Acta Univ. M. Belii
Math. no.4 (1996), pp. 25-38

VALUATIONS AND METRICS ON A POSET

Alfonz Haviar and Gabriela Monoszová

Abstract. The aim of the paper is to characterize metrics (pseudometrics) induced on connected posets by positive (isotone) valuations. Further, as an application it is shown that there exists a positive valuation on some posets of locally finite length.

Introduction

Several authors investigated valuations and metrics on posets (see, e.g. [1] - [3]). M. Kolibiar and J. Lihová [5] and J. Lihová [6] gave characterizations of metrics induced by positive (isotone) valuations on directed multilattices. In this paper we will present similar results for richier families of posets.

Recall some basic definitions. Let $\mathbf{F}_n = (\{a_1, \dots, a_n\}, \leq)$ be a poset. Let n be an odd integer, $n \geq 3$. The poset $\mathbf{F}_n$ is called *a fence* if

$$a_1 < a_2 > a_3 < \cdots < a_{n-1} > a_n$$

or

$$a_1 > a_2 < a_3 > \cdots > a_{n-1} < a_n$$

are the only comparability relations of the poset $\mathbf{F}_n$ (see Figures 1a, 1b). Let n be an even integer, $n \geq 4$. The poset $\mathbf{F}_n$ is called *a fence* if

$$a_1 < a_2 > a_3 < \cdots > a_{n-1} < a_n$$

are the only comparability relations of the poset $\mathbf{F}_n$ (see Fig. 1c).

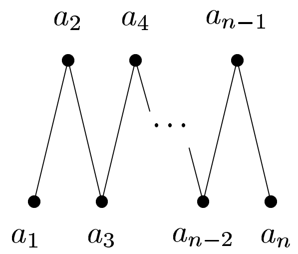

Fig. 1a

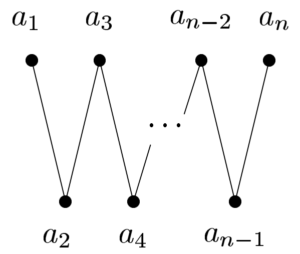

Fig. 1b

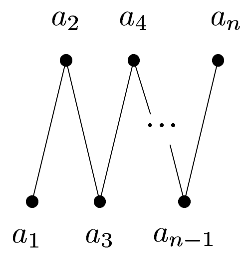

Fig. 1c

1991 *Mathematics Subject Classification.* Primary 06A99, 06B99.
Key words and phrases. Metric on a poset, valuation on a poset.

A poset $(P,\leq)$ is called *a connected poset* if for every elements $a,b\in P$ there is a fence $\mathbf{F}=(\{a_1,a_2,\dots,a_n\},\leq)$ such that $a=a_1$, $b=a_n$. In this case we say that $\mathbf{F}$ is *a fence from a to b.*

Let $\mathbf{C}=(\{a_0,a_1,\dots,a_n\},\leq)$ be a chain. The number n is called *the length* of the chain $\mathbf{C}$.

A poset $(P,\leq)$ is called *a poset of finite length* if all chains of the poset $(P,\leq)$ are finite and if there exists the maximum of their lengths. We write $l(P)=l_0$ if the maximum of all chains of the poset $(P,\leq)$ is l_0 and the number l_0 is called the *length of the poset* $(P,\leq)$. A poset $(P,\leq)$ is called *a poset of locally finite lenght* if all bounded chains in $(P,\leq)$ are finite. Note, it is possible that a poset of locally finite length has neither maximal nor minimal elements.

Let $(P,\leq)$ be a poset. A real valued function v defined on P is called

a) *a positive valuation* on a poset $(P,\leq)$ if

$$a<b \quad\Longrightarrow\quad v(a)<v(b),$$

b) *an isotone valuation* on a poset $(P,\leq)$ if

$$a\leq b \quad\Longrightarrow\quad v(a)\leq v(b).$$

Throughout the paper we will denote by Z and R the set of all integers and the set of all reals, respectively. We will denote by $|x|$ the absolute value of x.

1. Valuations and metrics on posets

The aim of this part is to characterize metrics induced by isotone (positive) valuations on posets. First we will give some definitions needed for our purposes.

Definition 1.1. Let $(P,\leq)$ be a poset. A finite sequence $(x_i)_{i=0}^{n}$ is said to be a way from a to b if

(a) $x_0=a$, $x_n=b$,

(b) $x_i\leq x_{i+1}$ or $x_i\geq x_{i+1}$ for each $i=0,1,\dots,n-1$.

Definition 1.2. Let $(P,\leq)$ be a poset and $\mathbf{C}_n=(a_1,b_1,\dots,a_n,b_n)$ be a 2n-element subposet of $(P,\leq)$. The subposet $\mathbf{C}_n$ is said to be a cycle-fence of $(P,\leq)$ if

$$a_1<b_1>a_2<b_2>\cdots>a_n<b_n>a_1. \tag{1.0}$$

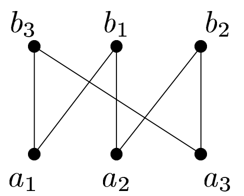

Fig. 2a

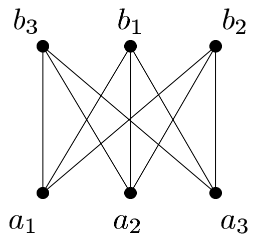

Fig. 2b

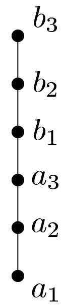

Fig. 2c

Note that the mentioned poset $\mathbf{C}_n$ is a crown if $n > 1$ and (1.0) are the only comparability relations on $\{a_1, b_1, \ldots, a_n, b_n\}$ (see Fig. 2a). The cycle-fences (by the above definition) are also the posets in Figures 2b and 2c.

Definition 1.3. Let $(P, \leq)$ be a poset, v an isotone valuation on $(P, \leq)$ and $(x_i)_{i=0}^n$ a way from a to b, $a, b \in P$. We define the length $l(x_0, x_1, \ldots, x_n)$ of the way $(x_i)_{i=0}^n$ by

$$(1.1) \qquad l(x_0, x_1, \ldots, x_n) =: \sum_{i=0}^{n-1} |v(x_i) - v(x_{i+1})|$$

A way $(x_0, x_1, \ldots, x_n)$ from a to b is said to be minimal if

$$l(x_0, x_1, \ldots, x_n) \leq l(y_0, y_1, \ldots, y_m)$$

for every way $(y_0, y_1, \ldots, y_m)$ from a to b.

Evidently, it is possible that there is no way from a to b or there exists a way from a to b but there is no minimal way from a to b.

Definition 1.4. An isotone valuation v on a connected poset $(P, \leq)$ is said to be a distance-valuation if there exists a minimal way from a to b for all $a, b \in P$.

Definition 1.5. Let v be a distance-valuation on a connected poset $(P, \leq)$. We define a non-negative real function $d_v : P \times P \longrightarrow R$ by

$$(1.2) \qquad d_v(a, b) =: l(x_0, x_1, \ldots, x_n)$$

where $(x_i)_{i=0}^n$ is a minimal way from a to b. The function d_v will be called the distance function induced by the distance-valuation v on the poset $(P, \leq)$.

Lemma 1.6. *Let d_v be a distance function induced by a distance-valuation v on a connected poset $(P, \leq)$. Then for all $a, b \in P$*

$$(1.3) \qquad a < b \implies d_v(a, b) = v(b) - v(a).$$

Proof. It is sufficient to prove that $(x_0, x_1) = (a, b)$ is a minimal way from a to b. Let $(x_i)_{i=0}^n$ be an arbitrary way from a to b. Then
$l(a, x_1, \ldots, x_{n-1}, b) = |v(a) - v(x_1)| + |v(x_1) - v(x_2)| + \cdots + |v(x_{n-1}) - v(b)| \geq$
$\geq |v(a) - v(x_1) + v(x_1) - v(x_2) + \cdots + v(x_{n-1}) - v(b)| = |v(a) - v(b)| = l(a, b)$, i.e. (a, b) is a minimal way from a to b. $\square$

Corollary 1.7. *Let $d_v : P \times P \longrightarrow R$ be a distance function induced by a distance-valuation v on a connected poset $(P, \leq)$. Let $(m_0, m_1, \ldots, m_{k-1}, m_k)$ be a minimal way from a to b. Then*

$$(1.4) \qquad d_v(a, b) = d_v(m_0, m_1) + d_v(m_1, m_2) + \cdots + d_v(m_{k-1}, m_k).$$

Lemma 1.8. *Let d_v be a distance function induced by a distance-valuation v on a connected poset $(P, \leq)$. Then*

$$\forall a, b, c \in P \quad a < b < c \quad \Longrightarrow \quad d_v(a, c) = d_v(a, b) + d_v(b, c) \tag{1.5}$$

and

$$\begin{aligned} d_v(a_1, b_1) + d_v(a_2, b_2) + \cdots + d_v(a_n, b_n) = \\ = d_v(b_1, a_2) + d_v(b_2, a_3) + \cdots + d_v(b_n, a_1) \end{aligned} \tag{1.6}$$

holds for every cycle-fence $(a_1, b_1, \ldots, a_n, b_n)$ of the poset $(P, \leq)$.

Proof. It is easy to verify that (1.5) and (1.6) follow from (1.3). □

Theorem 1.9. *Let d_v be a distance function induced by a distance-valuation v on a connected poset $(P, \leq)$. Then*
d_v is a metric on the poset $(P, \leq)$ if v is a positive valuation,
d_v is a pseudometric on the poset $(P, \leq)$ if v is a isotone valuation.

Proof. Let $a, b, c \in P$. Obviously, $d_v(a, a) = l(a, a) = 0$. If $a \neq b$ and v is a positive valuation then $d_v(a, b) > 0$. If $(x_0, x_1, \ldots, x_n)$ is a way from a to b then $(y_0, y_1, \ldots, y_n) = (x_n, x_{n-1}, \ldots, x_0)$ is the way from b to a, hence $d_v(a, b) = d_v(b, a)$. Let $(x_0, x_1, \ldots, x_n)$, $(y_0, y_1, \ldots, y_m)$ be minimal ways from a to b and from b to c, respectively. Then $(x_0, x_1, \ldots, x_n, y_1, \ldots, y_m)$ is the way from a to c and consequently, $d_v(a, c) \leq d_v(a, b) + d_v(b, c)$. □

Theorem 1.10. *Let d be a metric (pseudometric) on a poset $(P, \leq)$ satisfying the following three conditions*

(i) *for all $a, b \in P$ there exists a (minimal) way $(m_0, m_1, \ldots, m_k)$ from a to b for which*
$$d(a, b) = d(m_0, m_1) + d(m_1, m_2) + \cdots + d(m_{k-1}, m_k),$$

(ii) $a < b < c \quad \Longrightarrow \quad d(a, c) = d(a, b) + d(b, c)$ *for all $a, b, c \in P$,*

(iii)
$$\begin{aligned} & d(a_1, b_1) + d(a_2, b_2) + \cdots + d(a_n, b_n) = \\ & = d(b_1, a_2) + d(b_2, a_3) + \cdots + d(b_n, a_1) \end{aligned}$$
for every cycle-fence $(a_1, b_1, \ldots, a_n, b_n)$ of the poset $(P, \leq)$.

Then there exists a positive (isotone) distance-valuation v_d on $(P, \leq)$ and $d_{v_d} = d$.

Proof. By (ii) we can consider for noncomparable elements a, b only such ways $(x_0, x_1, \ldots, x_n)$ from a to b for which

$$x_i < x_{i+1} > x_{i+2} \quad \text{or} \quad x_i > x_{i+1} < x_{i+2} \tag{1.7}$$

for each $i = 0, 1, \ldots, n-2$, i.e. so-called fence ways.

Fix an element $c \in P$. Let $a \in P$. We define the valuation v_d on P by

$$(1.8) \qquad v_d(a) =: \begin{cases} v_d(c) + d(c, x_1) - d(x_1, x_2) + \cdots + (-1)^{n+1} d(x_{n-1}, a), \\ \text{if } c < x_1 \\ \\ v_d(c) - d(c, x_1) + d(x_1, x_2) - \cdots + (-1)^{n} d(x_{n-1}, a), \\ \text{if } c > x_1 \end{cases}$$

where $v_d(c)$ is any but fixed value from R and $(x_0, x_1, \ldots, x_{n-1}, x_n)$ is any fence way from c to a.

Now we will show that the function v_d is well-defined, i.e. $v_d(a)$ does not depend on a choice of the way from c to a and that v_d is the positive (isotone) valuation.

Let $(x_0, x_1, \ldots, x_{n-1}, x_n)$, $(y_0, y_1, \ldots, y_{m-1}, y_m)$ be two arbitrary fence ways from c to a (i.e. $c = x_0 = y_0$, $a = x_n = y_m$). Then one of the following subposet is a cycle-fence

1) $(c, x_1, \ldots, x_{n-1}, a, y_{m-1}, \ldots, y_1)$,
2) $(c, x_1, \ldots, x_{n-1}, y_{m-1}, \ldots, y_1)$,
3) $(x_1, \ldots, x_{n-1}, a, y_{m-1}, \ldots, y_1)$,
4) $(x_1, \ldots, x_{n-1}, y_{m-1}, \ldots, y_1)$.

In the first case we distinguish

1a) $c < x_1$ and $c < y_1$ and $a < x_{n-1}$ and $a < y_{m-1}$ (see Fig 3),
1b) $c < x_1$ and $c < y_1$ and $a > x_{n-1}$ and $a > y_{m-1}$,
1c) $c > x_1$ and $c > y_1$ and $a < x_{n-1}$ and $a < y_{m-1}$,
1d) $c > x_1$ and $c > y_1$ and $a > x_{n-1}$ and $a > y_{m-1}$.

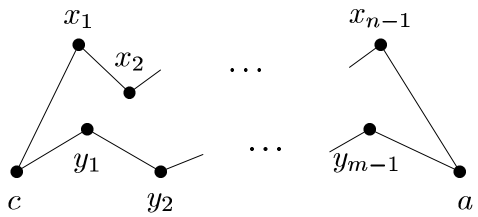

Fig. 3

(We illustrate in Fig. 3 only comparable relations among the elements of the fence ways $(x_0, x_1, \ldots, x_n)$ and $(y_0, y_1, \ldots, y_m)$, respectively.)

Let 1a) hold. By (iii) we get

$d(c, x_1) + d(x_2, x_3) + \cdots + d(x_{n-2}, x_{n-1}) + d(y_{m-1}, a) + d(y_{m-2}, y_{m-3}) +$
$+ \cdots + d(y_1, y_2) =$
$= d(x_1, x_2) + d(x_3, x_4) + \cdots + d(x_{n-1}, a) + d(y_{m-1}, y_{m-2}) + \cdots + d(y_1, c)$.

It implies

$$\begin{aligned} v_d(a) =& v_d(c) + d(c, x_1) - d(x_1, x_2) + d(x_2, x_3) - \cdots + d(x_{n-2}, x_{n-1}) - d(x_{n-1}, a) = \\ =& v_d(c) + d(c, y_1) - d(y_1, y_2) + d(y_2, y_3) - \cdots + d(y_{m-2}, y_{m-1}) - d(y_{m-1}, a), \end{aligned}$$

i.e. $v_d(a)$ does not depend on the choice of a way from c to a.
The other cases 1b), 1c), 1d) can be handled in the same way. Analogously to the case 1), we can distinguish four subcases for each of the cases 2), 3) and 4).
For example, let $(x_1, x_2, \ldots, x_{n-1}, y_{m-1}, \ldots, y_1)$ be a cycle-fence (the case 4)) and let $x_1 > c > y_1$ and $y_{m-1} > a > x_{n-1}$ (see Fig. 4). By (iii) we obtain

$$\begin{aligned} & d(x_1, x_2) + d(x_3, x_4) + \cdots + d(x_{n-2}, x_{n-1}) + d(y_{m-1}, y_{m-2}) + \cdots + d(y_2, y_1) = \\ & = d(y_1, x_1) + d(x_2, x_3) + \cdots + d(x_{n-1}, y_{m-1}) + d(y_{m-2}, y_{m-3}) + \cdots + d(y_3, y_2), \end{aligned}$$

where $d(y_1, x_1) = d(y_1, c) + d(c, x_1)$ and $d(x_{n-1}, y_{m-1}) =$ $= d(x_{n-1}, a) + d(y_{m-1}, a)$ by (ii).

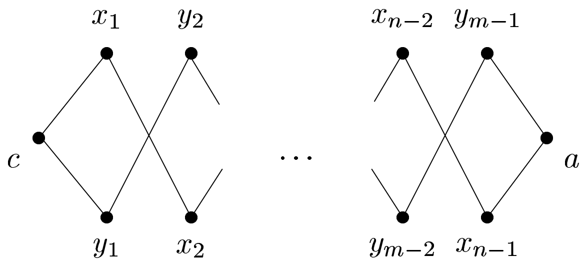

Fig. 4

Therefore,

$$\begin{aligned} v_d(a) =& v_d(c) + d(c, x_1) - d(x_1, x_2) + d(x_2, x_3) - \cdots + d(x_{n-1}, a) = \\ =& v_d(c) - d(c, y_1) + d(y_1, y_2) - d(y_2, y_3) + \cdots - d(y_{m-1}, a). \end{aligned}$$

Now we are going to prove that v_d is a positive (isotone) valuation.
Let $b < a$ and let $(x_0, x_1 \ldots, x_n)$, $(y_0, y_1, \ldots, y_m)$ be fence ways from c to a and from c to b, respectively. Similarly as above we can distinguish some cases.

1) $(c, x_1, \ldots, x_{n-1}, a, y_{m-1}, \ldots, y_1)$ is a cycle-fence and $c < x_1$, $c < y_1$, $x_{n-1} < a$, $y_{m-1} < b$ (Fig. 5).

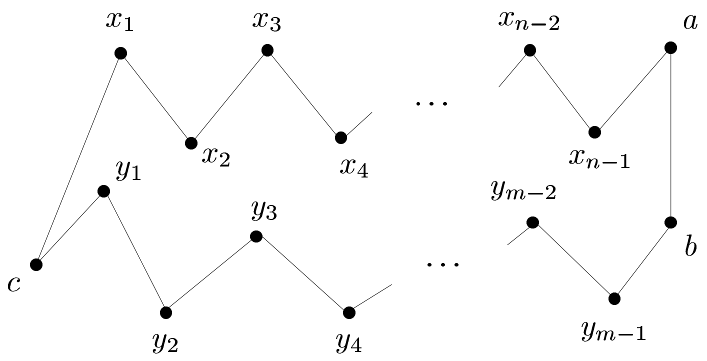

Fig. 5

Then we get by (iii)
$d(c,x_1)+d(x_2,x_3)+\cdots+d(x_{n-1},a)+d(y_{m-1},y_{m-2})+\cdots+d(y_3,y_4)+d(y_1,y_2)-$
$-d(x_1,x_2)-d(x_3,x_4)-\cdots-d(x_{n-2},x_{n-1})-d(a,y_{m-1})-d(y_{m-3},y_{m-2})-\cdots-d(y_2,y_3)-$
$d(y_1,c)=0$
and
$d(y_{m-1},a)=d(y_{m-1},b)+d(a,b)$ by (ii).
It implies
$v_d(a)-v_d(b)=v_d(c)+d(c,x_1)-d(x_1,x_2)+d(x_2,x_3)-\cdots-d(x_{n-2},x_{n-1})+d(x_{n-1},a)-$
$v_d(c)-d(c,y_1)+d(y_1,y_2)-d(y_2,y_3)+\cdots+d(y_{m-2},y_{m-1})-d(y_{m-1},b)=d(a,b)$ and
$d(a,b)>0$ if d is a metric and $d(a,b)\geq 0$ if d is a pseudometric. Thus $v_d(a)>v_d(b)$, $v_d(a)\geq v_d(b)$, respectively.
The other subcases of the case 1) can be handled in the same way.

2) Let $(x_1,\ldots,x_{n-1},y_{m-1},y_{m-2},\ldots,y_1)$ be a cycle-fence and $y_1<c<x_1$, $y_{m-1}<b<a<x_{n-1}$ (Fig. 6).

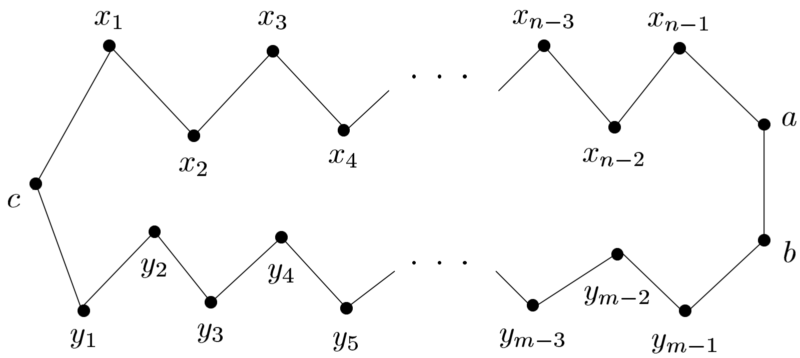

Fig. 6

By (iii) holds
$d(y_1,x_1)+d(x_2,x_3)+\cdots+d(x_{n-2},x_{n-1})+d(y_{m-1},y_{m-2})+d(y_{m-3},y_{m-4})+\cdots+$
$+d(y_2,y_3)-d(x_1,x_2)-d(x_3,x_4)-\cdots-d(x_{n-3},x_{n-2})-d(x_{n-1},y_{m-1})-d(y_{m-2},y_{m-3})-$

$\cdots - d(y_3, y_4) - d(y_1, y_2) = 0$

and by (ii)

$d(y_1, x_1) = d(y_1, c) + d(c, x_1)$ and $d(x_{n-1}, y_{m-1}) = d(x_{n-1}, a) + d(a, b) + d(b, y_{m-1})$.

Hence

$v_d(a) - v_d(b) = v_d(c) + d(c, x_1) - d(x_1, x_2) + d(x_2, x_3) - \cdots - d(x_{n-3}, x_{n-2}) + d(x_{n-2}, x_{n-1}) - d(x_{n-1}, a) - v_d(c) + d(c, y_1) - d(y_1, y_2) + d(y_2, y_3) - \cdots - d(y_{m-3}, y_{m-2}) + d(y_{m-2}, y_{m-1}) - d(y_{m-1}, b) = d(a, b)$,

i.e. $v_d(a) > v_d(b)$ if d is a metric and $v_d(a) \geq v_d(b)$ if d is a pseudometric.

The proofs in all other cases can be done analogously.

We are going to show that $d_{v_d} = d$.

Let $a, b \in P$ and let $(m_0, m_1, \ldots, m_k)$ be a way from a to b for which (i) holds. Let, for example, $a < m_1$ and $m_{k-1} < b$. Then

$$\begin{aligned} d_{v_d}(a, b) =& (v_d(m_1) - v_d(a)) + (v_d(m_1) - v_d(m_2)) + \cdots + (v_d(b) - v_d(m_{k-1})) = \\ =& (v_d(a) + d(a, m_1) - v_d(a)) + (v_d(a) + d(a, m_1) - v_d(a) - d(a, m_1) + \\ &+ d(m_1, m_2)) + \cdots + (v_d(a) + d(a, m_1) - d(m_1, m_2) + \cdots - \\ &- d(m_{k-2}, m_{k-1}) + d(m_{k-1}, b) - v_d(a) - d(a, m_1) + \\ &+ d(m_1, m_2) - \cdots + d(m_{k-2}, m_{k-1})) = \\ =& d(a, m_1) + d(m_1, m_2) + \cdots + d(m_{k-1}, b) = d(a, b) \end{aligned}$$

The proof is complete. □

Theorem 1.11. *Let d_v be a metric (pseudometric) induced by a positive (isotone) distance-valuation v on a poset $(P, \leq)$ and $c \in P$ be an arbitrary but fixed element. Let $a \in P$. We define (in the same way as in (1.8)) the valuation v_{d_v} on P by*

$v_{d_v}(c) =: v(c)$

$$v_{d_v}(a) =: \begin{cases} v(c) + d_v(c, x_1) - d_v(x_1, x_2) + d_v(x_2, x_3) - \cdots + (-1)^{n+1} d_v(x_{n-1}, a), \\ \text{if } c < x_1 \\ \\ v(c) - d_v(c, x_1) + d_v(x_1, x_2) - d_v(x_2, x_3) + \cdots + (-1)^{n} d_v(x_{n-1}, a), \\ \text{if } c > x_1 \end{cases}$$

where $(x_0, x_1, \ldots, x_{n-1}, x_n)$ is any fence way from c to a.
Then v_{d_v} is the positive (isotone) valuation on the poset $(P, \leq)$ and $v_{d_v} = v$.

Proof. We can show that v_{d_v} is correctly defined and v_{d_v} is a positive (isotone) valuation by the same way as in the proof of Theorem 1.10. It is immediate that $v_{d_v} = v$. □

2. Positive valuations on connected posets of locally finite length

In this section we apply the above results in order to show that there exist positive valuations on some connected graded posets of locally finite length. Particularly, we will deal with modular posets (multilattices).

Let $(P, \leq)$ be a poset. A graph $C(P) = (P, E)$ is called *the covering graph associated with the poset* $(P, \leq)$, if the edge set E consists of the pairs ab for which a covers b in $(P, \leq)$.

Let a, b be vertices of a covering graph $C(P)$ of a poset $(P, \leq)$. Let $W = (a_0, a_1, \ldots, a_n)$ be a finite sequence mutually different vertices of the graph $C(P)$. We call that W is *the way from a to b* (in $C(P)$) if

(j) $a = a_0$, $b = a_n$ and

(jj) a_i, a_{i+1} are adjacent vertices of the graph $C(P)$, for each $i = 0, 1, \ldots, n-1$ (i.e. a_i covers a_{i+1} or a_{i+1} covers a_i in the poset $(P, \leq)$).

The number n is called *the length of the way W*. *The distance* of vertices a and b in a covering graph $C(P)$ we mean the length of the shortes way from a to b (if it exists). We write $d(a, b) = d_0$ if the distance of the vertices a, b is d_0.

Let $(P, \leq)$ be a poset. In this section we will denote by d the distance function $d : P \times P \longrightarrow Z$ defined above (i.e. $d(a, b)$ is the distance of the vertices a, b in the covering graph $C(P)$ of the poset $(P, \leq)$).

Let $(P, \leq)$ be a connected poset of locally finite length. Since the set of all non-negative integers is well ordered, for every $a, b \in P$ there exists a fence way from a to b of shortest length. Thus, the function d is the metric on P and moreover the metric d satisfies the condition (i) (Theorem 1.10). On the other hand the metric d do not need satisfy the conditions (ii) and (iii) (see Fig. 7).

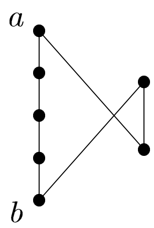

Fig. 7

Lemma 2.1. *Let $(P, \leq)$ be a connected poset of locally finite length. If the metric d satisfies the condition (ii) (Theorem 1.10) then the poset $(P, \leq)$ is graded.*

Proof. Let $a, b \in P$, $a < b$. The interval $[a, b]$ is the subposet of $(P, \leq)$ with the least element a. If $x, y \in [a, b]$ and y covers x we have

$$d(a, y) = d(a, x) + d(x, y) = d(a, x) + 1$$

by (ii). It implies that all maximal chains of the interval $[a, b]$ are of the same length. □

Remark. The poset depicted in Fig. 7 is graded but this poset does not satisfy the condition (ii).

Theorem 2.2. *Let $(P,\leq)$ be a connected poset of locally finite length. If the metric d the conditions (ii) and (iii) satisfies then there exists a positive distance-valuation v on the poset $(P,\leq)$.*

Proof. We can define the positive distance-valuation v on the poset $(P,\leq)$ by Theorem 1.10. $\square$

Let $(P,\leq)$ be a graded poset of locally finite length and $a<b,\ a,b\in P$. In this section we will denote by $l(a,b)$ the length of a maximal chain (i.e. of all maximal chains) of the interval $[a,b]$ of the graded poset $(P,\leq)$. Note it is possible that $l(a,b)\neq d(a,b)$ (for a, b in Fig. 7 we have $l(a,b)=4>d(a,b)=3$).

Theorem 2.3. *Let $(P,\leq)$ be a directed graded poset of locally finite length. There exits a positive distance-valuation v on the poset $(P,\leq)$ for which*

$$a<b \implies v(b)=v(a)+l(a,b)=v(a)+d(a,b). \tag{2.1}$$

Proof. Let $(a_1,b_1,\ldots,a_n,b_n)$ be a cycle-fence of the poset $(P,\leq)$. Let w and u be a lower bound and an upper bound of the poset $\{a_1,b_1,\ldots,a_n,b_n\}$, respectively. Since the poset $(P,\leq)$ is graded we have

$$\begin{aligned}&l(a_1,b_1)+\cdots+l(a_n,b_n)=\\ =&l(w,u)-l(w,a_1)-l(b_1,u)+\cdots+l(w,u)-l(w,a_n)-l(b_n,u)=\\ =&l(a_2,b_1)+l(a_3,b_2)+\cdots+l(a_1,b_n).\end{aligned}$$

Let $a<b$, $a,b\in P$. Let $(a,x_1,\ldots,x_{n-1},b)$ be a minimal fence way from a to b for which

$$d(a,b)=l(a,x_1)+l(x_1,x_2)+\cdots+l(x_{n-1},b).$$

The result of the first part of the proof implies (e.g. $x_1<a$)

$$l(a,b)=-l(a,x_1)+l(x_1,x_2)-l(x_2,x_3)+\cdots+(-1)^n l(x_{n-1},b)$$

(see Fig. 8).

Thus, $l(a,b)\leq d(a,b)$, i.e. $l(a,b)=d(a,b)$. Now it is immediate that (ii) and (iii) hold for the metric d.

We may now define the distance-valuation v by (1.8). Obviously, the valuation v is positive and (2.1) holds. $\square$

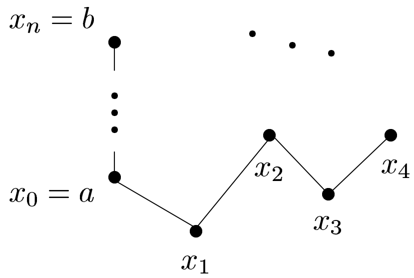

Fig. 8

Let $(M, \leq)$ be a poset and $a, b \in M$. Let $a \vee b$ be the set of all minimal elements of the set of all upper bounds of the set $\{a, b\}$ and $a \wedge b$ be the set of all maximal elements of the set of all lower bounds of the set $\{a, b\}$. We call a poset $(M, \leq)$ a multilattice if for every upper bound h the set $\{t;\, t \in a \vee b \;\;\&\;\; t \leq h\}$ is non-empty and for every lower bound k the set $\{t;\, t \in a \wedge b \;\;\&\;\; t \geq k\}$ is also non-empty for all $a, b \in M$. It is easy to verify that every poset of locally finite length is a multilattice.

A multilattice $(M, \leq)$ is said to be modular if whenever $a, b, c \in M$

$$(a \vee b) \cap (a \vee c) \neq \emptyset \;\;\&\;\; (a \wedge b) \cap (a \wedge c) \neq \emptyset \;\;\&\;\; b \leq c \quad \Longrightarrow \quad b = c.$$

Every modular multilattice of locally finite length is a graded poset.

Lemma 2.4. *Let* **M** *be a modular directed multilattice of locally finite length. If* $a, b \in M$, $u \in a \vee b$, $w \in a \wedge b$ *then* (a, u, b) *and* (a, w, b) *are minimal ways from* a *to* b *(i.e.* $d(a, b) = l(a, u) + l(b, u) = l(w, a) + l(w, b)$*).*

Proof. By Theorem 2.3 $d(a, b) = l(a, b)$ for any comparable elmements $a, b \in M$. Let a, b be two noncomparable elements and let $(a, x_1, \ldots, x_{n-1}, b)$ be a way from a to b. Let, for instance, $a > x_1 < x_2$ and let $x_1 \notin a \wedge x_2$ (Fig. 9).

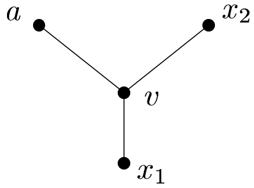

Fig. 9

Then there exists an element $v \in a \wedge x_2$, $v > x_1$ and the way $(a, v, x_2, \ldots, x_{n-1}, b)$ is shorter than the way $(a, x_1, \ldots, x_{n-1}, b)$. In the next part of the proof we will consider only fence ways $(a, x_1, \ldots, x_{n-1}, b)$ from a to b for which

$$x_i \in x_{i-1} \wedge x_{i+1} \quad \text{or} \quad x_i \in x_{i-1} \vee x_{i+1} \tag{2.2}$$

holds for each $i = 1, \ldots, n-1$. We will do the proof by induction.

Let $n = 2$. We distinguish two cases. Let $a > x_1 < b$. Then $x_1 \in a \wedge b$ and $l(a, x_1) = l(u, b)$ and $l(x_1, b) = l(a, u)$, because the multilattice is modular (see [5]). It implies that $d(a, u, b) = d(a, x_1, b)$. Let $a < x_1 > b$. Then $x_1 \in a \vee b$ and $u \in a \vee b$, too. The multilattice is modular, therefore $l(a, x_1) = l(w, b) = l(a, u)$ and $l(b, x_1) = l(w, a) = l(b, u)$, hence $l(a, x_1) + l(b, x_1) = l(a, u) + l(b, u)$, again.

Assume that for the lengths of all n-element fence ways from a to b the statement holds. We prove it for n+1-element fence ways.

Let $(a, x_1, \ldots, x_{n-1}, b)$ be an $(n+1)$-element fence way. For instance, let $a > x_1 < x_2 > \cdots < x_{n-1} > b$ (Fig. 10). If $y \in a \vee x_2$ then by induction hypothesis $l(a, y) + l(y, x_3) + \cdots + l(x_{n-1}, b) \geq d(a, u) + d(b, u)$. Because $l(a, x_1) = l(x_2, y)$ and $l(x_1, x_2) = l(a, y)$, we have

$$\begin{aligned} &l(a, x_1) + l(x_1, x_2) + l(x_2, x_3) + \cdots + l(x_{n-1}, b) = \\ =&l(a, y) + l(y, x_2) + l(x_2, x_3) + \cdots + l(x_{n-1}, b) = \\ =&l(a, y) + l(x_3, y) + + \cdots + l(x_{n-1}, b) \geq d(a, u) + d(b, u). \end{aligned}$$

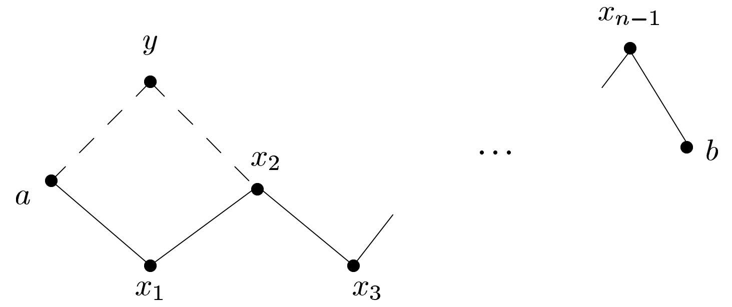

Fig. 10

Analogously, we can show the inequality in the other cases. The statement holds for a lower bound w by duality. □

Corollary 2.5. *Let* $\mathbf{M}$ *be a modular directed multilattice of locally finite length. There exists a positive valuation* v *on the multilattice* $\mathbf{M}$ *for which*

$$v(a) + v(b) = v(u) + v(w) \tag{2.3}$$

for all $a, b \in M$, $u \in a \vee b$, $w \in a \wedge b$.

Proof. We can define a positive distance-valuation v on the poset $\mathbf{M}$ by Theorem 2.3. The valuation v satifies (2.1). So, for every $a, b \in M$, $u \in a \vee b$, $w \in a \wedge b$ we have

$$\begin{aligned} v(a) + v(b) &= v(w) + l(w,a) + v(w) + l(w,b) = \\ &= v(w) + v(w) + l(w,a) + l(a,u) = \\ &= v(w) + v(u). \end{aligned}$$

□

In [2], by a valuation on a lattice L is meant a function $v : L \longrightarrow R$ for which (2.3) holds. This definition was accepted by authors of [5] and [6], too. If L is a modular multilattice of locally finite length, we have for the induced metric d_v

$$\begin{aligned} d_v(a,b) &= l(a,u) + l(b,u) = l(a,u) + l(w,a) = \\ &= v(u) - v(a) + v(a) - v(w) = v(u) - v(w) = \\ &= d_v(w,u) \end{aligned}$$

whenever $u \in a \vee b$, $w \in a \wedge b$.

Definition 2.6. Let $(P, \leq)$ be a poset. A positive valuation v on the poset $(P, \leq)$ is said to be a modular valuation if for every $a, b \in P$

$$\begin{aligned} & a \vee b \neq \emptyset \ \ \& \ \ a \wedge b \neq \emptyset \ \ \& \ \ u \in a \vee b \ \ \& \ \ w \in a \wedge b \\ & \qquad \Longrightarrow \quad v(a) + v(b) = v(u) + v(w). \end{aligned} \tag{2.4}$$

Two intervals $[a, b], [c, d]$ (of a poset $(P, \leq)$) are said to be *transposed* if either $a \in b \wedge c$, $d \in b \vee c$ or $c \in a \wedge d$, $b \in a \vee d$. The intervals I, J are *projective* if there exists a finite sequence of intervals $I = I_0, I_1, \ldots, I_n = J$ such that all adjoining intervals I_i, I_{i+1} are transposed.

Definition 2.7. Let $(P, \leq)$ be a connected poset of locally finite length. The poset $(P, \leq)$ is said to be a modular poset if the next two conditions are satisfied

(k) the lengths of every two projective intervals are equal

(kk) the metric d satisfies (ii) and (iii).

Theorem 2.8. *A connected graded poset $(P, \leq)$ of locally finite length is modular if and only if there exists a modular valuation v on $(P, \leq)$ satisfying (2.1).*

Proof. a) Let P be a modular connected poset of locally finite length. The metric d satisfies (i), (ii) and (iii), hence we can define a positive distance-valuation v on the poset $(P, \leq)$ by (1.8). If $u \in a \vee b$, $w \in a \wedge b$ then according to (1.8) we get

$$v(w) + v(u) = v(w) + v(w) + d(w, u) = 2v(w) + d(w, a) + d(a, u).$$

The intervals $[a, u]$, $[w, b]$ are projective, for this reason $d(a, u) = d(w, b)$ and we get

$$v(w) + v(u) = 2v(w) + d(w, a) + d(w, b) = v(a) + v(b)$$

Obviously, the valuation v satisfies (2.1).

b) Let there exist a modular valuation v satisfying (2.1) on a poset P. Let $[a, b]$, $[c, d]$ be two transposed intervals and let $a \in b \wedge c$, $d \in b \vee c$. From $v(b) + v(c) = v(a) + v(d)$ we get $v(b) - v(a) = v(d) - v(c)$, i.e. $d(a, b) = d(c, d)$ by (2.1). The transitivity of equality implies (k).

Let $a < b < c$. Using (2.1) we get

$$\begin{aligned} d(a, c) = v(c) - v(a) &= v(c) - v(b) + v(b) - v(a) = \\ &= d(a, b) + d(b, c). \end{aligned}$$

Let $(a_1, b_1, \ldots, a_n, b_n)$ be cycle fence of the poset $(P, \leq)$. By (2.1)

$$\begin{aligned} d(a_1, b_1) + \cdots + d(a_n, b_n) &= v(b_1) - v(a_1) + \cdots + v(b_n) - v(a_n) = \\ &= d(b_1, a_2) + \cdots + d(b_n, a_1). \end{aligned}$$

□

The authors thank the referee for several valuable remarks according to which the second section was rewritten.

References

1. Benado, M., *Les ensembles partiellement ordenés et le théoreme de raffinement de Schreier II*, Čechoslov. mat. ž. 5 (80) (1955), 308-344.
2. Birkhoff, G., *Lattice Theory, 3^{rd} edition*, Amer. Math. Soc., 1967.

3. McDiarmid, C.J.H., *Bisubmodular functions, distributive valuations and distance functions in lattice*, Algebra Univ. **5** (1975), 253-255.
4. Duffus, D. and Rival, I., *Path length in the covering graph of a lattice*, Discr. Math. **19** (1977), 139-158.
5. Kolibiar, M. and Lihová, J., *Modular and metric multilattices*, Math. Slovaca **45** (1995), 19-27.
6. Lihová, J., *Valuations and distance functions on directed multilattices*, Math. Slovaca **46** (1996), 143-155.

(Received October 20, 1996)

Dept. of Mathematics
Matej Bel University
Tajovského 40
974 01 Banská Bystrica
SLOVAKIA

E-mail address: haviar@fhpv.umb.sk
E-mail address: monosz@fhpv.umb.sk