Acta Univ. M. Belii
Math. no.4 (1996), pp. 3-15

SPECIAL ALMOST R-PARACONTACT CONNECTIONS

Andrew BUCKI

Abstract. For an almost r-paracontact manifold M_n with a structure Σ and any linear connection Γ on M_n, all almost r-paracontact connections (making all structure tensors parallel) have been found. Also, D-connections for which some distributions on M_n are parallel have been considered. Finally, pairs of connections compatible with a structure Σ have been discussed.

1. Almost r-paracontact manifolds. An almost r-paracontact structure is the generalization of an almost product structure on a differentiable manifold. The study of almost r-paracontact structures (utilizing, in part, certain distributions of tangent bundles of manifolds generated by these structures) and almost r-paracontact connections on manifolds provides a foundation for the investigation of geometric and topological properties of these manifolds.

In this section we recall the definition of an almost r-paracontact manifold [1] and present some of their properties.

Definition 1.1. Let M_n be an n-dimensional differentiable manifold. If on M_n there exist: a tensor field ϕ of type (1,1), r vector fields $\xi_1, \xi_2, \dots, \xi_r$ $(r < n)$, r 1-forms $\eta^1, \eta^2, \dots, \eta^r$ such that

$$\eta^\alpha(\xi_\beta) = \delta^\alpha_\beta, \qquad \alpha, \beta \in (r) = 1, 2, ..., r \tag{1.1}$$

$$\phi^2 = Id - \eta^\alpha \otimes \xi_\alpha, \quad \text{where} \quad a^\alpha b_\alpha \overset{\text{def}}{=} \sum_\alpha a^\alpha b_\alpha \tag{1.2}$$

$$\eta^\alpha \circ \phi = 0, \qquad \alpha \in (r) \tag{1.3}$$

then $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ is said to be an *almost r-paracontact structure* on M_n, and M_n is an *almost r-paracontact manifold*.

From (1.1), (1.2), and (1.3) we also have

$$\phi(\xi_\alpha) = 0, \qquad \alpha \in (r). \tag{1.4}$$

1991 *Mathematics Subject Classification.* 53C05, 53C15.

Key words and phrases. Almost r-paracontact structures, almost r-paracontact connections, D-connections, mixed covariant derivatives, pairs of connections compatible with a structure.

There exists a positive Riemannian metric g on M_n [1] such that

$$\eta^\alpha(X) = g(X, \xi_\alpha), \qquad \alpha \in (r) \tag{1.5}$$

$$g(\phi X, \phi Y) = g(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y). \tag{1.6}$$

Then, $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$ is called a *metric almost r-paracontact structure* on M_n, and g is said to be compatible Riemannian metric.

From (1.1) through (1.6) we get

$$g(\phi X, Y) = g(X, \phi Y). \tag{1.7}$$

Remark 1.1. On an almost r-paracontact manifold M_n with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ the tensor ϕ has constant eigenvalues $1, -1$, and 0. Let p, q, and r be their multiplicities, respectively, with $p + q + r = n = \dim M_n$. M_n is said to be of type (p, q).

Proposition 1.1 [1]. *An almost r-paracontact manifold M_n admits the following complementary distributions*

$$D^+ = \{X;\ \ \phi X = X\} \tag{1.8}$$

$$D^- = \{X;\ \ \phi X = -X\} \tag{1.9}$$

$$D^0 = \{X;\ \ \phi X = 0\} \tag{1.8}$$

with $\dim D^+ = p$, $\dim D^- = q$, $\dim D^0 = r$, *and* $p + q + r = n$.

Theorem 1.1. *A necessary and sufficient condition for M_n to admit an almost r-paracontact structure is that there exist three complementary distributions D_1, D_2, and D_3 of dimensions p, q, and r, respectively, with $p + q + r = n$.*

Proof. The necessary condition follows immediately from Proposition 1.1. Now, let D_1, D_2, and D_3 be three complementary distributions of dimensions p, q, and r, respectively, with $p + q + r = n$. For any $p \in M_n$ we have $T_pM_n = D_{1p} \oplus D_{2p} \oplus D_{3p}$. Let $\{e_1, \ldots, e_p, e_{p+1}, \ldots, e_{p+q}, e_{p+q+1} = \xi_1, \ldots, e_n = \xi_r\}$ be a basis for T_pM_n, and let $\{e^1, \ldots, e^p, e^{p+1}, \ldots, e^{p+q}, e^{p+q+1} = \eta^1, \ldots, e^n = \eta^r\}$ be the dual basis for the cotangent space $T^*_pM_n$, i.e., $e^i(e_j) = \delta^i_j,\ \ i, j \in (n)$. Hence, we get $e^a \otimes e_a + e^{p+\lambda} \otimes e_{p+\lambda} + \eta^\alpha \otimes \xi_\alpha = Id,\ \ a \in (p), \lambda \in (q), \alpha \in (r)$. Let $\phi = e^a \otimes e_a + \varepsilon\, e^{p+\lambda} \otimes e_{p+\lambda},\ \ \varepsilon = \pm 1$. Then, the structure $(\phi, \xi_\alpha, \eta^\alpha)$ is an almost r-paracontact structure on M_n. □

Definition 1.2. If M_n is an almost r-paracontact manifold with the structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$, then Σ is said to be normal if an almost product structure F defined on $M_n \times R^r$ by $F(X, f^\alpha \frac{d}{dt^\alpha}) = (\phi X + f^\alpha \xi_\alpha, \eta^\alpha(X)\frac{d}{dt^\alpha})$ is integrable, i.e., its Nijenhuis tensor field N_F vanishes.

Theorem 1.2 [1]. *An almost r-paracontact structure Σ on M_n is normal if and only if $N(X, Y) = N_\phi(X, Y) - 2d\eta^a(X, Y)\xi_\alpha = 0$, where N_ϕ is the Nijenhuis tensor for ϕ.*

2. Projection operators on almost r-paracontact manifolds. In this section various projection operators on an almost r-paracontact manifold are defined.

Definition 2.1. A multilinear operator Φ on an appropriate space is said to be a *projection operator* if $\Phi^2 = \Phi$.

Definition 2.2. A set of operators $\{\Phi_i\}$ is said to be the *set of complementary projection operators* if $\sum_i \Phi_i = Id, \quad \Phi_i^2 = \Phi_i, \quad \Phi_i \Phi_j = 0, \quad i \neq j$.

Proposition 2.1 [3]. *If Φ is a projection operator and $\Psi = Id - \Phi$, then Φ and Ψ are complementary projection operators and all solutions of the equation $\Phi x = y$ are of the form $x = y + \Psi w$, where w is an arbitrary element.*

Remark 2.1. If operators Φ and Ψ are tensor fields of type (2,2) and S, ϕ, X are tensor fields of type (1,2), (1,1), and (0,1) respectively, then the operations $\Phi\Psi, \Phi S, \Phi\phi, \Phi X$ are expressed locally as follows: $\Phi^{ij}_{kl}\Psi^{ml}_{in}, \Phi^{ij}_{kl}S^{l}_{mi}, \Phi^{ij}_{kl}\phi^{l}_{i}, \Phi^{ij}_{kl}X^{k}$.

Let $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ be an almost r-paracontact structure on M_n. We define the following operators on M_n.

$$\phi_1 = \tfrac{1}{2}(Id + \phi - \eta^\alpha \otimes \xi_\alpha) = \tfrac{1}{2}(\phi^2 + \phi) \tag{2.1}$$

$$\phi_2 = \tfrac{1}{2}(Id - \phi - \eta^\alpha \otimes \xi_\alpha) = \tfrac{1}{2}(\phi^2 - \phi) \tag{2.2}$$

$$\phi_3 = \eta^\alpha \otimes \xi_\alpha = Id - \phi^2 \tag{2.3}$$

$$\phi_4 = \phi^2 = Id - \phi_3 \tag{2.4}$$

with the properties

$$\begin{array}{c} \phi = \phi_1 - \phi_2, \quad \phi\phi_1 = \phi_1\phi = \phi_1, \quad \phi\phi_2 = \phi_2\phi = -\phi_2, \\ \phi\phi_3 = \phi_3\phi = 0, \quad \phi\phi_4 = \phi_4\phi = \phi \end{array}. \tag{2.5}$$

Proposition 2.2. *The operators ϕ_1, ϕ_2, ϕ_3 are complementary projection operators on an almost r-paracontact manifold M_n.*

Proposition 2.3. *The operators ϕ_3 and ϕ_4 are complementary projection operators on an almost r-paracontact manifold M_n.*

The distributions (1.8), (1.9), and (1.10) can be expressed as follows

$$D^+ = \{X; \ \ \phi_1 X = X\} \tag{2.6}$$

$$D^- = \{X; \ \ \phi_2 X = X\} \tag{2.7}$$

$$D^0 = \{X; \ \ \phi_3 X = X\}. \tag{2.8}$$

Proposition 2.4. *The distributions D^+, D^-, and D^0 are generated by the projection operators ϕ_1, ϕ_2, and ϕ_3, respectively.*

Let A, B, and C be tensor fields of type (2,2) defined as

$$A = \tfrac{1}{2}(Id \otimes Id - \phi \otimes \phi) \tag{2.9}$$

$$B = \tfrac{1}{2}(Id \otimes Id + \phi \otimes \phi) \tag{2.10}$$

$$C = \tfrac{1}{2}(\phi_3 \otimes Id + Id \otimes \phi_3 - \phi_3 \otimes \phi_3). \tag{2.11}$$

The operators A, B, and C possess the following properties

$$\begin{gathered} A + B = Id \otimes Id, \quad AA = A - \frac{1}{2}C, \quad BB = B - \frac{1}{2}C, \\ AB = BA = AC = CA = BC = CB = CC = \frac{1}{2}C \end{gathered}. \tag{2.12}$$

Define two operators

$$F = A + C \tag{2.13}$$

$$H = B - C. \tag{2.14}$$

Proposition 2.5. *The operators F and H are complementary projection operators on an almost r-paracontact manifold M_n.*

Remark 2.2. The operators F and H can be expressed in the following form

$$F = Id \otimes Id - \phi_1 \otimes \phi_1 - \phi_2 \otimes \phi_2 \tag{2.15}$$

$$H = \phi_1 \otimes \phi_1 + \phi_2 \otimes \phi_2. \tag{2.16}$$

Define two more operators

$$P = F - \phi_3 \otimes \phi_3 \tag{2.17}$$

$$Q = H + \phi_3 \otimes \phi_3 \tag{2.18}$$

or

$$P = Id \otimes Id - \phi_1 \otimes \phi_1 - \phi_2 \otimes \phi_2 - \phi_3 \otimes \phi_3 \tag{2.19}$$

$$Q = \phi_1 \otimes \phi_1 + \phi_2 \otimes \phi_2 + \phi_3 \otimes \phi_3. \tag{2.20}$$

Proposition 2.6. *The operators P and Q are complementary projection operators on an almost r-paracontact manifold M_n.*

3. Almost r-paracontact connections. In this section the definition of an almost r-paracontact connection on an almost r-paracontact manifold has been given and all such connections are found.

Definition 3.1 [2]. For an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ a linear connection Γ, given by its covariant derivative ∇, is said to be an *almost r-paracontact* if

$$\nabla_X \phi = 0 \tag{3.1}$$

$$\nabla_X \eta^\alpha = 0, \qquad \alpha \in (r) \tag{3.2}$$

for any vector field X.

From (3.1) and (3.2) it follows

$$\nabla_X \xi_\alpha = 0, \qquad \alpha \in (r). \tag{3.3}$$

If M_n is a metric almost r-paracontact manifold and an almost r-paracontact connection Γ satisfies

$$\nabla_X g = 0 \tag{3.4}$$

then it is called a *metric almost r-paracontact* on M_n.

Now, assume that Γ, given by its covariant derivative ∇, is a linear connection on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$. We are going to find an almost r-paracontact connection $\overline{\Gamma}$ given by $\overline{\nabla}$ in the form

$$\overline{\nabla}_X = \nabla_X + S_X \tag{3.5}$$

where S is a tensor field of type (1.2) with $S_X(Y) = S(X,Y)$.

For tensors fields f, Y, and ω of type (1,1), (0,1), and (1,0) respectively, we have

$$\overline{\nabla}_X f = \nabla_X f + S_X \circ f - f \circ S_X \tag{3.6}$$

$$\overline{\nabla}_X Y = \nabla_X Y + S_X(Y) \tag{3.7}$$

$$\overline{\nabla}_X \omega = \nabla_X \omega - \omega \circ S_X. \tag{3.8}$$

For the structure tensors ϕ, ξ_α, and η^α, we have

$$\overline{\nabla}_X \phi = \nabla_X \phi + S_X \circ \phi - \phi \circ S_X \tag{3.9}$$

$$\overline{\nabla}_X \xi_\alpha = \nabla_X \xi_\alpha + S_X(\xi_\alpha), \qquad \alpha \in (r) \tag{3.10}$$

$$\overline{\nabla}_X \eta^\alpha = \nabla_X \eta^\alpha - \eta^\alpha \circ S_X, \qquad \alpha \in (r). \tag{3.11}$$

Since $\overline{\Gamma}$ is an almost r-paracontact connection, so S_X satisfies

$$\nabla_X \phi = \phi \circ S_X - S_X \circ \phi \tag{3.12}$$

$$\nabla_X \xi_\alpha = -S_X(\xi_\alpha), \qquad \alpha \in (r) \tag{3.13}$$

$$\nabla_X \eta^\alpha = \eta^\alpha \circ S_X, \qquad \alpha \in (r). \tag{3.14}$$

Applying ϕ on the right to (3.12) and making use of (3.13) and (1.2) we obtain

$$S_X - \phi S_X \phi = \phi \nabla_X \phi + \nabla_X \eta^\alpha \otimes \xi_\alpha \tag{3.15}$$

or using the projection operator A from (2.9)

$$AS_X = \tfrac{1}{2}(\phi \nabla_X \phi + \nabla_X \eta^\alpha \otimes \xi_\alpha) \tag{3.16}$$

or making use of (2.5), (2.1)–(2.4), and Proposition 2.2

$$AS_X = \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \tfrac{1}{2}\phi_3 \nabla_X \phi_3 + \tfrac{1}{2}\eta^\alpha \otimes \nabla_X \xi_\alpha - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha. \tag{3.17}$$

Acting with the operator C from (2.11) on (3.16) and using (2.12) we get

$$\begin{aligned} CS_X = \tfrac{1}{2}(&\phi \nabla_X \phi \phi_3 + \nabla_X \eta^\alpha \circ \phi_3 \otimes \xi_\alpha + \phi_3 \phi \nabla_X \phi + \nabla_X \eta^\alpha \otimes \phi_3(\xi_\alpha) \\ &-\phi_3 \phi \nabla_X \phi \phi_3 - \nabla_X \eta^\alpha \circ \phi_3 \otimes \phi_3(\xi_\alpha)). \end{aligned}$$

Making use of (2.5), (2.1)–(2.4), and Proposition 2.2 we get

$$CS_X = \tfrac{1}{2}(\phi_3 \nabla_X \phi_3 - \eta^\alpha \otimes \nabla_X \xi_\alpha) \tag{3.18}$$

Adding (3.17) and (3.18) up, and using (2.13) we get

$$FS_X = \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha. \tag{3.19}$$

Making use of Propositions 2.5 and 2.1 we obtain from (3.19)

$$S_X = \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha + HW_X \tag{3.20}$$

where W is an arbitrary tensor field of type (1,2) with $W_X(Y) = W(X, Y)$, or

$$S_X = \tfrac{1}{2}\phi \nabla_X \phi + \nabla_X \eta^\alpha \otimes \xi_\alpha - \tfrac{1}{2}\eta^\alpha \otimes \nabla_X \xi_\alpha + \tfrac{1}{2}\eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha + HW_X. \tag{3.21}$$

Theorem 3.1 [2]. *The general family of the almost r-paracontact connections $\overline{\Gamma}$ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ is given by*

$$\overline{\nabla}_X = \nabla_X + \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha + HW_X \tag{3.22}$$

where H is defined by (2.16), W is an arbitrary tensor field of type (1,2) with $W_X(Y) = W(X, Y)$, and ∇ is the covariant derivative of an arbitrary linear connection Γ on M_n.

Corollary 3.1. *If the initial connection Γ is an a.r-p.-connection on M_n, then the general family of a.r-p.-connections $\overline{\Gamma}$ on M_n is given by*

$$\overline{\nabla}_X = \nabla_X + HW_X \tag{3.23}$$

where W is an arbitrary tensor field.

Now, suppose that M_n is an almost r-paracontact Riemannian manifold with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$. Let Γ, given by its covariant derivative ∇, be the Riemannian connection generated by the Riemannian metric g on M_n. Of all almost r-paracontact connections $\overline{\Gamma}$ given by (3.22) we will find metric connections, i.e., those satisfying $\overline{\nabla} g = 0$. For the Riemannian metric g and $\overline{\Gamma}$ given by (3.5) we have $(\overline{\nabla}_Z g)(X,Y) = -g(S_Z X, Y) - g(X, S_Z Y)$. $\overline{\Gamma}$ is a metric almost r-paracontact connection if and only if

$$g(S_Z X, Y) + g(X, S_Z Y) = 0. \tag{3.24}$$

From (1.5) and (1.1) we have

$$\eta^\alpha(\nabla_Z \xi_\beta) + \eta^\beta(\nabla_Z \xi_\alpha) = 0 \tag{3.25}$$

$$(\nabla_Z \eta^\alpha)(X) = g(X, \nabla_Z \xi_\alpha). \tag{3.26}$$

From (1.6), using (1.5) and (1.2), we get

$$\begin{aligned} &g(\nabla_Z(\phi X), \phi Y) + g(\phi X, \nabla_Z(\phi Y)) - g(\phi^2(\nabla_Z X), Y) \\ &-g(X, \phi^2(\nabla_Z Y)) + \sum_i (\nabla_Z \eta^\alpha)(X)\eta^\alpha(Y) + \sum_i (\nabla_Z \eta^\alpha)(Y)\eta^\alpha(X) = 0. \end{aligned} \tag{3.27}$$

From (3.24), using (3.21), (1.7), (3.26), and then (3.27) and (3.25), we get

$$\begin{aligned} &\tfrac{1}{2} g(\phi\nabla_Z \phi X, Y) - \tfrac{1}{2}\eta^\alpha(X) g(\nabla_Z \xi_\alpha, Y) + \nabla_Z \eta^\alpha(X) g(\xi_\alpha, Y) + \tfrac{1}{2} g(X, \phi\nabla_Z \phi Y) \\ &-\tfrac{1}{2}\eta^\alpha(Y) g(X, \nabla_Z \xi_\alpha) + \nabla_Z \eta^\alpha(Y) g(X, \xi_\alpha) + \eta^\alpha(\nabla_Z \xi_\beta)\eta^\beta(X) g(\xi_\alpha, Y) \\ &+\eta^\alpha(\nabla_Z \xi_\beta)\eta^\beta(Y) g(X, \xi_\alpha) + g(H_X W_Z, Y) + g(X, H_Y W_Z) = \\ &= \tfrac{1}{2} g(\nabla_Z(\phi X), \phi Y) - \tfrac{1}{2} g(\phi^2(\nabla_Z X), Y) + \tfrac{1}{2} g(\phi X, \nabla_Z(\phi Y)) - \tfrac{1}{2} g(X, \phi^2(\nabla_Z Y)) \\ &+\tfrac{1}{2}\sum_\alpha (\nabla_Z \eta^\alpha)(X)\eta^\alpha(Y) + \tfrac{1}{2}\sum_\alpha (\nabla_Z \eta^\alpha)(Y)\eta^\alpha(X) \\ &+\tfrac{1}{2}\sum_{\alpha,\beta} \eta^\alpha(X)\eta^\beta(Y)[\eta^\alpha(\nabla_Z \xi_\beta) + \eta^\beta(\nabla_Z \xi_\alpha)] + g(H_X W_Z, Y) + g(X, H_Y W_Z) = \\ &= g(H_X W_Z, Y) + g(X, H_Y W_Z) = 0. \end{aligned}$$

Hence,

Theorem 3.2. *Let M_n be an almost r-paracontact Riemannian manifold with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$. An almost r-paracontact connection $\overline{\Gamma}$ given by* (3.22) *on M_n is metric if and only if there exists a tensor field W of type* (1,2) *with $W(X,Y) = W_X(Y)$ satisfying*

$$g(H_X W_Z, Y) + g(X, H_Y W_Z) = 0. \tag{3.28}$$

From Theorem 3.1 we get

Theorem 3.3. *The linear connection $\overline{\Gamma}$ given by*

$$\overline{\nabla}_X = \nabla_X + \phi_1 \nabla_X \phi_1 + \phi_2 \nabla_X \phi_2 + \phi_3 \nabla_X \phi_3 - \eta^\alpha(\nabla_X \xi_\beta)\eta^\beta \otimes \xi_\alpha \tag{3.29}$$

is a metric almost r-paracontact connection on an almost r-paracontact Riemannian manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha, g)_{\alpha\in(r)}$, where ∇ is the Riemannian connection, i.e., $\overline{\nabla}\phi = 0$, $\overline{\nabla}\eta^\alpha = 0$, $\overline{\nabla}\xi_\alpha = 0$, $\overline{\nabla} g = 0$.

4. ***D*-connections**. In this section the definition of a D-connection on an almost r-paracontact manifold is given and all such connections are determined.

Definition 4.1. Let D be a distribution on a manifold M_n. A linear connection Γ given by its covariant derivative ∇ on M_n is said to be a *D-connection*, or D is said to be *parallel with respect to* Γ, if for any vector field Y and a vector field X from D the vector field $\nabla_Y X$ belongs to D.

Theorem 4.1. *The distribution D^+ given by* (1.8) *or* (2.6) *is parallel with respect to a linear connection Γ given by its covariant derivative ∇ on an almost r-paracontact manifold M_n, or Γ is a D^+-connection, if and only if*

$$\nabla\phi_1 \circ \phi_1 = 0. \tag{4.1}$$

Proof. For a vector field X from D^+ one obtains

$$\nabla_Y X = \nabla_Y(\phi_1 X) = (\nabla_Y \phi_1)X + \phi_1(\nabla_Y X). \tag{4.2}$$

If D^+ is parallel, then from (4.2) and (2.6) $(\nabla_Y\phi_1)X = 0$ for any vector field X in D^+, or $(\nabla_Y\phi_1)(\phi_1 Z) = 0$ for any vector field Z. Hence, (4.1) is obtained. Conversely, if (4.1) is satisfied, then for any $X \in D^+$, $0 = (\nabla\phi_1\circ\phi_1)X = (\nabla\phi_1)X$, so by (4.2) $\nabla_Y X = \phi_1(\nabla_Y X)$, and $\nabla_Y X$ is in D^+. □

In similar way we obtain

Theorem 4.2. *The distribution D^- given by* (1.9) *or* (2.7) *is parallel with respect to a linear connection Γ given by its covariant derivative ∇ on an almost r-paracontact manifold M_n, or Γ is a D^--connection, if and only if*

$$\nabla\phi_2 \circ \phi_2 = 0. \tag{4.3}$$

Theorem 4.3. *The distribution D^0 given by* (1.10) *or* (2.8) *is parallel with respect to a linear connection Γ given by its covariant derivative ∇ on an almost r-paracontact manifold M_n, or Γ is a D^0-connection, if and only if*

$$\nabla\phi_3 \circ \phi_3 = 0. \tag{4.4}$$

If Γ is an almost r-paracontact connection, then from (2.1), (2.2), and (2.3) we get $\nabla\phi_1 = \nabla\phi_2 = \nabla\phi_3 = 0$, and on account of Theorems 4.1, 4.2, and 4.3 we obtain

Theorem 4.4. *If a linear connection Γ on an almost r-paracontact manifold M_n is an almost r-paracontact connection, then the distributions D^+, D^-, and D^0 given by* (1.8), (1.9) *and* (1.10) *are parallel with respect to this connection, or Γ is a D^+-connection, and a D^--connection, and a D^0-connection.*

Definition 4.2. A linear connection Γ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ is said to be a *D_Σ-connection* if it is a D^+-connection, and a D^--connection, and a D^0-connection.

Theorem 4.5. *Every almost r-paracontact connection* Γ *on an almost r-paracontact manifold* M_n *with a structure* Σ *is a* D_Σ*-connection.*

If a linear connection Γ is a D_Σ-connection, then from Proposition 2.2 and Theorems 4.1, 4.2, and 4.3 we have

$$\nabla\phi_i = \phi_i \nabla\phi_i, \qquad i = 1,2,3 \tag{4.5}$$

and since $\phi_1 + \phi_2 + \phi_3 = Id$, we get

$$\sum_i \phi_i \nabla\phi_i = 0. \tag{4.6}$$

Acting with ϕ_j on the left to (4.6) we obtain

$$\phi_j \nabla\phi_j = 0, \qquad j = 1,2,3 \tag{4.7}$$

then, from (4.5) we get

Theorem 4.6. *A linear connection* Γ *on an almost r-paracontact manifold* M_n *with a structure* $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ *is a* D_Σ*-connection if and only if*

$$\nabla\phi_i = 0, \qquad i = i,2,3. \tag{4.8}$$

Now, we shall find all D_Σ-connections on an almost r-paracontact manifold M_n which are of the form

$$\hat{\nabla}_X = \nabla_X + J_X \tag{4.9}$$

where ∇ is the covariant differentiation operator of arbitrary linear connection Γ on M_n, and J is a tensor field of type (1,2) with $J_X(Y) = J(X,Y)$. From (3.6) we have for ϕ_i, $i = 1,2,3$

$$\hat{\nabla}_X \phi_i = \nabla_X \phi_i + J_X \circ \phi_i - \phi_i \circ J_X \qquad i = 1,2,3. \tag{4.10}$$

Since $\hat{\Gamma}$ is a D_Σ-connection, then from (4.10) and Theorem 4.6 we have

$$\nabla_X \phi_i + J_X \circ \phi_i - \phi_i \circ J_X = 0 \qquad i = 1,2,3. \tag{4.11}$$

Applying ϕ_i on the left to (4.11) and using Proposition 2.2 we get

$$\phi_i \nabla_X \phi_i + \phi_i J_X \phi_i - \phi_i J_X = 0, \qquad i = 1,2,3. \tag{4.12}$$

Using Proposition 2.2 we get from (4.12)

$$\phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3 + \phi_1 J_X \phi_1 + \phi_2 J_X \phi_2 + \phi_3 J_X \phi_3 - J_X = 0. \tag{4.13}$$

Using (2.19) we obtain

$$P J_X = \phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3. \tag{4.14}$$

Hence, in virtue of Propositions 2.6 and 2.1 we obtain

$$J_X = \phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3 + QV_X \tag{4.15}$$

where V is an arbitrary tensor field of type (1,2) with $V_X(Y) = V(X,Y)$.

Hence,

Theorem 4.7. *The general family of the* D_Σ*-connections* $\hat{\Gamma}$ *on an almost r-paracontact manifold* M_n *with a structure* $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ *is given by*

$$\hat{\nabla}_X = \nabla_X + \phi_1\nabla_X\phi_1 + \phi_2\nabla_X\phi_2 + \phi_3\nabla_X\phi_3 + QV_X \tag{4.16}$$

where Q *is defined by* (2.20), V *is an arbitrary tensor field of type* (1,2) *with* $V_X(Y) = V(X,Y)$, *and* ∇ *is the covariant derivative of an arbitrary linear connection* Γ *on* M_n.

Corollary 4.1. *If the initial connection* Γ *is a* D_Σ*-connection on* M_n*, then the general family of* D_Σ*-connections* $\hat{\Gamma}$ *on* M_n *is given by*

$$\hat{\nabla}_X = \nabla_X + QV_X \tag{4.17}$$

where V *is an arbitrary tensor field.*

5. Pairs of connections compatible with a structure. In this section a definition of a pair of connections compatible with an almost r-paracontact structure on an almost r-paracontact manifold is given and all such pairs are found.

Let $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$ be two linear connections, given by their covariant derivatives $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$, on an almost r-paracontact manifold M_n.

For a function f, a vector field Y, a 1-form ω, and a tensor field ψ of type (1,1), (0,2), or (2,0), we define the following *mixed covariant derivatives*

$$\overset{ij}{\nabla}_Z f = \overset{ji}{\nabla}_Z f = Zf \tag{5.1}$$

$$\overset{ij}{\nabla}_Z Y = \overset{i}{\nabla}_Z Y \tag{5.2}$$

$$\overset{ij}{\nabla}_Z \omega = \overset{i}{\nabla}_Z \omega \tag{5.3}$$

$$(\overset{ij}{\nabla}_Z \psi)(A,B) = Z\psi(A,B) - \psi(\overset{i}{\nabla}_Z A, B) - \psi(A, \overset{j}{\nabla}_Z B) \tag{5.4}$$

where $i,j = 1,2;\ i \neq j$.

Proposition 5.1. $\overset{m}{\nabla}_Z = \frac{1}{2}(\overset{1}{\nabla}_Z + \overset{2}{\nabla}_Z)$ *is a covariant differentiation operator of a certain connection* $\overset{m}{\Gamma}$ *on* M_n.

Definition 5.1. The connection $\overset{m}{\Gamma}$ on M_n is called a *mean connection* of $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$ if its *mean covariant derivative* is

$$\overset{m}{\nabla}_Z = \tfrac{1}{2}(\overset{1}{\nabla}_Z + \overset{2}{\nabla}_Z). \tag{5.5}$$

Proposition 5.2. $d_Z = \overset{2}{\nabla}_Z - \overset{1}{\nabla}_Z$ *is a tensor field of type* (1,1) *on* M_n for any vector field Z.

Definition 5.2. The tensor field d of type (1,2) defined by

$$d_Z = \overset{2}{\nabla}_Z - \overset{1}{\nabla}_Z \tag{5.6}$$

with $d_Z(X) = d(Z, X)$ on M_n is called a *deformation tensor field* of connections $\overset{1}{\Gamma}$ and $\overset{2}{\Gamma}$.

For the structure tensors ϕ and η^α we obtain

$$d_Z(\eta^\alpha) = -\eta^\alpha \circ d_Z \tag{5.7}$$

$$d_Z(\phi) = d_Z \circ \phi - \phi \circ d_Z \tag{5.8}$$

$$\overset{12}{\nabla}_Z \phi = \overset{2}{\nabla}_Z \phi + \phi \circ d_Z \tag{5.9}$$

$$\overset{21}{\nabla}_Z \phi = \overset{1}{\nabla}_Z \phi - \phi \circ d_Z. \tag{5.10}$$

Making use of these formulas we get for the structure tensors ϕ, η^α, and ξ_α

$$\tfrac{1}{2}(\overset{12}{\nabla}_Z \eta^\alpha + \overset{21}{\nabla}_Z \eta^\alpha) = \overset{m}{\nabla}_Z \eta^\alpha \tag{5.11}$$

$$\overset{12}{\nabla}_Z \eta^\alpha - \overset{21}{\nabla}_Z \eta^\alpha = \eta^\alpha \circ d_Z \tag{5.12}$$

$$\tfrac{1}{2}(\overset{12}{\nabla}_Z \phi + \overset{21}{\nabla}_Z \phi) = \overset{m}{\nabla}_Z \phi \tag{5.13}$$

$$\overset{12}{\nabla}_Z \phi - \overset{21}{\nabla}_Z \phi = d_Z \circ \phi + \phi \circ d_Z \tag{5.14}$$

$$\tfrac{1}{2}(\overset{12}{\nabla}_Z \xi_\alpha + \overset{21}{\nabla}_Z \xi_\alpha) = \overset{m}{\nabla}_Z \xi_\alpha \tag{5.15}$$

$$\overset{12}{\nabla}_Z \xi_\alpha - \overset{21}{\nabla}_Z \xi_\alpha = -d_Z \xi_\alpha. \tag{5.16}$$

For (1,1) tensor fields ψ and χ, using (5.8), (5.14) and (5.9), (5.10) we get

$$\overset{1}{\nabla}_Z(\psi\chi) = (\overset{21}{\nabla}_Z \psi)\chi + \psi \overset{12}{\nabla}_Z \chi \tag{5.17}$$

$$\overset{2}{\nabla}_Z(\psi\chi) = (\overset{12}{\nabla}_Z \psi)\chi + \psi \overset{21}{\nabla}_Z \chi. \tag{5.18}$$

Definition 5.3. A pair of linear connections $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ is said to be *compatible with* Σ if

$$\overset{12}{\nabla} \phi = 0, \qquad \overset{1}{\nabla} \eta^\alpha = 0, \qquad \overset{2}{\nabla} \xi_\alpha = 0, \qquad \alpha \in (r). \tag{5.19}$$

Theorem 5.1. *A pair of linear connections* $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ *on an almost r-paracontact manifold* M_n *with a structure* $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha \in (r)}$ *is compatible with* Σ *if and only if all structure tensor fields are parallel with respect to both mixed covariant derivatives* $\overset{12}{\nabla}$ *and* $\overset{21}{\nabla}$.

Proof. We have to show that the following conditions are satisfied

$$(5.20)\qquad \overset{21}{\nabla}\phi = 0, \qquad \overset{2}{\nabla}\eta^\alpha = 0, \qquad \overset{1}{\nabla}\xi_\alpha = 0, \qquad \alpha \in (r).$$

From (1.1) we get

$$(5.21)\qquad (\overset{i}{\nabla}\eta^\alpha)(\xi_\beta) + \eta^\alpha(\overset{i}{\nabla}\xi_\beta) = 0, \qquad i = 1, 2$$

From (1.3), using (5.9) and $(5.19)_1$, we get $\overset{2}{\nabla}\eta^\alpha \circ \phi = 0$. Hence $\overset{2}{\nabla}\eta^\alpha - (\overset{2}{\nabla}\eta^\alpha)(\xi_\beta)\eta^\beta = 0$, and using (5.21) we obtain $(5.20)_2$. From (1.4), using $(5.19)_1$ and (5.9), we get $\phi\overset{1}{\nabla}\xi_\alpha = 0$. Hence $\overset{1}{\nabla}\xi_\alpha - \eta^\beta(\overset{1}{\nabla}\xi_\alpha)\xi_\beta = 0$. Again, using (5.21) we obtain $(5.20)_3$. Now, making use of (5.17) and (5.18) with $\psi = \chi = \phi$ we obtain from (1.2), after using $(5.19)_2$, $(5.19)_3$, $(5.20)_2$, $(5.20)_3$, and $(5.19)_1$, $(\overset{21}{\nabla}\phi)\phi = 0$, $\phi\overset{21}{\nabla}\phi = 0$. Hence $\overset{21}{\nabla}\phi = \overset{21}{\nabla}\phi^3 = \phi\overset{21}{\nabla}\phi\phi + \overset{21}{\nabla}\phi\phi^2 + \phi^2\overset{21}{\nabla}\phi = 0$. □

Remark 5.1. From Theorem 5.1 we obtain the symmetry of compatibility, i.e., a pair $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ is compatible with Σ if and only if $(\overset{2}{\Gamma}, \overset{1}{\Gamma})$ is compatible with Σ.

Now, we obtain the following

Theorem 5.2. *A pair of linear connections $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ on an almost r-paracontact manifold M_n with a structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ is compatible with Σ if and only if*

(i) *the mean connection $\overset{m}{\Gamma}$ given by* (5.5) *is an almost r-paracontact connection on M_n*

(ii) $Bd_Z = 0$, *where B is given by* (2.10)

Proof. Using (5.11) through (5.16) the conditions (5.19) and (5.20) are equivalent to

$$(5.22)\qquad \overset{m}{\nabla}\phi = 0, \qquad \overset{m}{\nabla}\eta^\alpha = 0, \qquad \overset{m}{\nabla}\xi_\alpha = 0, \qquad \alpha \in (r)$$

$$(5.23)\qquad d_Z\phi + \phi d_Z = 0, \qquad \eta^\alpha d_Z = 0, \qquad d_Z(\xi_\alpha) = 0, \qquad \alpha \in (r).$$

The conditions (5.22) are equivalent to (i) and (5.23) to (ii). □

Remark 5.2. The condition (ii) of Theorem 5.2 implies

$$(5.24)\qquad Hd_Z = 0$$

where H is given by (2.14).

Hence we get

Theorem 5.3. *If a pair of linear connections $(\overset{1}{\Gamma}, \overset{2}{\Gamma})$ is compatible with an almost r-paracontact structure $\Sigma = (\phi, \xi_\alpha, \eta^\alpha)_{\alpha\in(r)}$ on a manifold M_n, then*

$$(5.25)\qquad \overset{1}{\nabla}_Z = \nabla_Z + S_Z - \tfrac{1}{2}FV_Z$$

$$(5.26)\qquad \overset{2}{\nabla}_Z = \nabla_Z + S_Z + \tfrac{1}{2}FV_Z$$

where ∇ is an arbitrary linear connection on M_n, S_Z is given by (3.21), *F is defined by* (2.15), *and V is an arbitrary tensor field of type* (1,2) *with $V_Z(X) = V(Z, X)$.*

Proof. From (5.24), using Proposition 2.1, we obtain

$$d_Z = FV_Z. \tag{5.27}$$

From Theorem 5.2(i), using Theorem 3.1, we get

$$\overset{m}{\nabla}_Z = \nabla_Z + S_Z. \tag{5.28}$$

From (5.27), (5.28), (5.5), and (5.6) we obtain (5.25) and (5.26). □

REFERENCES

[1]. A. Bucki, A. Miernowski, *Almost r-paracontact structures*, Ann. Univ. Mariae Curie-Sklodowska, Sect. A **39** (1985), 13–26.

[2]. A. Bucki, A. Miernowski, *Almost r-paracontact connections*, Acta Math. Hungar. **45** (1985), 327–336.

[3]. M. Obata, *Hermitian manifolds with quaternion structures*, Tohoku Math. J. **10** (1958), 11–18.

[4]. A. P. Norden, *Spaces with Affine Connections (in Russian)*, Nauka, Moscow, 1970.

[5]. K. Yano, *Differential Geometry on Complex and Almost Complex Space*, Pergamon Press, Oxford, 1965.

(Received October 27, 1996)

Dept. of Mathematics
Oklahoma School of Science and Mathematics
Oklahoma City, OK 73104
USA

E-mail address: abucki@ossm.edu