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ON NON-VERTICAL LINEAR (1,1)-TENSOR FIELDS AND CONNECTIONS ON TANGENT BUNDLES

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ABSTRACT. Special (1,1)-tensor fields α on tangent bundles TM which do not preserve the vertical subbundles VTM and satisfy some linearity conditions determine linear connections Γ_α on TM. The relationships between α and Γ_α are studied in this paper.

INTRODUCTION

In the papers [1], [2] we have dealt with connections on TM which are canonicaly determined by (1,1)-tensor fields on tangent bundles TM, especially by almost complex structures on TM. This contribution completes our previous considerations by some (1,1)-tensor fields which do not preserve the subbundle VTM of all vertical tangent vectors on TM and satisfy some linearity conditions. We suppose that all manifolds and maps are infinitely differentiable.

Let (x^i, x_1^i) be the local chart on the tangent bundle $\pi : TM \to M$ induced by a local chart (x^i) on a manifold M. Then the coordinate form of an arbitrary (1,1)-tensor α is as follows

$$\alpha = (a_j^i dx^j + b_j^i dx_1^j) \otimes \partial/\partial x^i + (c_j^i dx^j + h_j^i dx_1^j) \otimes \partial/\partial x_1^i \ ,$$

where $a_j^i, b_j^i, c_j^i, h_j^i$ are functions of the variables x^k, x_1^k.

A connection Γ on TM can be considered as a special (1,1)-tensor field $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma_j^i(x, x_1) dx^j \otimes \partial/\partial x_1^i$ such that $h_\Gamma(VTM) = 0$ and $T\pi \cdot h_\Gamma = T\pi$ where Tf denotes the tangent prolongation of a map f. Then $v_\Gamma = Id_{TM} - h_\Gamma$ is the vertical form of Γ. Denote by $H\Gamma := h_\Gamma(TM) \subset TM$ the vector subbundle of the Γ-horizontal vectors on TM, i.e. such vectors $(x^i, x_1^i, dx^i, dx_1^i)$ on TM which satisfy the equation $dx_1^i = \Gamma_j^i dx^j$. The functions $\Gamma_j^i(x, x_1)$ will be called local functions of Γ. Recall that a connection Γ is linear iff $\Gamma_j^i = \Gamma_{jk}^i(x) x_1^k$.

Let us introduce a short survey of the main results published in [1] we will need.

Let $Y = \eta^i \partial/\partial x_1^i$ be an arbitrary vertical vector field on TM and $X = \xi^i \partial/\partial x^i + \Gamma_j^i \xi^j \partial/\partial x_1^i$ be a horizontal vector field of a given connection Γ on TM. Then $\alpha(Y) =$

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$b^i_j\eta^j\partial/\partial x^i + h^i_j\eta^j\partial/\partial x^i_1$ or $\alpha(X) = (a^i_j + b^i_t\Gamma^t_j)\xi^j\partial/\partial x^i + (c^i_j + h^i_t\Gamma^t_j)\xi^j\partial/\partial x^i_1$, is Γ-horizontal for any vertical vector field Y or vertical for any Γ-horizontal vector field X iff

$$\Gamma^i_k b^k_j = h^i_j \quad \text{or} \quad a^i_j + b^i_t\Gamma^t_j = 0\ , \tag{1}$$

respectively.

Let $J = dx^i \otimes \partial/\partial x^i_1$ be the canonical morphism (almost tangent structure) determined by Id_{TM} and the canonical identification $VTM = TM \times_M TM$. Then the (1, 1)-tensor field $B := J\alpha J = b^i_j dx^j \otimes \partial/\partial x^i_1$ can be interpreted as the vector bundle morphism $T\pi \cdot \alpha|_{VTM} = b^i_j dx^j_1 \otimes \partial/\partial x^i : VTM \to TM$ over $\pi : TM \to M$ or as a vector bundle morphism $VTM \to VTM$, $B = b^i_j dx^j_1 \otimes \partial/\partial x^i_1$.

If $B = 0$ or $B \neq 0$ i.e. if $\alpha(VTM) \subset VTM$ or $\alpha(VTM) \not\subset VTM$ we say that α is vertical or non vertical, respectively.

It is evident from (1) that in the case when α is non vertical and B is an isomorphism then there is a unique connection Γ^1_2 and a unique connection Γ^2_α on TM such that $\alpha(H\Gamma^1_\alpha) = VTM$ and $\alpha(VTM) = H\Gamma^2_\alpha$. Then $\Gamma^i_j = -\tilde{b}^i_k a^k_j$ or $\Gamma^i_j = h^i_k\tilde{b}^k_j$, $\tilde{b}^i_k b^k_j = \delta^i_j$, are local functions of Γ^1_α or Γ^2_α, respectively.

The following coordinate conditions

$$a^i_s a^s_j + b^i_s c^s_j = -\delta^i_j, \quad a^i_s b^s_j + b^i_s h^s_j = 0, \quad c^i_s a^s_j + h^i_s c^s_j = 0, \quad c^i_s b^s_j - h^i_s h^s_j = -\delta^i_j \tag{2}$$

under which a (1,1)-tensor field α is an almost complex structure on TM and the equalities (1) immediately give

Lemma 1. *Let Γ be a connection on TM. Let $B = b^i_j dx^j_1 \otimes \partial/\partial x^i : VTM \to TM$ be a vector bundle isomorphism over $\pi : TM \to M$. Then there exists a unique almost complex structure $\alpha(\Gamma, B)$ on TM such that $T\pi \cdot \alpha|_{VTM} = B$ and $\Gamma^1_\alpha = \Gamma^2_\alpha = \Gamma$.*

Remark 1. It is easy to see that in the case when B is regular then the third and fourth equations of (2) are the consequence of the first and the second ones. The second equality of (2) is the coordinate condition for α^2 to be vertical and for the equality $\Gamma^1_\alpha = \Gamma^2_\alpha$.

NON VERTICAL LINEAR (1,1)-TENSOR FIELD ON TM

Definition 1. A non vertical (1,1)-tensor field α on TM is said to be v-projectable if the (1,1)-tensor field $B = J\alpha J$ is the v-lift of a (1,1)-tensor field $\overline{B} = b^i_j(x)dx^j \otimes \partial/\partial x^i$ on M.

Definition 2. A v-projectable (1,1)-tensor field α is called l-linear if $T\pi \cdot \alpha X : TM \to TM$ is a vector bundle morphism over Id_M for any projectable and linear vector field $X : TM \to TTM$. Analogously a v-projectable (1,1)-tensor field α is called r-linear if $\alpha(Y^v)$ is a projectable and linear vector field on TM for any vertical lift Y^v of any vector field Y on M.

In coordinates, let $X = \xi^i(x)\partial/\partial x^i + \eta^i_j(x)x^j_1\partial/\partial x^i_1$ be a projectable and linear vector field on TM and α be v-projectable. Then the map $T\pi \cdot \alpha X$ is given by the equations

$\overline{x}^i = x^i$, $\overline{x}_1^i = a^i_j(x,x_1)\xi^j + b^i_k(x)\eta^k_j x^j_1$. So it is a vector bundle morphism, i.e. α is l-linear iff $a^i_j(x,x_1) = a^i_{jk}(x)x^k_1$.

Analogously for a field $Y = \xi^i(x)\partial/\partial x^i$ on M its v-lift is $Y^v = \xi^i(x)\partial/\partial x^i_1$ and so $\alpha(Y^v) = b^i_j(x)\xi^j(x)\partial/\partial x^i + h^i_j(x,x_1)\xi^j(x)\partial/\partial x^i_1$ is projectable and linear, i.e. α is r-linear iff $h^i_j(x,x_1) = h^i_{jk}(x)x^k_1$.

Definition 3. A non vertical (1,1)-tensor field α is called linear if it is r- and l-linear.

The following lemma is evident from the equalities (1) and (2).

Lemma 2. *Let α be such a (1,1)-tensor field on TM that B is regular. Then the connection Γ^1_α or Γ^2_α is linear if α is l-linear or r-linear, respectively. Both connections Γ^1_α and Γ^2_α are linear if α is linear. If moreover α^2 is vertical then if α is l-linear or r-linear then α is linear. If connection Γ is linear and B is v-lift of a regular (1,1)-tensor field $\overline{B}$ on M then the almost complex structure $\alpha(\Gamma, B)$ is linear.*

Recall that a semispray S on TM is such a vector field on TM that $J(S) = V$ where $V = x^i_1\partial/\partial x^i_1$ is the canonical (Liouville) vector field the flows of which are the homotheties on individual fibres T_xM. S is a spray if $[S,V] = S$. In local coordinates $S = x^i_1\partial/\partial x^i + \eta^i(x,x_1)\partial/\partial x^i_1$ and it is a spray if $\eta = \eta^i_k(x)x^k_1$.

Every connection Γ on TM with local functions Γ^i_j determines a unique semispray $S_\Gamma = x^i_1\partial/\partial x^i + \Gamma^i_j x^j\partial/\partial x^i_1$ (called Γ-horizontal).

Lemma 3. *Let α be such a (1,1)-tensor field on TM that B is regular. Then $\alpha(V)$ is Γ^2_α-horizontal and $\alpha(B^{-1}(V))$ is the Γ^2_α-horizontal semispray.*

Proof in coordinates. $\alpha(V) = b^i_j x^j_1\partial/\partial x^i + h^i_j x^j_1\partial/\partial x^i_1$ and considering B as a morphism on VTM we have $B^{-1}(V) = \tilde{b}^i_j x^j_1\partial/\partial x^i$, $\alpha(B^{-1}(V)) = x^i_1\partial/\partial x^i + h^i_k\tilde{b}^k_j x^j_1\partial/\partial x^i_1$. The proof is finished.

In concordance with many authors (for example Modugno [4], Yano [5]) using the Nijenhuis-Frölicher bracket $[\alpha,\beta]$ of two vector valued tensor fields we introduced the following notions

Definition 4. We will say that a (1,1)-tensor field α is quasi-symmetric or semi-symmetric or symmetric if $[\alpha, J]$ is a vertical valued or a semi-basic vertical valued (1,2)-form on TM or vanishes, respectively. We will say that a connection Γ is symmetric, (equivalently without torsion), when its torsion $T_\Gamma = [h_\Gamma, J]$ vanishes.

Throughout this paper we will use the denotations $f_i := \frac{\partial f}{\partial x^i}$, $f_{i_1} := \frac{\partial f}{\partial x^i_1}$.

By direct computations we get in the case of a v-projectable (1,1)-tensor field:

$$[\alpha, J] = a^i_{sj_1}dx^j \wedge dx^s \otimes \partial/\partial x^i + [(c^i_{sj_1} + a^i_{js})dx^j \wedge dx^s + (b^i_{js} - h^i_{js_1} - a^i_{sj_1})dx^j_1 \wedge dx^s]\otimes \otimes\, \partial/\partial x^i_1$$

$$[h_\Gamma, J] = \Gamma^i_{sj_1}dx^j \wedge dx^s \otimes \partial/\partial x^i_1\ , \tag{3}$$

where Γ^i_j are the local functions of a connection Γ. So

$$a^i_{sj_1} = a^i_{js_1} \quad \text{or} \quad a^i_{sj_1} = a^i_{js_1},\ b^i_{js} - h^i_{js_1} - a^i_{sj_1} = 0 \tag{4}$$

are the coordinate conditions of α to be quasi- or semi-symmetric.

Proposition 1. *Let α be a v-projectable (1,1)-tensor field on TM such that $\overline{B}$ is regular. Then the connection Γ^1_α is symmetric if and only if α is quasi-symmetric.*

Proof follows from the relationships (3), (4) and from the local functions $-b^i_k(x)a^k_j$ of the connection Γ^1_α.

Remark 2. If α is l-linear and $\overline{B}$ is regular then $a^i_j = a^i_{jk}(x)x^k_1$ and so α is quasi-symmetric iff $a^i_{jk} = a^i_{kj}$.

Let $i_2 : TTM \to TTM, (x^i, x^i_1, dx^i, dx^i_1) \to (x^i, dx^i, x^i_1, dx^i_1)$ be the canonical involution on TTM. Let $T\overline{B} : (x^i, x^i_1, dx^i, dx^i_1) \to (x^i, b^i_j x^j_1, dx^i, b^i_{kj}x^k_1 dx^j + b^i_j dx^j_1)$ denote the tangent prolongation of the map $\overline{B} : TM \to TM, (x^i, x^i_1) \to (x^i, b^i_j x^j_1)$ given by a (1,1)-tensor field $\overline{B}$ on M. Consider the following maps:

$$\begin{aligned} i_2 \cdot T\overline{B} \cdot i_2 :&(x^i, x^i_1, dx^i, dx^i_1) \to (x^i, x^i_1, b^i_j dx^j, b^i_{kj} dx^k x^j_1 + b^i_j dx^j_1) \\ J\alpha :&(x^i, x^i_1, dx^i, dx^i_1) \to (x^i, x^i_1, 0, a^i_j dx^j + b^i_j dx^j_1) \\ \alpha J :&(x^i, x^i_1, dx^i, dx^i_1) \to (x^i, x^i_1, b^i_j dx^j, h^i_j dx^j) \ . \end{aligned}$$

Then the map

$$\begin{aligned} i_2 \cdot T\overline{B} \cdot i_2 - J\alpha - \alpha J :&(x^i, x^i_1, dx^i, dx^j_1) \to (x^i, x^i_1, 0, b^i_{kj} dx^k x^j_1 - (a^i_j + h^i_j)dx^j) = \\ &= (x^i, b^i_{kj} dx^k x^j_1 - (a^i_j + h^i_j)dx^j) \end{aligned}$$

determines in the case of a linear (1,1)-tensor field α, (i.e. $a^i_j = a^i_{jk}x^k_1,\ h^i_j = h^i_{jk}x^k_1$), a (1,2)-tensor α_B on M,

$$\alpha_B = (b^i_{jk} - a^i_{jk} - h^i_{jk})dx^j \otimes dx^k \otimes \partial/\partial x^i \ . \tag{5}$$

Proposition 2. *Let α be a quasi-symmetric and linear (1,1)-tensor field on TM. Then α is semi-symmetric if and only if $\alpha_B = 0$.*

Proof. Using $a^i_{jk} = a^i_{kj}$ and comparing (4) with (5) we complete our proof.

By the map $T\overline{B}$ we can construct another connection $\tilde{\Gamma}$ on TM as follows: We have

$$i_2 \cdot T\overline{B} \cdot i_2 - \alpha J = [(b^i_{jk}x^k_1 - h^i_j)dx^j + b^i_j dx^j_1] \otimes \partial/\partial x^i_1 \ .$$

Considering B as a vector bundle isomorphism $VTM \to VTM$ over Id_{TM} we see that the following (1,1)-tensor field

$$B^{-1}[i_2 \cdot T\overline{B} \cdot i_2 - \alpha J] = [\tilde{b}^i_t(b^t_{jk}x^k_1 - h^t_j)dx^j + dx^j_1] \otimes \partial/\partial x^i_1$$

is the vertical form of the connection $\tilde{\Gamma}$ the local functions of which are $\Gamma^i_j = -\tilde{b}^i_t(b^t_{jk}x^k_1 - h^t_j)$.

Proposition 3. *Let α be a linear quasi-symmetric (1,1)-tensor field on TM the morphism B of which is regular. Then α is semi-symmetric if and only if the equality $\tilde{\Gamma} = \Gamma^1_\alpha$ is satisfied.*

Proof. Under our suppositions the equality $\tilde{\Gamma} = \Gamma^1_\alpha$ has the following coordinate form

$$-\tilde{b}^i_t(b^t_{jk} - h^t_{jk})x^k_1 = -\tilde{b}^i_t a^t_{jk} x^k_1, \quad \text{i.e.} \quad b^i_{jk} - h^i_{jk} = a^i_{jk} \ .$$

Comparing it with (4) we finish our proof.

Recall that if Γ is a linear connection on TM with local functions $\Gamma^i_j = \Gamma^i_{jk}x^k_1$, $\overline{B} = b^i_j dx^j \otimes \partial/\partial x^i$ is a (1,1)-tensor field on M and $X = \xi^i \partial/\partial x^i$ is a vector field on M then the covariant derivative $\nabla^\Gamma_X \overline{B}$ is the (1,1)-tensor field on M the coordinate form of which is as follows

$$\nabla^\Gamma_X \overline{B} = (b^i_{kj} + b^i_s \Gamma^s_{jk} - \Gamma^i_{js} b^s_k)\xi^j dx^k \otimes \partial/\partial x^i \ .$$

Then $\nabla^\Gamma \overline{B} = (b^i_{kj} + b^i_s \Gamma^s_{jk} - \Gamma^i_{js} b^s_k) dx^j \otimes dx^k \otimes \partial/\partial x^i$ is a (1,2)-tensor field on M and the equality

$$b^i_{kj} + b^i_s \Gamma^s_{jk} - \Gamma^i_{js} b^s_k = 0 \tag{6}$$

is the coordinate condition for $\overline{B}$ to be constant under the covariant derivative with respect to the connection Γ.

Remark 3. The covariant derivative $\nabla^\Gamma \overline{B}$ can be interpreted as follows. If $\overline{B} = b^i_j dx^j \otimes \partial/\partial x^i$ and if $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma^i_{jk} x^k_1 dx^j \otimes \partial/\partial x^i_1$ then the map $TB \cdot h_\Gamma - h_\Gamma \cdot TB \cdot h_\Gamma$: $\overline{x}^i = x^i$, $\overline{x}^i_1 = b^i_j x^j_1$, $d\overline{x}^i = 0$, $d\overline{x}^i_1 = (b^i_{kj} + b^i_s \Gamma^s_{jk} - \Gamma^i_{js} b^s_k) x^k_1 dx^j$ determines the tensor field $\nabla^\Gamma \overline{B}$ on M.

Lemma 4. *Let $\overline{B} = b^i_j dx^j \otimes \partial/\partial x^i$ be a (1,1)-tensor field on M and $B = b^i_j dx^j \otimes \partial/\partial x^i_1$ be its vertical lift on TM. Let Γ be a symmetric linear connection on TM. Then the Nijenhuis-Frölicher bracket $[h_\Gamma, B]$ is the skew-symmetrization of the v-lift of the tensor field $\nabla^\Gamma \overline{B}$.*

Proof follows immediately from the coordinate form $[h_\Gamma, B] = (b^i_{kj} + \Gamma^i_{ku} b^u_j) dx^j \wedge dx^k \otimes \partial/\partial x^i_1$.

Let $S = x^i_1 \partial/\partial x^i + \eta(x, x_1)\partial/\partial x^i_1$ be a semi-spray on TM and $B = b^i_j(x) dx^j \otimes \partial/\partial x^i_1$ be the v-lift of a regular (1,1)-tensor field $\overline{B}$ on M. Then the Lie derivative

$$L_S B = -b^i_j dx^j \otimes \partial/\partial x^i + [(b^i_{jk} x^k_1 - \eta^i_{k_1} b^k_j) dx^j + b^i_j dx^j_1] \otimes \partial/\partial x^i_1$$

is the so-called skew 2-projectable (1,1)-tensor field on TM, see [2]. Such a (1,1)-tensor field β determines a unique connection Γ_{BS} the horizontal subbundle $H\Gamma_{BS}$ of which spannes the vectors Y on TM for which $L_{\overline{S}} L_S B(Y) \in VTM$, where $\overline{S}$ is an arbitrary semi-spray on TM on which the connection Γ_{BS} is independent. It is easy to calculate that

$$\Gamma^i_j = -\frac{1}{2}\tilde{b}^i_s(2b^s_{jk} x^k_1 - \eta^s_{k_1} b^k_j) \tag{7}$$

are its local functions.

Denote by $A : \alpha \to \Gamma^1_\alpha$ and by $H : \alpha \to \Gamma^2_\alpha$ the operators from the space α_v of all v-projectable (1,1)-tensor fields α on TM with regular $B = J\alpha J$ into the space Conn TM of all connections on TM. Evidently A, H are zero- order natural operators in the category $\mathcal{M}_m$ of all smooth manifolds and smooth local diffeomorphisms.

Let $DB : V(1,1) \times S \to$ Conn TM denote the operator from the product space of all v-lifts of all regular (1,1)-tensor fields on M and of all semisprays on TM into the space Conn TM given by the above described rule $DB : (S, B) \to \Gamma_{BS}$. It is easy to see that DB is a natural operator of first order in the category $\mathcal{M}_n$. Reader is kindly refered to [3] for the theory of natural operators.

Proposition 4. *Let B be the v-lift of a regular (1,1)-tensor field $\overline{B}$ on M. Let Γ be a linear symmetric connection on TM. Then the following conditions are equivalent*

a) $\nabla^\Gamma \overline{B} = 0$
b) $DB(S_\Gamma, B) = \Gamma$

where S_Γ is the Γ-horizontal spray on TM.

Proof. Let $S_\Gamma = x^i_1 \partial/\partial x^i + \eta^i \partial/\partial x^i_1$, $\eta^i = \Gamma^i_{jk} x^j_1 x^k_1$, be Γ-horizontal spray. Then by (7) $\Gamma^i_j = -\tilde{b}^i_s (b^s_{jk} - \Gamma^s_{su} b^u_j) x^k_1$ are the local functions of the connection $DB(S_\Gamma, B)$. So locally $DB(S_\Gamma, B) = \Gamma$ if and only if

$$b^i_{kj} - \Gamma^i_{js} b^s_k + b^i_s \Gamma^s_{kj} = 0$$

which coincides with (6), i.e. with the condition $\nabla^\Gamma \overline{B} = 0$. Proof is finished.

It is evident from the definitions of operators A and DB that

Proposition 5. $A(L_S L_S B) = DB(S, B)$.

Remember that if α^2 is vertical then $\Gamma^1_\alpha = \Gamma^2_\alpha$, i.e., $H(\alpha) = A(\alpha)$.

Proposition 6. *Let α be a l-linear and quasi-symmetric (1,1)-tensor field on TM such that α^2 is vertical and B is regular. Then the following conditions are equivalent*

i) $\nabla^{A(\alpha)} B = 0$,
ii) *α is semi-symmetric.*

Proof. The equality (6) for the connection $\Gamma^1_\alpha = A(\alpha)$ yields $b^i_{kj} - b^i_s \tilde{b}^s_t a^t_{jk} + \tilde{b}^i_t a^t_{js} b^s_k = 0$. Then using the second relationship of (2) we have $b^i_{kj} - a^i_{jk} - h^i_{kj} = 0$. So the equalities (4) are satisfied, i.e. α is semi-symmetric iff $\nabla^{A(\alpha)} B = 0$.

Corollary. *Under the conditions of Proposition 6 α is semi-symmetric if and only if $DB(S_{\Gamma^1_\alpha}, B) = \Gamma^1_\alpha$.*

Let Γ be a given connection on TM. Recall that the Γ-lift of a vector $X \in T_x M$ at $u \in TM$ is the Γ-horizontal vector $\Gamma(X) \in T_u TM$ such that $T\pi(\Gamma(X)) = X$. Let $\overline{B}$ be a (1,1)-tensor field on M. Then $\overline{B}$ acts on the Γ-horizontal vectors on TM by the rule: If $Y \in T_u TM$ then $\overline{B}Y = \Gamma(\overline{B}(T\pi Y)) \in T_u TM$, i.e. $\overline{B}Y$ is Γ-lift of the vector $\overline{B}(T\pi Y)$ at u. In coordinates, $\overline{B}(\xi^i \partial/\partial x^i + \Gamma^i_j \xi^j \partial/\partial x^i_1) = b^i_j \xi^j \partial/\partial x^i + \Gamma^i_t b^t_j \xi^j \partial/\partial x^i_1$.

Let α be a r-linear (1,1)-tensor field on TM. Recall that $\overline{B}$ is a (1,1)-tensor on M determined by $B = J\alpha J$. Let Γ be a linear connection on TM then the map $\overline{B}h_\Gamma - \alpha J : (x^i, x_1^i, dx^i, dx_1^i) \to (x^i, x_1^i, 0, (\Gamma^i_{sk} b^s_j - h^i_{jk}) x_1^k dx^j)$ determines the (1,2)-tensor field $\overline{B}h_\Gamma - \alpha J = (\Gamma^i_{sk} b^s_j - h^i_{jk}) dx^j \otimes dx^k \otimes \partial/\partial x^i$ on M.

Proposition 7. *Let α be a quasi-symmetric and l-linear (1,1)-tensor field on TM such that $\overline{B}$ is regular, α^2 is vertical and α_B is symmetric (1,2)-tensor. Then just the connection $\Gamma^2_\alpha = \Gamma^1_\alpha = \Gamma_\alpha$ is such a linear symmetric connection on TM that the tensor $\nabla^\Gamma B$ is symmetric and the tensor $\overline{B}h_{\Gamma_\alpha} - \alpha J$ is skew-symmetric.*

Proof. By our assumptions α is linear and (see (2), (5)),

$$a^i_{jk} = a^i_{kj}, \quad a^i_{sk} b^s_j = -b^i_s h^s_{jk}, \quad b^i_{jk} - h^i_{jk} = b^i_{kj} - h^i_{kj} \ . \tag{8}$$

Then the tensor $\nabla^\Gamma B$ is symmetric and $\overline{B}h_{\Gamma_\alpha} - \alpha J$ is skew-symmetric iff

$$b^i_{jk} - b^i_{kj} - \Gamma^i_{ku} b^u_j + \Gamma^i_{ju} b^u_k = 0 \quad \text{i.e.,} \quad h^i_{jk} - h^i_{kj} - \Gamma^i_{uk} b^u_j + \Gamma^i_{uj} b^u_k = 0 \ , \tag{9}$$

$$\Gamma^i_{uk} b^u_j - h^i_{jk} + \Gamma^i_{uj} b^u_k - h^i_{kj} = 0 \ , \tag{10}$$

where Γ is a linear symmetric connection. If the relationships (9), (10) are satisfied then

$$\Gamma^i_{uj} b^u_k = h^i_{kj}, \quad \text{i.e.} \quad \Gamma^i_{jk} = h^i_{tk} \tilde{b}^t_j, \quad \text{i.e.} \quad \Gamma = \Gamma^2_\alpha = \Gamma_\alpha \ .$$

Conversely, it easy to show that under the conditions (8) the connection Γ_α is symmetric and satisfies the equalities (9) and (10). The proof is finished.

References

[1] Dekrét, A., *Almost complex structures and connections on TM*, Proc. Conf. Diff. Geometry and Appl. Brno, Masaryk. Univ., Brno (1996), 133 – 140.

[2] Dekrét, A., *On skew 2-projectable almost complex structures on TM*, Proc. Conf. Diff. Geometry (1996), Budapest, to appear.

[3] Kolář, J, - Michor, P.W. - Slovák, J., *Natural operations in differential geometry*, Springer-Verlag, 1993.

[4] Modugno, M., *New results on the theory of connections: systems, over connections and prolongations* (1987), Diff. Geometry and Its Applications, Proceedings, D. Reider Publishing Company, 243 – 269.

[5] Yano, K., *Differential Geometry on complex and almost complex spaces* (1965), Pergamon Press, New York.

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