinality greater than 1 and the operation $*$ on M be defined by $x * y = x$ for all $x, y \in M$. Then $\mathrm{Eq}_*(x,y,z,w) \iff x = z$. We cannot define the elements of M (as constants) in the structure $\langle M; \mathrm{Eq}_* \rangle$. Nevertheless, the operation $*$ is definable in this structure. (Of course, we need not use Eq_* at all.)

3. Some results for more classical structures

Theorem 3.1. *Let $*$ be an idempotent operation on a nonempty set M and let Eq_* be its associated equivalence relation. Then the operation $*$ is first order definable in the structure $\langle M; \mathrm{Eq}_* \rangle$.*

Proof. The operation $*$ can be defined by the formula

$$z = x * y \iff \mathrm{Eq}_*(z, z, x, y).$$

Indeed, since always $z * z = z$ the equations $z = x * y$ and $z * z = x * y$ are equivalent. □

Remark. Theorem 3.1 is formulated for a single algebra. We can reformulate it for the class of all (nonempty) idempotent groupoids without any difficulties. However, it cannot be extended to the class of all groupoids, as Example 2.5 shows.

For the next theorem remember that *the center* of a group is its subset consisting of all elements which commute with every element. The center of a group is always nonempty because it contains its neutral element.

Theorem 3.2. *Let $\langle M; * \rangle$ be a group and let Eq_* be the associated equivalence relation of the operation $*$. Then $*$ is definable in the structure $\langle M; \mathrm{Eq}_* \rangle$ if and only if the center of the group $\langle M; * \rangle$ consists of unique element.*

Proof. If the center of $\langle M; * \rangle$ contains only the neutral element of G then we can define this element and then the operation $*$ in $\langle M; * \rangle$ as follows:

$$x = 1 \iff \forall y, u, v\big(\mathrm{Eq}_*(x, y, u, v) \implies \mathrm{Eq}_*(y, x, u, v)\big),$$
$$z = x * y \iff \mathrm{Eq}_*(z, 1, x, y).$$

If the center contains an element $a \neq 1$ then we shall consider the operation $\otimes$ defined by $x \otimes y = a * x * y$. The structure $\langle M; \otimes \rangle$ is a group isomorphic with $\langle M; * \rangle$ (and distinct from it). However, both operations have the same associated equivalence relations Eq_*, and hence none of them can be definable in $\langle M; \mathrm{Eq}_* \rangle$. □

Theorem 3.3. (i) *The operation $+$ (on the set $\mathbb{N}$) is definable in the structure $\langle \mathbb{N}; \mathrm{EqPlus} \rangle$, where* $\mathrm{EqPlus} = \{(x, y, u, v) \in \mathbb{N}^4 \mid x + y = u + v\}$.

(ii) *The operation $\times$ (on the set $\mathbb{N}$) is definable in the structure $\langle \mathbb{N}; \mathrm{EqTimes} \rangle$, where* $\mathrm{EqTimes} = \{(x, y, u, v) \in \mathbb{N}^4 \mid xy = uv\}$.

Proof. In $\langle \mathbb{N}; \mathrm{EqPlus} \rangle$ we can define

$$x = 0 \iff \forall y, z\big(\mathrm{EqPlus}(x, x, y, z) \implies x = y \wedge x = z\big).$$

Then we have $z = x + y \iff \mathrm{EqPlus}(z, 0, x, y)$. The proof of (ii) is similar; we shall define 1 at first. □

Theorem 3.4. (i) *The operation* $+$ *(on the set* $\mathbb{Z}$*) is not definable in the structure* $\langle \mathbb{Z}, \mathrm{EqPlus} \rangle$*, where* $\mathrm{EqPlus} = \{(x,y,u,v) \in \mathbb{Z}^4 \mid x+y = u+v\}$.

(ii) *The operation* $\times$ *(on the set* $\mathbb{Z}$*) is not definable in the structure* $\langle \mathbb{Z}; \mathrm{EqTimes} \rangle$*, where* $\mathrm{EqTimes} = \{(x,y,u,v) \in \mathbb{Z}^4 \mid xy = uv\}$.

Proof. For (i) we can apply Theorem 3.2 because the center of the commutative group $\langle \mathbb{Z}; + \rangle$ is $\mathbb{Z}$.

For (ii) let us consider the mapping $f : x \mapsto -x$. We can see that f is an automorphism of $\langle \mathbb{Z}; \mathrm{EqTimes} \rangle$, and f is not an automorphism of $\langle \mathbb{Z}, \times \rangle$. Therefore $\times$ cannot be definable in $\langle \mathbb{N}; \mathrm{EqTimes} \rangle$. (Remarks: 1. f is the only nontrivial automorphism of the considered structure. 2. We can define the set $\{-1, 1\}$ (as a unary relation), but we cannot distinguish the element 1.) $\square$

4. Auxiliary results about Pascal's triangles modulo n

Here we shall present some notions and results useful for the next section. The results will be given without proofs; they are either classical or easy or contained in [Bo90] or [Ko93]. We shall start with n-adic masking relation for arbitrary $n > 1$, although it is closely related to Pascal's triangle modulo n only for n prime. If a number $x \in \mathbb{N}$ is given by its n-adic digits $a_r, a_{r-1}, \dots, a_0$ we shall write $x = [a_r a_{r-1} \dots a_0]_n$. Leading zeros are allowed if necessary (e. g., to obtain equal numbers of digits in two integers). For $x = [a_r \dots a_1 a_0]_n$, $y = [b_r \dots b_1 b_0]_n$ we shall write $x \sqsubseteq_n y$ if it holds $a_i \le b_i$ for all $i = 0, 1, \dots, r$. The relation $\sqsubseteq_n$ will be called *n-adic masking relation.*

Claim 4.1. *For every integer* $n > 1$ *the relation* $\sqsubseteq_n$ *is a partial order on the set* $\mathbb{N}$*. In the structure* $L = \langle \mathbb{N}; \sqsubseteq_n \rangle$ *we can define:*

$x \sqsubset_n y$	*proper masking relation;*
$x \sqcap_n y$	*meet operation in* L*;*
$x \sqcup_n y$	*join operation in* L*;*
0	*the constant* 0 *(zero) as the smallest element of* L*;*
$\mathrm{Pow}_p(x)$	x *is a power of* p*;*
$\mathrm{CFAdd}_p(x,y,z)$	*carry-free addition:* $x + y = z$*, and no carry occurs when* $x + y$ *is computed.*

The structure $\langle \mathbb{N}; \sqcup_n, \sqcap_n \rangle$ *is a distributive lattice with the smallest element* 0.

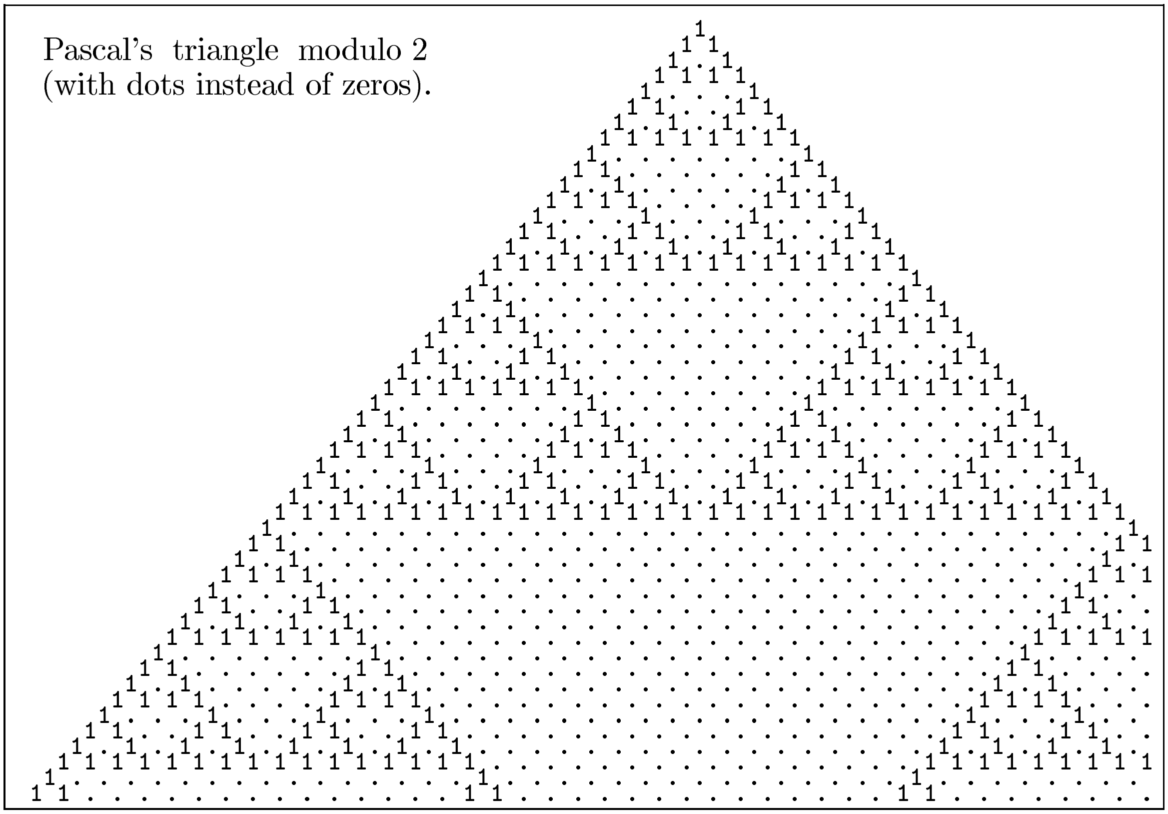
Figure 4 contains Pascal's triangles modulo 2 and modulo 3. The coordinate system with the axes oriented right downward and left downward is used, and the elements 0 are replaced by dots. (The same system is used in further figures, too.) We can see their rather simple "fractal" structure, which is common for all Pascal's triangles modulo prime numbers. A very useful tool in investigating them is Lucas' theorem, see e. g. [Bo90]. We shall give it in a slightly modified form, with $\binom{x+y}{x}$ instead of $\binom{x}{y}$.

Theorem 4.2. *If* n *is a prime and*

$$x = [a_r \dots a_1 a_0]_n, \qquad y = [b_r \dots b_1 b_0]_n \tag{4.2.1}$$

then

$$\binom{x+y}{x} \equiv \binom{a_0+b_0}{a_0} \cdot \binom{a_1+b_1}{a_1} \cdot \dots \cdot \binom{a_r+b_r}{a_r} \pmod{n}. \tag{4.2.2}$$


Pascal's triangle modulo 2
(with dots instead of zeros).

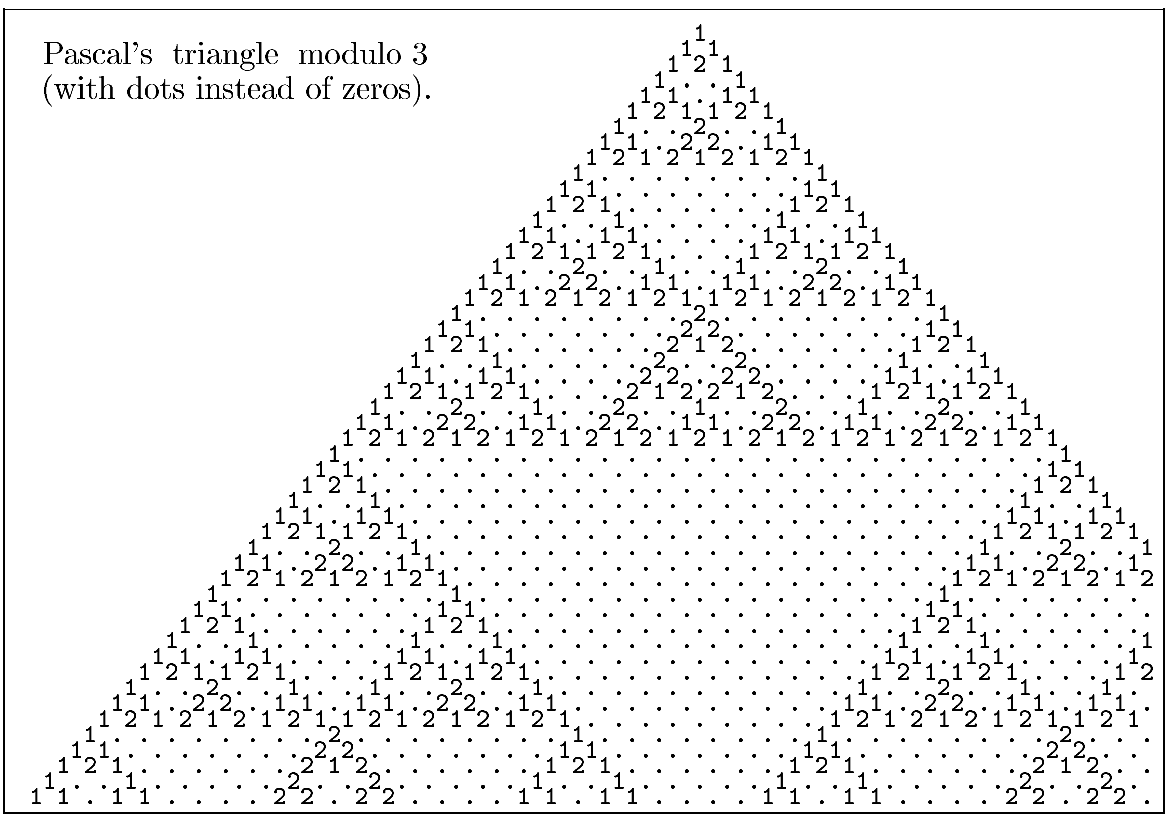

Pascal's triangle modulo 3
(with dots instead of zeros).

Figure 4.

Corollary 4.3. *For n, x, y as in Theorem 4.2 we have*

$$B_n(x, y) = 0 \text{ if and only if } a_i + b_i \geq n \text{ for some } i \in \{0, 1, \ldots, r\}.$$

Theorem 4.4. *For every prime n the relation $\sqsubseteq_n$ is first order definable in the structure $\langle \mathbb{N}; B_n \rangle$.*

Proof. The defining formula can be

$$x \sqsubseteq_n y \iff \forall z \big(B_n(x, z) = 0 \implies B_n(y, z) = 0\big). \tag{4.4.1}$$

To prove that, some considerations about n-adic digits must be done and Corollary 4.3 applied. □

Claim 4.5. *For every prime n and $x, y, e \in \mathbb{N}$, the integer $\binom{x+y}{x}$ is divisible by n^e if and only if at least e carries occur in the addition of x, y in n-adic number system.*

For the proof binomial coefficients must be expressed by factorials, and the exponents of n in factorizations of the factorials ought to be computed.

5. Definability of Pascal's triangles modulo n

Now we shall investigate definability of the operations B_n (i.e., Pascal's triangles modulo n) from their associated equivalence relations EqB_n defined by

$$\mathrm{EqB}_n = \big\{(x, y, u, v) \in \mathbb{N}^4 \mid B_n(x, y) = B_n(u, v)\big\}.$$

In the present paper we shall consider only few small values of n.

Let us define $VB_n(x) = \big\{B_n(x, y) \mid y \in \mathbb{N}\big\}$ and let $\mathrm{CVB}_n^k(x)$ mean that $\mathrm{card}(VB_n(x)) = k$. The functions VB_n cannot be (first order) defined in $\langle \mathbb{N}; B_n \rangle$ simply because their values are *subsets* of $\mathbb{N}$, and not elements of $\mathbb{N}$. However, the predicates CVB_n^k (for $1 \leq k \leq n$) can be defined:

$$\mathrm{CVB}_n^k(x) \iff \exists y_1, \ldots, y_k \bigwedge_{i=2}^{k} \bigwedge_{j=1}^{i-1} \neg \mathrm{EqB}_n(x, y_i, x, y_j) \wedge$$
$$\wedge \forall y_1, \ldots, y_{k+1} \bigvee_{i=2}^{k+1} \bigvee_{j=1}^{i-1} \mathrm{EqB}_n(x, y_i, x, y_j);$$

for $k = 1$ the first member no the right can be deleted. Further, let $\mathrm{EB}_n^{i_1 \ldots i_k}(x, y)$ mean $B_n(x, y) \in \big\{i_1, \ldots, i_k\big\}$. In particular, let $\mathrm{EB}_n^i(x, y)$ mean $B_n(x, y) = i$.

Now we shall investigate Pascal's triangle modulo 4; its structure is more complicated than that of B_2 but there is an obvious relationship between them; it is expressed by the formula $B_2(x, y) = B_4(x, y) \,\mathrm{MOD}\, 2$. The function B_4 is displayed in Figure 5.

Theorem 5.1. *The operation B_4 (Pascal's triangle modulo 4) is first order definable in the structure $\langle \mathbb{N}; \mathrm{EqB}_4 \rangle$, where EqB_4 is the equivalence relation associated to B_4.*

Proof. Pascal's triangle modulo 4 is displayed in Figure 2. We can see that it contains only 1's on its margin (i.e. for $x = 0$ or $y = 0$) and (with exception on the top) only even

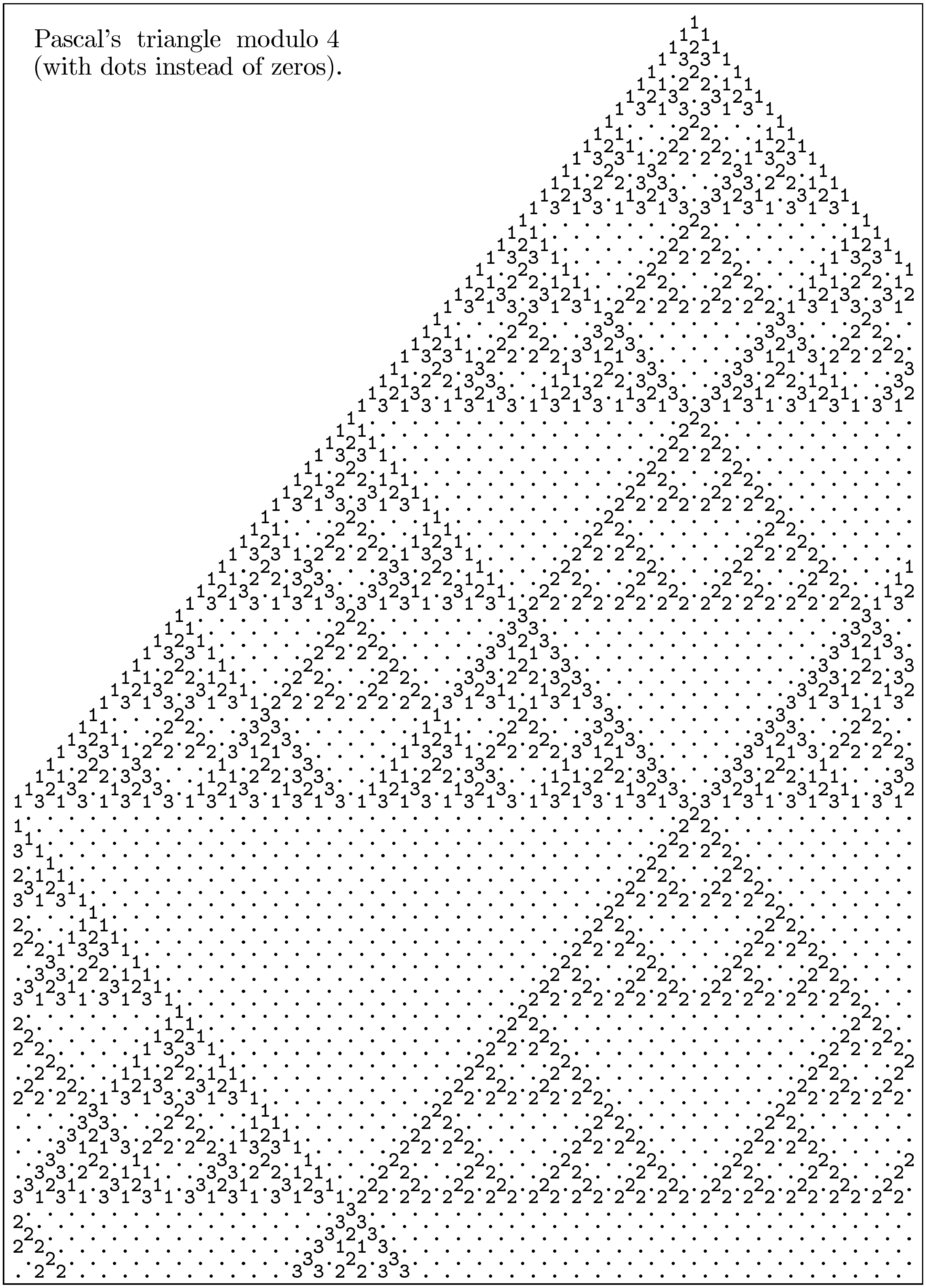


Figure 5.

elements on its axis. The first property is obvious. The second one can be easily proved from Lucas' Theorem (used for the modulus 2 and $x = y$). Therefore we can define

$$\begin{aligned} x = 0 &\iff \forall y, z\, \mathrm{EqB}_4(x, y, x, z), \\ \mathrm{EB}_4^1(x, y) &\iff \mathrm{EqB}_4(x, y, 0, 0), \\ \mathrm{EB}_4^{02}(x, y) &\iff \exists z\big(z \neq 0 \land \mathrm{EqB}_4(x, y, z, z)\big). \end{aligned}$$

The formula $\mathrm{EB}_4^{02}(x, y)$ obviously corresponds to $B_2(x, y) = 0$. Therefore we can define the masking relation $\sqsubseteq_2$ by a formula similar to (4.4.1) as follows:

$$x \sqsubseteq_2 y \iff \forall z\big(\mathrm{EB}_4^{02}(x, z) \implies \mathrm{EB}_4^{02}(y, z)\big).$$

The structure $\langle \mathbb{N}; \sqsubseteq_2 \rangle$ is a partially ordered set with the smallest element 0; we can consider it as a distributive lattice in the usual way (see Claim 4.1); let the lattice operations be $\sqcup_2$, $\sqcap_2$. We can define the set Pow_2 as the set of atoms of the lattice. Further we can define

$$\begin{aligned} \mathrm{EB}_4^2(x, y) &\iff \exists z\big(\mathrm{Pow}_2(z) \land \mathrm{EqB}_4(x, y, z, z)\big), \\ \mathrm{EB}_4^0(x, y) &\iff \mathrm{EB}_4^{02}(x, y) \land \neg\, \mathrm{EB}_4^2(x, y), \\ \mathrm{EB}_4^3(x, y) &\iff \neg\, \mathrm{EB}_4^{02}(x, y) \land \neg\, \mathrm{EB}_4^1(x, y). \end{aligned}$$

Now we shall use that $B_4(2^x, 2^x + 2^y) = 0$ only if $y = x + 1$; indeed, only in this case two carries occur in the addition of 2^x and $2^x + 2^y$ (in binary number system; see Claim 4.5). This enables us to define the constants 1, 2, 3. (We could also define addition, but we need not that now.)

$$\begin{aligned} x = 1 &\iff \mathrm{Pow}_2(x) \land \forall y\big(\mathrm{Pow}_2(y) \implies \mathrm{EB}_4^2(y, x \sqcup_2 y)\big), \\ x = 2 &\iff \mathrm{Pow}_2(x) \land \mathrm{EB}_4^0(1, x \sqcup_2 1)\big), \\ 3 &= 1 \sqcup_2 2. \end{aligned}$$

Finally, the function B_4 can be defined by

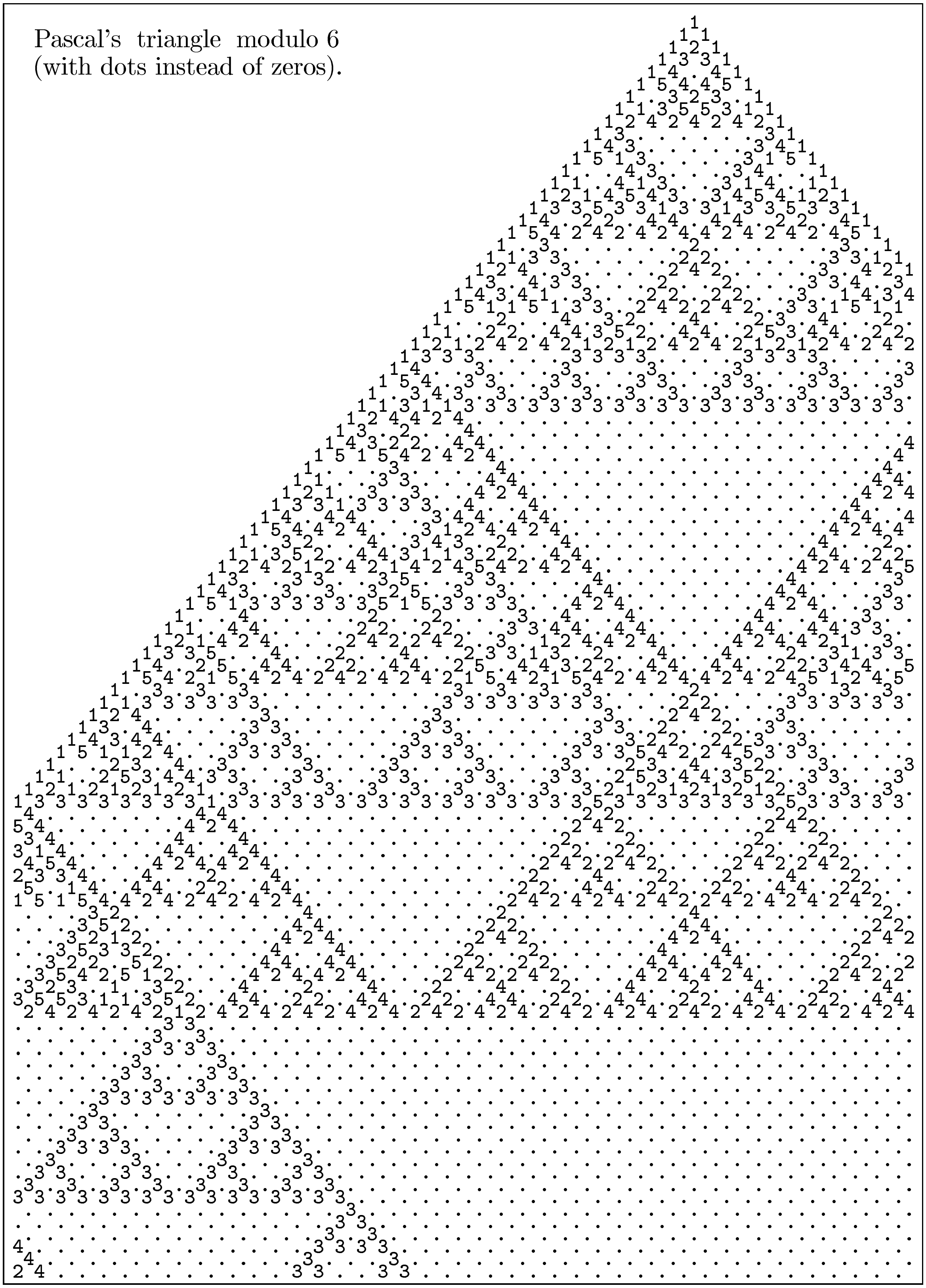
$$z = B_4(x, y) \iff z = 0 \land \mathrm{EqB}_4(x, y, 1, 3) \lor \bigvee_{i=1}^{3} \big(z = i \land \mathrm{EqB}_4(x, y, 1, i - 1)\big),$$

and the proof is finished. □

Now we shall deal with Pascal's triangle modulo 6; it is displayed in Figure 6. Notice that B_6 is connected with B_2 and B_3 by the formulas

$$B_2(x, y) = B_6(x, y) \,\mathrm{MOD}\, 2, \qquad B_3(x, y) = B_6(x, y) \,\mathrm{MOD}\, 3.$$

They obviously enable us to compute the values of B_2 and B_3. However, since 2, 3 are relatively prime we can also compute the values of B_6 from the values of B_2 and B_3. We can also use results about rather simple B_2, B_3 in the investigation of B_6.



Pascal's triangle modulo 6
(with dots instead of zeros).

Figure 6.

Theorem 5.2. *The operation B_6 (Pascal's triangle modulo 6) is first order definable in the structure $\langle \mathbb{N}; \mathrm{EqB}_6 \rangle$, where EqB_6 is the equivalence relation associated to B_6.*

Proof. For every $x \in \mathbb{N}$ we have $VB_6(x) = \{1\}$, $VB_6(x) = \{0,1,2,3,4,5\}$ or $VB_6(x) = \{0,1,3,4\}$. The second case is the most frequent. The first case takes place only for $x = 0$. The third case takes place for those $x > 0$ which have no 1's in their 3-adic expansion. Indeed, congruence considerations show that $VB_6(x) = \{0,1,3,4\}$ is equivalent with the conjunction of $VB_2(x) = \{0,1\}$ and $VB_3(x) = \{0,1\}$. For $x > 0$ the first condition is always satisfied, hence it need not be considered. For the second condition we have to use Lucas' theorem for $n = 3$, and that $B_3(2,a) \in \{0,1\}$ for every $a \in \{0,1,2\}$.

Using the above facts we can define:

$$\begin{aligned}
x = 0 &\iff \forall y, z\, \mathrm{EqB}_6(x,y,x,z),\\
\mathrm{EB}_6^1(x,y) &\iff \mathrm{EqB}_6(x,y,0,0),\\
\mathrm{EB}_6^{024}(x,y) &\iff \exists z\big(z \neq 0 \wedge \mathrm{EqB}_6(x,y,z,z)\big),\\
\mathrm{EB}_6^{0134}(x,y) &\iff \exists u,v\big(\,\mathrm{CVB}_6^4(u) \wedge \mathrm{EqB}_6(x,y,u,v)\big),\\
\mathrm{EB}_6^{0}(x,y) &\iff \exists u\big(\,\mathrm{CVB}_6^4(u) \wedge \mathrm{EqB}_6(x,y,u,u)\big),\\
\mathrm{EB}_6^{03}(x,y) &\iff \mathrm{EB}_6^{0134}(x,y) \wedge \big(\mathrm{EB}_6^{0}(x,y) \vee \neg\, \mathrm{EB}_6^{024}(x,y)\big) \wedge \neg\, \mathrm{EB}_6^{1}(x,y).
\end{aligned}$$

(The meaning of the defined predicates was explained above; now we have to check that these defining formulas correspond to the intended meaning. It is not difficult.)

The formulas $\mathrm{EB}_6^{024}(x,y)$ and $\mathrm{EB}_6^{03}(x,y)$ obviously correspond to $B_2(x,y) = 0$ and $B_3(x,y) = 0$. Therefore we can define the masking relations for the bases 2, 3 as follows:

$$\begin{aligned}
x \sqsubseteq_2 y &\iff \forall z\big(\,\mathrm{EB}_6^{024}(x,z) \implies \mathrm{EB}_6^{024}(y,z)\big),\\
x \sqsubseteq_3 y &\iff \forall z\big(\,\mathrm{EB}_6^{03}(x,z) \implies \mathrm{EB}_6^{03}(y,z)\big),
\end{aligned}$$

Now we could use the result of [Ko93] that for distinct primes p, q the operations $+$, $\times$ are definable in $\langle \mathbb{N}; \sqsubseteq_p, \sqsubseteq_q \rangle$. Then all arithmetical operations and relations, hence B_6, too are definable in such structures. However, we shall use a more elementary consideration.

Using $\sqsubseteq_2$ we can define Pow_2 and $\sqcup_2$. Using $\sqsubseteq_3$ we can define Pow_3 and $\sqcup_3$. Further we can define

$$\begin{aligned}
&x = 1 \iff \mathrm{Pow}_2(x) \wedge \mathrm{Pow}_3(x),\\
&x = 2 \iff 1 \sqsubseteq_3 x \wedge x \neq 1 \wedge \forall y\big(y \sqsubseteq_3 x \implies y \sqsubseteq_3 1 \vee y = x\big),\\
&\quad 3 = 1 \sqcup_2 2, \qquad 4 = 1 \sqcup_3 3, \qquad 5 = 2 \sqcup_3 3.
\end{aligned}$$

If we have the constant $0, 1, \ldots, 5$ we can define B_6 as follows:

$$z = B_6(x,y) \iff z = 0 \wedge \mathrm{EqB}_6(x,y,1,5) \vee \bigvee_{i=1}^{5} \big(z = i \wedge \mathrm{EqB}_6(x,y,1,i-1)\big). \quad \square$$

Corollary 5.3. *The operations* $+$, $\times$ *are first order definable in the structure* $\langle \mathbb{N}; \mathrm{EqB}_6 \rangle$, *where* EqB_6 *is the equivalence relation associated to* B_6.

In the corollary we cannot replace 6 by 4 because multiplication is not definable in $\langle \mathbb{N}; B_4 \rangle$; moreover, the elementary theory of $\langle \mathbb{N}; \mathrm{EqB}_4 \rangle$ is decidable.

In the theorems we cannot replace 6 (or 4) by any other positive integer $n \leq 6$. The function B_1 is a constant function (with the value 0), hence $\mathrm{EqB}_1 = \mathbb{N}^4$ is trivial, and we cannot define neither 0 nor B_1 in the structure $\langle \mathbb{N}; \mathrm{EqB}_4 \rangle$. The other cases are covered by the next theorem.

Theorem 5.4. *If* n *is prime then* B_n *is not definable in* $\langle \mathbb{N}; \mathrm{EqB}_n \rangle$.

Proof. Let n be prime. Every permutation of the set Pow_p induces an automorphism of the structure $\langle \mathbb{N}; \mathrm{EqB}_n \rangle$. However, only permutations which preserve 1 induce automorphisms of $\langle \mathbb{N}; B_n \rangle$. In particular, the mapping $f : \mathbb{N} \to \mathbb{N}$ defined by

$$f(xn^2 + yn + z) = xn^2 + zn + y \quad \text{for all} \quad x \in \mathbb{N},\ 0 \leq y < n,\ 0 \leq z < n$$

is an automorphism of $\langle \mathbb{N}; \mathrm{EqB}_n \rangle$, but it is not an automorphism of $\langle \mathbb{N}; B_n \rangle$. (The corresponding permutation interchanges 1 and n, and preserves the other powers of n.) Therefore B_n cannot be definable in $\langle \mathbb{N}; \mathrm{EqB}_n \rangle$. $\square$

Remarks. 1. The proof of Theorem 5.4 shows the unique reason of non-definability of B_n in $\langle \mathbb{N}; \mathrm{EqB}_n \rangle$. In the structure $\langle \mathbb{N}; \mathrm{EqB}_n, 1 \rangle$ the function B_n is definable.

2. The relation $\sqsubseteq_n$ is definable in $\langle \mathbb{N}; \mathrm{EqB}_n \rangle$. Conversely EqB_2 is definable in $\langle \mathbb{N}; \sqsubseteq_2 \rangle$. However, if n is an odd prime then EqB_n is not definable in $\langle \mathbb{N}; \sqsubseteq_n \rangle$.

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A MEASURE EXTENSION WITH RESPECT TO A MEASURE PRESERVING MAPPING

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Abstract. Abstract The present paper shows that a measure defined on a σ-algebra and invariant with respect to a measurable mapping may be extended onto a greater σ-algebra such that the mapping is strict measurable with respect to this greater σ-algebra.

1. Preliminaries.

It is known that the continuous image of a Borel set need not be Borel. This result belongs to M. Souslin, which in [5] corrected an error of H. Lebesgue, which in [4] stated that the continuous image of a measurable set is measurable. Let J be the set of all irrational numbers with the standard topology. Then J is homeomorphic to the product of countable many copies of the countable discrete topological space [3, p. 32]. Using results of Exercise 6 of [1, pp. 152 – 153] it may be proved the existence of a continuous mapping $T : J \to J$ such that the set $T(J)$ is not Borel.

In the whole paper we consider a quadruple $(X, \mathcal{A}, m, T)$, where X is a set, $\mathcal{A}$ is a σ-algebra on X, m is a σ-finite measure on $\mathcal{A}$ and $T : X \to X$ is an $\mathcal{A}$-measurable m-preserving mapping, i.e. $T^{-1}(A) \in \mathcal{A}$ and $m(T^{-1}(A)) = m(A)$ for any $A \in \mathcal{A}$, see [6, p. 19]. In this case the measure m is said to be T-invariant.

The mapping T preserving the measure m is almost surjective in the following sense. For $A \in \mathcal{A}$ with $A \cap T(X) = \emptyset$ we have $m(A) = m(T^{-1}(A)) = m(\emptyset) = 0$. Particularly, $m(X \setminus T(X)) = 0$ whenever $T(X) \in \mathcal{A}$. As it was said, in a general case $\mathcal{A}$-measurability of a mapping T does not imply $\mathcal{A}$-measurability of the set $T(X)$, i.e. $T(X) \in \mathcal{A}$. However, for a strict $\mathcal{A}$-measurable mapping $T : X \to X$ we can guarantee $T(X) \in \mathcal{A}$ and more generally $T^n(X) \in \mathcal{A}$ for all natural n. The definition of a strict measurable mapping follows.

Definition 1.1. Let X be a set, $\mathcal{A}$ be a σ-algebra on X and $T : X \to X$ be a mapping. Then T is said to be strict $\mathcal{A}$-measurable if $A \in \mathcal{A}$ if and only if $T^{-1}(A) \in \mathcal{A}$ for any $A \subset X$.

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The present paper constructs a natural extension $\tilde{m}$ of the measure m onto a greater σ-algebra $\tilde{\mathcal{A}}$, such that T is strict $\tilde{\mathcal{A}}$-measurable and $\tilde{m}$-preserving.

2. One step extension and extension by induction.

Let and $T : X \to X$ be a mapping and $\mathcal{A}$ be a σ-algebra on X. Put $\mathcal{A}^T = \{A : A \subset X \text{ and } T^{-1}(A) \in \mathcal{A}\}$.

Proposition 2.1.

(i) *$\mathcal{A}^T$ is a σ-algebra.*
(ii) *If T is $\mathcal{A}$-measurable, then $\mathcal{A} \subset \mathcal{A}^T$ and T is $\mathcal{A}^T$-measurable.*
(iii) *$T(X) \in \mathcal{A}^T$.*
(iv) *T is strict $\mathcal{A}$-measurable, if and only if $\mathcal{A} = \mathcal{A}^T$.*

Example 2.1. Let $T : X \to X$ be an $\mathcal{A}$-measurable mapping such that $T(A) \in \mathcal{A}$ for all $A \in \mathcal{A}$. Then $\mathcal{A}^T$ consists of the sets of the form $C = A \cup B$, where $A \in \mathcal{A}$ and $B \cap T(X) = \emptyset$. If moreover $T(X) = X$ then T is strict $\mathcal{A}$-measurable.

Proposition 2.2. *Let m be a measure on $\mathcal{A}$. Put $m^T(A) = m(T^{-1}(A))$ for $A \in \mathcal{A}^T$.*

(i) *m^T is a measure on $\mathcal{A}^T$.*
(ii) *If the measure space $(X, \mathcal{A}, m)$ is complete, then so is $(X, \mathcal{A}^T, m^T)$.*
(iii) *If T is $\mathcal{A}$-measurable and m-preserving, then m^T is a unique T-invariant extension of m onto $\mathcal{A}^T$; if T is not strict $\mathcal{A}$-measurable then m^T is a nontrivial extension of m.*

Proof. Part (i) is obvious. We shall prove (ii).
Let $B \subset A \in \mathcal{A}^T$ and $m^T(A) = 0$. Then $T^{-1}(A) \in \mathcal{A}$ and $m(T^{-1}(A))$. Since $(X, \mathcal{A}, m)$ is complete, we have $T^{-1}(B) \in \mathcal{A}$ and $B \in \mathcal{A}^T$. It proves (ii). Now, let T be $\mathcal{A}$-measurable and m-preserving. Take $A \in \mathcal{A}$. Then $m^T(A) = m(T^{-1}(A)) = m(A)$, because T is m-preserving. In means that the measure m^T is an extension of m. Take $A \in \mathcal{A}^T$, then $T^{-1}(A) \in \mathcal{A}$ and as we have shown $m^T(T^{-1}(A)) = m(T^{-1}(A))$. The definition of m^T yields $m^T(A) = m(T^{-1}(A))$. Therefore $m^T(T^{-1}(A)) = m^T(A)$ and T is m^T-preserving. Let μ be another T-invariant extension of m. For $A \in \mathcal{A}^T$ we have $\mu(A) = \mu(T^{-1}(A)) = m(T^{-1}(A)) = m^T(A)$, because $T^{-1}(A) \in \mathcal{A}$ and μ is an extension of m. If T is not strict $\mathcal{A}$-measurable, then $\mathcal{A} \subset \mathcal{A}^T$ but $\mathcal{A} \neq \mathcal{A}^T$ by Proposition 2.1.

Obviously, we can continue extension procedure by induction. Put $\mathcal{A}_0 = \mathcal{A}$, $m_0 = m$ and $\mathcal{A}_{m+1} = \mathcal{A}_n^T$, $m_{n+1} = m_n^T$. Then the measure m_{n+1} is an extension of m_n onto $\mathcal{A}_{n+1}$. The union $\bigcup_{n=1}^{\infty} \mathcal{A}_n$ is an algebra. For $A \in \mathcal{A}_n$ put $\mu(A) = m_n(A)$. Then we obtain a measure μ defined on the algebra $\bigcup_{n=1}^{\infty} \mathcal{A}_n$. Denote by $\mathcal{A}_\omega$ the σ-algebra generated by $\bigcup_{n=1}^{\infty} \mathcal{A}_n$. The measure μ may by uniquely extended onto $\mathcal{A}_\omega$ [2, p. 40]. Denote this extension by m_ω. It is clear that the mapping $T : X \to X$ is $\mathcal{A}_n$-measurable for all natural n and $T^{-1}(A) \in \mathcal{A}_n$ implies $A \in \mathcal{A}_{n+1}$ for any $A \subset X$. However, we are not able to prove that the mapping

$T: X \to X$ is strict $\mathcal{A}_\omega$-measurable. The problem of a strict measurability of the mapping T will be solved in the following section. We shall show that the σ-algebra $\mathcal{A}_\omega$ has some interesting properties.

Proposition 2.3. *The σ-algebra $\mathcal{A}_\omega$ contains $T^n(X)$ for all natural n and their intersection $\bigcap_{n=1}^{\infty} T^n(X)$ as well.*

Proof. Note that for any $A \subset X$ the iterated preimage $(T^{-1})^n(A)$ and the preimage under iterated mapping $(T^n)^{-1}(A)$ coincide. This set will be denoted by $T^{-n}(A)$. For any natural n we have $T^{-n}(T^n(X)) = X \in \mathcal{A} = \mathcal{A}_0$. By induction $T^{-(n-k)}(T^n(X)) \in \mathcal{A}_k$ for all natural k, $0 \le k \le n$. (The equality $T^{-(n-k)}(T^n(X)) = T^k(X)$ does not hold generally, but it is true for $k = n$.) It means $T^n(X) \in \mathcal{A}_n$ and $T^n(X) \in \mathcal{A}_\omega$. Therefore $\mathcal{A}_\omega$ contains the set $\bigcap_{n=1}^{\infty} T^n(X)$.

Corollary 2.1. *If T is strict $\mathcal{A}$-measurable then $T^n(X) \in \mathcal{A}$ for all natural n.*

Corollary 2.2. *Let $\bigcap_{n=1}^{\infty} T^n(X) \in \mathcal{A}$. Then $m(X \setminus \bigcap_{n=1}^{\infty} T^n(X)) = 0$.*

Proof. (Note, that we suppose nothing about the sets $T^n(X)$.) Since all degrees T^n preserve the measure m, they preserve also the measure m_ω. Therefore $m_\omega(X \setminus T^n(X)) = m_\omega(T^{-n}(X \setminus T^{-n}(X))) = m_\omega(\emptyset) = 0$ and $m(X \setminus \bigcap_{n=1}^{\infty} T^n(X)) = m_\omega(X \setminus \bigcap_{n=1}^{\infty} T^n(X)) = 0$.

3. Extension of $\mathcal{A}$ by transfinite induction.

Now, consider only a σ-algebra $\mathcal{A}$ on X and an $\mathcal{A}$-measurable mapping $T : X \to X$. Denote by $Ext(\mathcal{A})$ the class of all σ-algebras $\mathcal{B}$ on X such that:

(i) $\mathcal{A} \subset \mathcal{B}$.
(ii) T is strict $\mathcal{B}$-measurable.

We shall show, that the class $Ext(\mathcal{A})$ contains the smallest σ-algebra $\tilde{\mathcal{A}}$, which may be described by transfinite induction. Let ω_1 be the first uncountable ordinal. Put

$$\mathcal{A}_0 = \mathcal{A},$$
$$\mathcal{A}_\alpha = \mathcal{A}_{\alpha-1}^T \text{ , when } \alpha < \omega_1 \text{ is an unlimit ordinal}$$
$$\mathcal{A}_\alpha = \sigma\left(\bigcup_{\beta<\alpha} \mathcal{A}_\beta\right) \text{ , when } \alpha \text{ is a limit ordinal and}$$
$$\tilde{\mathcal{A}} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha \text{ .}$$

Proposition 3.1. *The σ-algebra $\tilde{\mathcal{A}}$ is the smallest element of the class $Ext(\mathcal{A})$. The mapping $T : X \to X$ is strict $\tilde{\mathcal{A}}$-measurable.*

Proof. Let $\mathcal{B}$ be the element of the class $Ext(\mathcal{A})$. All inclusions $\mathcal{A}_\alpha \subset \mathcal{B}$ for $\alpha < \omega_1$ follows immediately by transfinite induction from the strict $\mathcal{B}$-measurability of the mapping T and the inclusion $\mathcal{A} \subset \mathcal{B}$. We shall show that T is $\tilde{\mathcal{A}}$-measurable. It suffices to prove that T is $\mathcal{A}_\alpha$-measurable for all $\alpha < \omega_1$. The case $\alpha = 0$ is obvious. If $\alpha < \omega_1$ is an unlimit ordinal, then $\mathcal{A}_\alpha$-measurability follows from $\mathcal{A}_{\alpha-1}$-measurability and Proposition 2.1. If $\alpha < \omega_1$ is a limit ordinal then $\mathcal{A}_\alpha$ contains $T^{-1}(A)$ for all $A \in \bigcup\limits_{\beta<\alpha} \mathcal{A}_\beta$ by the inductive assumption. Since $\mathcal{A}_\alpha$ is a σ-algebra, it contains $T^{-1}(A)$ for all $A \in \sigma\left(\bigcup\limits_{\beta<\alpha} \mathcal{A}_\beta\right) = \mathcal{A}_\alpha$. It shows $\tilde{\mathcal{A}}$-measurability of T. Finally, if $T^{-1}(A) \in \tilde{\mathcal{A}}$ then $T^{-1}(A) \in \mathcal{A}_\alpha$ and $A \in \mathcal{A}_{\alpha+1}$. It completes the proof.

4. Extension of a measure onto $\tilde{\mathcal{A}}$.

Put $m_0 = m$, $m_\alpha = m^T_{\alpha-1}$ for any unlimit countable ordinal $\alpha > 0$. For a limit countable ordinal α the measure m_α will be defined in the following way. Note that $\bigcup\limits_{\beta<\alpha} \mathcal{A}_\beta$ is an algebra on X. Take $A \in \bigcup\limits_{\beta<\alpha} \mathcal{A}_\beta$. Then $A \in \mathcal{A}_\beta$ for some $\beta < \alpha$ and put $\mu_\alpha(A) = m_\beta(A)$. Then we obtain a σ-finite measure defined on the algebra $\bigcup\limits_{\beta<\alpha} \mathcal{A}_\beta$. The measure μ_α may be uniquely extended onto σ-algebra $\sigma\left(\bigcup\limits_{\beta<\alpha} \mathcal{A}_\beta\right)$, [2, p. 40]. This extension will be denoted by m_α. Finally put $\tilde{m}(A) = m_\alpha(A)$ whenever $A \in \mathcal{A}_\alpha$ for some $\alpha < \omega_1$.

Theorem 4.1. *The measure $\tilde{m}$ is a unique T-invariant extension of the measure m onto $\tilde{\mathcal{A}}$.*

Proof. Since $\tilde{\mathcal{A}} = \bigcup\limits_{\alpha<\omega_1} \mathcal{A}_\alpha$, it suffices to prove that m_α is a unique T-invariant extension of m onto $\mathcal{A}_\alpha$.
The case $\alpha = 0$ is trivial.
Let $\alpha > 0$ be an unlimit countable ordinal. Suppose that $m_{\alpha-1}$ is a unique T-invariant extension of m onto $\mathcal{A}_{\alpha-1}$. By Proposition 2.2. m_α is a unique T-invariant extension of $m_{\alpha-1}$ onto $\mathcal{A}_\alpha$. Therefore m_α is a unique T-invariant extension of m onto $\mathcal{A}_\alpha$. Now, let α be a limit countable ordinal. Suppose that for all $\beta < \alpha$ the measure m_β is a unique T-invariant extension of m onto $\mathcal{A}_\beta$. Then μ_α is a unique T-invariant extension of m onto the algebra $\bigcup\limits_{\beta<\alpha} \mathcal{A}_\beta$. The measure m_α is a unique extension of μ_α onto $\mathcal{A}_\alpha$ in the realm of measures. It suffices to prove that m_α is T-invariant. To see this for $A \in \mathcal{A}_\alpha$ put $v_\alpha(A) = m_\alpha(T^{-1}(A))$. Then v_α is a measure, which coincide with μ_α on the algebra $\bigcup\limits_{\beta<\alpha} \mathcal{A}$. It means that v_α coincide with μ_α on $\mathcal{A}_\alpha$ and the measure m_α is T-invariant.

5. Complete extension.

In this section we describe an extension $\hat{m}$ of the measure m onto a σ-algebra $\hat{\mathcal{A}}$, such that the measure space $(X, \hat{\mathcal{A}}, \hat{m})$ will be also complete.

Let us recall the notion of the completion of a measure space. Let $(X, \mathcal{A}, m)$ be a measure space. Denote be $\overline{\mathcal{A}}$ the system of all sets A of the form $A = A_1 \cup A_0$, where $A_1 \in \mathcal{A}$ and $A_0 \subset B_0$ for some $B_0 \in \mathcal{A}$ with $m(B_0) = 0$, and for such a set A define $\overline{m}(A) = m(A_1)$. Then $(X, \overline{A}, \overline{m})$ is a complete measure space and $\overline{m}$ is an extension of m. It is easy to see, that any $\mathcal{A}$-measurable m-preserving mapping $T : X \to X$ is also $\overline{\mathcal{A}}$-measurable and $\overline{m}$-preserving. It means that T is also $\overline{\tilde{\mathcal{A}}}$-measurable and $\overline{\tilde{m}}$-preserving ($\tilde{\mathcal{A}}$ and $\tilde{m}$ has been constructed in the preceding sections.) Unfortunately, there are no arguments that T is strict $\overline{\tilde{\mathcal{A}}}$-measurable.

So we modify the construction of $\tilde{\mathcal{A}}$ and $\tilde{m}$ in the following way.
Put $\mathcal{A}_0 = \overline{\mathcal{A}}$ and $= \overline{m}$. When $\alpha > 0$ is unlimit countable ordinal put $\mathcal{A}_\alpha = \mathcal{A}^T_{\alpha-1}$ and $m_\alpha = m^T_{\alpha-1}$. (If $m_{\alpha-1}$ is complete then m_α is complete by Proposition 2.2.) For a limit countable ordinal α the σ-algebra $\mathcal{A}_\alpha$ and the measure m_α defined in preceding sections must be replace by their completions $\overline{\mathcal{A}}_\alpha$ and $\overline{m}_\alpha$. Put $\hat{\mathcal{A}} = \bigcup_{\alpha<\omega_1} \mathcal{A}_\alpha$ and $\hat{m}(A) = m_\alpha(A)$ for $A \in \mathcal{A}_\alpha$.

Theorem 5.1. *The measure space $(X, \hat{\mathcal{A}}, \hat{m})$ is complete and the mapping $T : X \to X$ is strict $\hat{\mathcal{A}}$-measurable and $\hat{m}$-preserving.*

The proof of the last theorem is simiral to the proof of Theorem 4.1.

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SURELY COMPLETE MATRICES

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Abstract. The "sure completness" of the 3x3 matrices over the set $\{0,1,2,*\}$ is defined and it is found an example of such matrix with only 4 numbers. (No such matrix with less than 4 numbers can be surely complete.) Using surely complete matrices, a lot of functionally complete algebras can be generated.

Studying the functional completness and other properties of the algebras of the type (2) on the set $\{0,1,2,\}$, we will use the matrix denotation by [3]. Similarly, the unary functions we will write in the vector form.

Definition. Let G be a 3x3 matrix over the set $\{0,1,2,*\}$ and let H be a 3x3 matrix over the set $\{0,1,2\}$. The matrix H will be called a **specification** of the matrix G iff the following implication is satisfied:

$$G(i,j) \in \{0,1,2\} \Rightarrow H(i,j) = G(i,j)$$

Example 1. The matrix

$$H = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 2 \\ 1 & 1 & 2 \end{pmatrix}$$

is a specification of the matrix

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & * \end{pmatrix}.$$

Definition. Let G be a 3x3 matrix over the set $\{0,1,2,*\}$. The matrix G will be called **surely complete** iff the following condition is satisfied: For every specification H of the matrix G, the algebra $(\{0,1,2\},H)$ of the type (2) is funcionally complete.

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Example 2. Put

$$G = \begin{pmatrix} 1 & 0 & 2 \\ * & 2 & * \\ * & * & 0 \end{pmatrix}, \; H = \begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{pmatrix}.$$

The matrix G is not surely complete. In fact, its specification H can be described by the formula

$$H(x, y) = 2x + 2y + 1 \; modulo \; 3.$$

and every polynomial of the algebra $(\{0, 1, 2, \}, H)$ is a polynomial of the algebra $(\{0, 1, 2, \}, +)$, too. On the other hand, the last algebra is not functionally complete.

In [3], the following theorem is proved.

Theorem 1. *Assume that* $A = \{0, 1, 2\}$ *and that the algebra* (A, F) *has the following properties:*

1) *among unary polynomial functions there exist at least one transposition, at least one 3-cycle and at least one function with exactly 2 values,*
2) *among binary polynomial functions there exists a function* G *and there exist* $a, b, c, d \in A$ *such that*

$$\{G(a, c), \; G(a, d), \; G(b, c), \; G(b, d)\} = A.$$

Then (A, F) *is functionally complete.*

Example 3. Put

$$G = \begin{pmatrix} 1 & 0 & 2 \\ * & 2 & 0 \\ * & * & 1 \end{pmatrix}.$$

This matrix G is surely complete. In fact, the polynomial fuction $G(0, x) = (1, 0, 2)$ is a transposition, $G(x, 2) = (2, 0, 1)$ is a 3-cycle, $G(x, x) = (1, 2, 1)$ has exactly 2 values and

$$\{G(0, 0), \; G(0, 1), \; G(1, 0), \; G(1, 1)\} = \{0, 1, 2\}.$$

Lemma 1. *The matrix*

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & 0 \end{pmatrix}$$

is surely complete.

Proof. Following unary functions are polynomial:

$$p_1(x) = G(x, x) = (1, 0, 0) \quad (it \; has \; 2 \; values),$$

$$p_2(x) = G(x, 2) = (0, 2, 0),$$

$$p_3(x) = p_1(p_1(x)) = (0, 1, 1),$$

$$p_4(x) = G(p_3(x), x) = (1, 0, 2) \quad (a \; transposition),$$

$$p_5(x) = p_2(p_3(x)) = (0, 2, 2),$$

$$p_6(x) = G(x, p_5(x)) = (1, 2, 0) \quad (a \; 3-cycle).$$

Moreover, $\{G(0, 0), \; G(0, 2), \; G(1, 0), \; G(1, 2)\} = \{0, 1, 2\}$. Now apply Theorem 1.

Lemma 2. *The matrix*

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & 1 \end{pmatrix}$$

is surely complete.

Proof. Following unary functions are polynomial:

$$p_1(x) = G(x,x) = (1,0,1) \quad (\textit{it has 2 values}),$$

$$p_2(x) = G(x,2) = (0,2,1) \quad (\textit{a transposition}),$$

$$p_3(x) = p_1(p_1(x)) = (0,1,0),$$

$$p_4(x) = G(p_3(x),x) = (1,0,0),$$

$$p_5(x) = p_1(p_4(x)) = (0,1,1),$$

$$p_6(x) = G(p_5(x),x) = (1,0,2),$$

$$p_7(x) = p_2(p_6(x)) = (2,0,1) \quad (\textit{a } 3-\textit{cycle}).$$

Moreover, $\{G(0,0),\ G(0,2),\ G(1,0),\ G(1,2)\} = \{0,1,2\}$. Now apply Theorem 1.

Lemma 3. *The matrix*

$$G = \begin{pmatrix} 1 & 0 & 0 \\ * & 0 & 2 \\ * & * & 2 \end{pmatrix}$$

is surely complete.

Proof. Following unary functions are polynomial:

$$p_1(x) = G(x,2) = (0,2,2) \quad (\textit{it has 2 values}),$$

$$p_2(x) = G(x,x) = (1,0,2) \quad (\textit{a transposition}),$$

$$p_3(x) = p_1(p_2(x)) = (2,0,2),$$

$$p_4(x) = G(0,p_3(x)) = (0,1,0),$$

$$p_5(x) = G(p_4(x),p_1(x)) = (1,2,0) \quad (\textit{a } 3-\textit{cycle}).$$

Moreover, $\{G(0,0),\ G(0,2),\ G(1,0),\ G(1,2)\} = \{0,1,2\}$. Now apply Theorem 1.

Lemma 4. *The matrix*

$$G = \begin{pmatrix} 1 & 1 & 0 \\ * & 0 & 2 \\ * & * & 2 \end{pmatrix}$$

is surely complete.

Proof. Following unary functions are polynomial:

$$p_1(x) = G(0, x) = (1, 1, 0) \quad (it\ has\ 2\ values),$$

$$p_2(x) = G(x, x) = (1, 0, 2) \quad (a\ transposition),$$

$$p_3(x) = G(x, 2) = (0, 2, 2),$$

$$p_4(x) = p_3(p_1(x)) = (2, 2, 0),$$

$$p_5(x) = p_1(p_3(x)) = (1, 0, 0),$$

$$p_6(x) = G(p_5(x), p_4(x)) = (2, 0, 1) \quad (a\ 3-cycle).$$

Moreover, $\{G(0, 0),\ G(0, 2),\ G(1, 0),\ G(1, 2)\} = \{0, 1, 2\}$. Now apply Theorem 1.

Lemma 5. *The matrix*

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & 2 \end{pmatrix}$$

is surely complete.

Proof. For the value $G(0, 1)$ we have only 3 possibilities. In the case $G(0, 1) = 0$, it suffices to apply Lemma 3. In the case $G(0, 1) = 1$, it suffices to apply Lemma 4. In the case $G(0, 1) = 2$, the following unary functions are polynomial:

$$G(x, 2) = (0, 2, 2) \quad (it\ has\ 2\ values),$$

$$G(x, x) = (1, 0, 2) \quad (a\ transposition),$$

$$G(0, x) = (1, 2, 0) \quad (a\ 3-cycle).$$

Moreover, $\{G(0, 0),\ G(0, 2),\ G(1, 0),\ G(1, 2)\} = \{0, 1, 2\}$. Now apply Theorem 1.

Theorem 2. *The matrix*

$$G = \begin{pmatrix} 1 & * & 0 \\ * & 0 & 2 \\ * & * & * \end{pmatrix}$$

is surely complete.

Proof. Apply Lemma 1, Lemma 2 and Lemma 5.

Theorem 3. *The algebra* $(\{0, 1, 2\}, F)$ *is functionally complete iff there exists a binary polynomial function* G *such that*

$$G(0, 0) = 1,\ G(0, 2) = 0,\ G(1, 1) = 0,\ G(1, 2) = 2.$$

Lemma 6. *Let* G *be a surely complete 3x3 matrix. Then at least one row of the matrix* G *contains at least 2 different numbers.*

Proof. Assume that no row of the matrix G contains different numbers. Then the matrix G has a specification H of the form

$$H = \begin{pmatrix} a & a & a \\ b & b & b \\ c & c & c \end{pmatrix}.$$

Trivially, the algebra $(\{0,1,2\}, H)$ is not functionally complete.

Lemma 7. *Let G be a surely complete 3x3 matrix. Then at least one column of the matrix G contains at least 2 different numbers.*

Lemma 8. *Let G be a 3x3 matrix over the set $\{0,1,2,*\}$. Assume that at least one row and at least one column of the matrix G are "number-free". Then the matrix G is not surely complete.*

The idea of the proof. For example, the matrix

$$G = \begin{pmatrix} * & a & b \\ * & c & d \\ * & * & * \end{pmatrix}$$

has a specification

$$H = \begin{pmatrix} b & a & b \\ d & c & d \\ b & a & b \end{pmatrix}.$$

The algebra $(\{0,1,2\}, H)$ is not functionally complete. In fact, it has a non-trivial congruence $cg(0,2)$.

Theorem 4. *Let G be a surely complete 3x3 matrix over the set $\{0,1,2\}$. Then G contains at least 4 numbers.*

Proof. Apply Lemma 6, Lemma 7 and Lemma 8.

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