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AFFINE COMPLETE ALGEBRAS ABSTRACTING DOUBLE STONE AND KLEENE ALGEBRAS

Miroslav Haviar

Abstract. In this paper we generalize R. Beazer's characterization of affine complete double Stone algebras with a non-empty bounded core [1] to the class of double K_2-algebras with a non-empty bounded core. These algebras have appeared in the literature as a common generalization of double Stone and Kleene algebras. We show that Post algebras of order 3 are the only locally affine complete (in a stronger sense of [12]) double K_2-algebras with a non-empty bounded core and the only finite affine complete double K_2-algebras. Then introducing some extension properties for congruence-preserving functions we characterize (infinite) affine complete double K_2-algebras with a non-empty bounded core. We finally derive the Beazer result for double Stone algebras.

1. Introduction. The problem of characterizing affine complete algebras was posed by G. Grätzer in [6] (Problem 6). Recall that an n-ary function f on an algebra A is *compatible* if for any congruence θ on A, $a_i \equiv b_i$ (θ) $(a_i, b_i \in A)$, $i = 1, \dots, n$ yields $f(a_1, \dots, a_n) \equiv f(b_1, \dots, b_n)$ (θ). A *polynomial* function of A is a function that can be obtained by composition of the basic operations of A, the projections and the constant functions. Clearly, all polynomial functions of A are compatible. An algebra A is called *affine complete* if the polynomial functions of A are the only compatible functions. Hence in general, affine complete algebras have 'many' congruences.

In [4] G. Grätzer proved that every Boolean algebra is affine complete. In [5] he showed that affine complete bounded distributive lattices are those which do not have proper Boolean subintervals. A list of particular varieties in which all affine complete members were characterized can be found in [3] and its up-to-date version in [10].

Two 'local' versions of affine completeness have been studied in the literature. A weaker notion of local affine completeness can be found e.g. in [16]. According to the stronger meaning of this concept, which we adopt here, an algebra A is *locally affine complete* if any finite partial function in $A^n \to A$ (i.e. function whose domain is a finite subset of A^n) which is compatible (where defined) can be interpolated by a polynomial of A (see e.g. [12]).

In [1] R. Beazer characterized affine complete algebras in the class of double Stone algebras with a non-empty bounded core. A generalization of this result, to the class of

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so-called quasi-modular double p-algebras, was presented in [9] where also locally affine complete Stone algebras (in the stronger sense of [12]) were characterized. Recently, affine complete Kleene algebras were successfully described in [11]. This allows us to investigate affine completeness in the class of double K_2-algebras which are known as a common generalization of double Stone and Kleene algebras [2]. This investigation, which uses techniques similar to those in [8] and [9], is the object of this paper.

First we show that for any double K_2-algebra L with a non-empty bounded core $K(L)$, (locally) affine completeness of L yields (locally) affine completeness of $K(L)$ as a bounded distributive lattice (Theorem 3.3). Consequently, we get that Post algebras of order 3 are the only locally affine complete double K_2-algebras with a non-empty bounded core and the only affine complete algebras among the double K_2-algebras with a finite skeleton and a finite non-empty core (Corollaries 3.6 and 3.9). Then we introduce some extension properties for compatible functions and with their use we reduce the question of characterizing (infinite) affine complete double K_2-algebras with non-empty bounded core to those questions for Kleene algebras and distributive lattices (Theorem 3.17) where the answers are already known. We finally derive from our characterization the Beazer result for double Stone algebras.

2. Preliminaries.

MS-algebras were introduced by T.S. Blyth and J.C. Varlet in the beginning of eighteees as a nice generalization of de Morgan and Stone algebras and have shown a fruitful development during the previous decade (cf. [2]).

Let us recall that an *MS-algebra* is an algebra $(L;\vee,\wedge,^\circ,0,1)$ of type (2,2,1,0,0) where $(L;\vee,\wedge,0,1)$ is a bounded distributive lattice and $^\circ$ is a unary operation such that for all $x,y\in L$

(1) x $\leq$ $x^{\circ\circ}$,

(2) $(x\wedge y)^\circ = x^\circ\vee\ y^\circ$,

(3) $1^\circ = 0$.

One can show that the following rules of computation hold further in L:

$(x\vee y)^\circ = x^\circ\wedge y^\circ$,

$x^{\circ\circ\circ} = x^\circ$,

$0^\circ = 1$.

The class of all MS-algebras is equational. The subvariety K_2 of MS-algebras is defined by two additional identities

(4) $\quad x\wedge x^\circ = x^{\circ\circ}\wedge x^\circ$ and

(5) $\quad (x\wedge x^\circ)\vee y\vee y^\circ = y\vee y^\circ$,

The subvariety M of de Morgan algebras is defined by the identity (6) $x = x^{\circ\circ}$. Another important subvarieties of MS-algebras are the subvarieties B, S, K of Boolean, Stone and Kleene algebras, respectively which are characterized by the identities $B:\ x\vee x^\circ = 1$, $S:$ $x\wedge x^\circ = 0$ and $K:(5),(6)$, respectively.

Let L be an MS-algebra from the subvariety K_2. Then

(i) $L^{\circ\circ} = \{x\in L;\ x = x^{\circ\circ}\}$ is a Kleene subalgebra of L ;

(ii) $L^\wedge = \{x\wedge x^\circ;\ x\in L\}$ is an ideal of L;

(iii) $L^\vee = \{x\vee x^\circ;\ x\in L\}$ is a filter of L.

Now we recall basic facts about *double* MS-algebras. A double MS-algebra is an algebra $(L; \vee, \wedge, ^\circ, ^+, 0, 1)$ of type (2,2,1,1,0,0) such that $(L; \vee, \wedge, ^\circ, 0, 1)$ is an MS-algebra, $(L; \vee, \wedge, ^+, 0, 1)$ is a dual MS-algebra, and the unary operations are linked by the identities

$x^{\circ +} = x^{\circ\circ}$ and $x^{+\circ} = x^{++}$.

Obviously, every de Morgan algebra $(L; \vee, \wedge, ^-, 0, 1)$ can be made into a double MS-algebra if one defines $x^\circ = x^+ = \overline{x}$. Conditions under which an MS-algebra can be made into a double MS-algebra are known (cf. [2]). It is proved that the subvarieties B, K, M of MS-algebras are *dense*, i.e. all algebras in these subvarieties can be made into double MS-algebras. Further, *bistable* subvarieties of MS-algebras are defined as those V that for every double MS-algebra $(L; \vee, \wedge, ^\circ, ^+, 0, 1)$, whenever $(L; \vee, \wedge, ^\circ, 0, 1) \in V$, then $(L; \vee, \wedge, ^+, 0, 1) \in V$ too. It is known which subvarieties of MS-algebras are bistable and which fail (cf. [2]). Among first, the subvarieties B, S, K, S∨K and M are included, among non-bistable one can find the subvariety K_2. It is true that the identity (5) implies the dual one, (5^d) $(x \vee x^+) \wedge y \wedge y^+ = y \wedge y^+$, however for the identity (4), which defines the subvariety $K_2 \vee M$, this is not the case. Therefore the variety of double K_2-algebras is defined by the identities (4), (5) and (4^d) $x \vee x^+ = x^{++} \vee x^+$.

It is known that there are precisely 22 non-isomorphic subdirectly irreducible double MS-algebras. The lattice of subvarieties of double MS-algebras has cardinality 381 (cf. [2]).

Some subsets of double MS-algebras play a significant role in investigations. By the *skeleton* $S(L)$ of a double MS-algebra L is meant a de Morgan algebra $L^{\circ\circ} = \{x \in L;\ x^{\circ\circ} = x\} = L^{++} = \{x \in L;\ x^{++} = x\}$. If L is a double K_2-algebra then $S(L)$ is a Kleene algebra. Further, in a double MS-algebra L, $x^\circ \leq x^+$ and consequently $x^{++} \leq x \leq x^{\circ\circ}$ hold for any element x. Therefore, the notion of the *core* $K(L)$, known for double Stone algebras, can be generalized for double K_2-algebras as follows:

$K(L) = \{x \vee x^\circ;\ x \in L\} \cap \{x \wedge x^+;\ x \in L\}$.

Double Stone algebras L which have a one-element core, $|K(L)| = 1$, are named *Post algebras of order 3*. They form a subclass (not a subvariety) of the variety of so-called three-valued Lukasiewicz algebras which are double Stone algebras defined by the identity $(x \wedge x^+) \vee (y \vee y^\circ) = y \vee y^\circ$ (cf. [1]). The variety of three-valued Lukasiewicz algebras is known to be arithmetical (i.e. congruence-distributive and congruence-permutable).

An important role is played by so-called *determination congruence* which is defined as follows:

$x \equiv y\ (\Phi^\circ_+)$ iff $x^\circ = y^\circ$ and $x^+ = y^+$.

For other properties of MS-algebras and double MS-algebras we refer the reader to [2].

We finally mention few facts concerning the affine completeness. We start with basic Grätzer's results.

2.1 Theorem ([4]). *Any Boolean algebra is affine complete.*

Let us recall that a function $f : L^n \to L$ on a lattice L is *order-preserving* if $x_i \leq y_i$ $(x_i, y_i \in L,\ i = 1, \dots, n)$ implies $f(x_1, \dots, x_n) \leq f(y_1, \dots, y_n)$ where $\leq$ is the lattice order. It is well-known that every polynomial function on a lattice is order-preserving.

2.2 Theorem ([5; Corollaries 1,3]). *Let L be a bounded distributive lattice. The following conditions are equivalent.*

(1) L is affine complete;

(2) every compatible function on L is order-preserving;

(3) L contains no proper Boolean interval.

Now we present the Beazer result for double Stone algebras and its immediate consequences.

2.3 Proposition ([1; Theorem 5]). *Let L be a double Stone algebra with a non-empty bounded core $K(L)$. The following conditions are equivalent.*

(1) L is affine complete;

(2) $K(L)$ is an affine complete distributive lattice;

(3) No proper interval of $K(L)$ is Boolean.

2.4 Corollary ([1; Corollary 6]). *Any Post algebra of order 3 is an affine complete double Stone algebra.*

2.5 Corollary ([1; Corollary 7]). *A finite double Stone algebra having a non-empty core is affine complete if and only if it is a Post algebra of order 3.*

In [8] it was shown that if a K_2-algebra L is affine complete then the filter $L^\vee$ is (as a lattice) affine complete, too. Since by 2.2 a finite distributive lattice $L^\vee$ is affine complete if and only if $|L^\vee| = 1$, we immediately get

2.6 Proposition ([8; Corollary 4]). *Let L be a K_2-algebra such that $L^\vee$ is finite. Then L is affine complete if and only if L is a Boolean algebra.*

In a Kleene algebra L, the filter $L^\vee$ is isomorphic to the ideal $L^\wedge$. Hence we have

2.7 Corollary. *Let L be a Kleene algebra such that $L^\wedge$ is finite. Then L is affine complete if and only if L is a Boolean algebra.*

The following few facts are considered to be a part of 'folklore':

2.8 Proposition. *If a lattice L contains a Boolean interval $[a, b]$ $(a < b)$, then L is not affine complete.*

Proof. Define a function $f : L \to [a, b]$ by $f(x) = ((x \vee a) \wedge b)'$, where $'$ denotes the complement in the Boolean interval [a,b]. For any non-trivial congruence $\theta \in \mathrm{Con}(L)$ and $x \equiv y\ (\theta)$ $(x,\ y \in L)$ we have $((x \vee a) \wedge b)' \equiv ((y \vee a) \wedge b)'\ (\theta)$, i.e. f is a compatible function of L. But f is not order-preserving because $f(a) = b,\ f(b) = a$, therefore f cannot be represented by a lattice polynomial. Hence L is not affine complete. □

2.2 implies that a finite distributive lattice L is affine complete if and only if $|L| = 1$. Now 2.8 yields that the assumption about distributivity of L can even be dropped.

2.9 Corollary. *A finite lattice L is affine complete if and only if $|L| = 1$.* □

2.10 Proposition. *For any lattice L the following are equivalent:*

(1) *L is locally affine complete;*
(2) *every finite partial compatible function of L is order-preserving;*
(3) $|L| = 1$.

Proof. (1) $\Longrightarrow$ (3): Let L be locally affine complete and let $a, b \in L,\ a < b$. The function $f = \{(a, b), (b, a)\}$ is a finite partial compatible function on L, thus by hypothesis it can be interpolated on $\{a, b\}$ by a polynomial of L, which is an order-preserving function. But we have $f(a) = b,\ f(b) = a$, a contradiction.

(2) $\Longrightarrow$ (3): If $|L| \neq 1$ then we can define the same partial function f as above which contradicts (2).

The rest of the proof is trivial. $\square$

The following result (see [13] or [14]) characterizes those varieties of which all members are locally affine complete as arithmetical.

2.11 Theorem. *A variety V is arithmetical if and only if for each algebra $A \in V$, a finite partial function f on A can be interpolated by a polynomial function of A just in the case f is Con(A)-compatible.*

Since the class of the Post algebras of order 3 is contained in the arithmetical variety of the three-valued Lukasiewicz algebras, we immediately get

2.12 Corollary. *Every Post algebra of order 3 is locally affine complete.*

We conclude with a technical lemma which will be applied in Section 3 (for D being a Boolean algebra and a bounded distributive lattice, respectively; its proof, which can be found in [7] or [9], will be repeated here as it is not long and we want this paper to be self-contained.)

2.13 Lemma. *Let D be any algebra such that its reduct is a bounded distributive lattice $(D, \vee, \wedge, 0, 1)$ and the algebra D is a subdirect product of 2-element algebras. Let $f', g' : D^n \to D$ be (partial) compatible functions with domains F and G ($F, G \subseteq D^n$), respectively, let $S := F \cap G$ and let $S \cap \{0, 1\}^n \neq \varnothing$. For any $(0, 1)$-homomorphism $h : D \to \{0, 1\}$ between the algebra D and a 2-element algebra $\{0, 1\}$, denote $h(S) := \{(h(x_1), \ldots, h(x_n)) \in \{0, 1\}^n;\ (x_1, \ldots, x_n) \in S\}$ and let $h(S) = h(S \cap \{0, 1\}^n)$ hold for every such h. Then $f' \equiv g'$ identically on S if and only if $f' \equiv g'$ identically on $S \cap \{0, 1\}^n$.*

Proof. Let $f' \equiv g'$ identically on $S \cap \{0, 1\}^n$. Suppose on the contrary that there exists an n-tuple $(d_1, \ldots, d_n) \in S$ such that $f'(d_1, \ldots, d_n) = a \neq b = g'(d_1, \ldots, d_n)$. Since $a \neq b$ in D which is a subdirect product of 2-element algebras, there exists a 'projection map' $h : D \to \{0, 1\}$, which is a $(0, 1)$-homomorphism between the algebra D and some algebra $\{0, 1\}$, such that $h(a) \neq h(b)$. Define functions $f'_2,\ g'_2 : h(S) \to \{0, 1\}$ by the following rules:

$f'_2(h(x_1), \ldots, h(x_n)) = h(f'(x_1, \ldots, x_n))$,

$g'_2(h(x_1), \ldots, h(x_n)) = h(g'(x_1, \ldots, x_n))$ where $(x_1, \ldots, x_n) \in S$.

Obviously, $f'_2,\ g'_2$ are well-defined, since f', g' preserve the kernel congruence of the homomorphism h. Obviously, $f'_2 \equiv g'_2$ identically on $h(S)$, because $h(S) = h(S \cap \{0, 1\}^n)$, $h(0) = 0,\ h(1) = 1$ and $f' \equiv g'$ identically on $S \cap \{0, 1\}^n$. Therefore

$h(a) = h(f'(d_1, \ldots, d_n)) = f_2'(h(d_1), \ldots, h(d_n)) = g_2'(h(d_1), \ldots, h(d_n)) = h(g'(d_1, \ldots, d_n)) = h(b)$, a contradiction. Hence $f' \equiv g'$ identically on S. The proof is complete. $\square$

We finally mention that in order to abbreviate some expressions, we shall often use the notation $\tilde{x}$ for an n-tuple $(x_1, \ldots, x_n)$, and $f(\tilde{x})$ for $f(x_1, \ldots, x_n)$ in the next section. Further, $\tilde{x}^\circ$ and $\tilde{x}^+$ will denote $(x_1^\circ, \ldots, x_n^\circ)$ and $(x_1^+, \ldots, x_n^+)$, respectively, $(\tilde{x} \vee k) \wedge l$ will abbreviate $((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l)$, etc.

3. Affine completeness.

We start with a canonical form of any polynomial function on a double MS-algebra.

3.1 Lemma. *Any polynomial function $f(x_1, \ldots, x_n)$ on a double MS-algebra L can be represented in the form*

$$f(x_1, \ldots, x_n) = \bigvee_{\tilde{i}, \tilde{j} \in \{-2,-1,0,1,2,3\}^n,\ \tilde{i} < \tilde{j}} (\alpha(i_1, j_1, \ldots, i_n, j_n) \wedge x_1^{i_1} \wedge x_1^{j_1} \wedge \cdots \wedge x_n^{i_n} \wedge x_n^{j_n})$$

and dually, in the form

$$f(x_1, \ldots, x_n) = \bigwedge_{\tilde{i}, \tilde{j} \in \{-2,-1,0,1,2,3\}^n,\ \tilde{i} < \tilde{j}} (\beta(i_1, j_1, \ldots, i_n, j_n) \vee x_1^{i_1} \vee x_1^{j_1} \vee \cdots \vee x_n^{i_n} \vee x_n^{j_n})$$

where $\bigvee$ and $\bigwedge$ are taken over all vectors $\tilde{i} = (i_1, \ldots, i_n)$, $\tilde{j} = (j_1, \ldots, j_n) \in \{-2, -1, 0, 1, 2, 3\}^n$, the coefficients $\alpha(i_1, j_1, \ldots, i_n, j_n), \beta(i_1, j_1, \ldots, i_n, j_n) \in L$ and $x^{-2}, x^{-1}, x^0, x^1, x^2$ and x^3 denote $x^{\circ\circ}, x^\circ, x, x^+, x^{++}$, and 1, respectively.

Proof. It follows from the following facts:

(i) for every $x \in L$ $\ x^{\circ+} = x^{\circ\circ}$, $x^{+\circ} = x^{++}$, $x^{\circ\circ\circ} = x^\circ$, $x^{+++} = x^+$ and $x^\circ \leq x^+$, $x^{++} \leq x \leq x^{\circ\circ}$;

(ii) for every $x, y \in L$ $\ (x \vee y)^\circ = x^\circ \wedge y^\circ$, $(x \wedge y)^\circ = x^\circ \vee y^\circ$ and $(x \wedge y)^+ = x^+ \vee y^+$, $(x \vee y)^+ = x^+ \wedge y^+$;

(iii) the lattice L is distributive. $\square$

3.2 Lemma. *Let L be a double K_2-algebra with a non-empty core $K(L)$. Then for any $x, y \in K(L)$ $\ x^\circ \leq y$ and $x^+ \geq y$.*

Proof. It follows from the facts that any element of $K(L)$ can be represented in the form $a \vee a^\circ$ as well as $b \wedge b^+$ for some $a, b \in L$, and that the identities (4),(5), (5^d) hold in L. $\square$

3.3 Theorem. *Let L be a double K_2-algebra with a non-empty bounded core $K(L)$. If L is (locally) affine complete then $K(L)$ is a (locally) affine complete distributive lattice.*

Proof. Let L be (locally) affine complete . Let f' be an n-ary (finite partial) compatible function of the lattice $K(L) = [k, l]$. Define a (finite partial) function $f : L^n \to L$ by

$$f(x_1, \ldots, x_n) = f'((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l).$$

Obviously, $f' = f \restriction K(L)^n$ and f is a (finite partial) compatible function of the algebra L (in local version we can always take $f \equiv f'$ to assure a finite domain of f). Indeed, if θ is a congruence of L and $x_i \equiv y_i\ (\theta)$, $i = 1, \ldots, n$, then $(x_i \vee k) \wedge l \equiv (y_i \vee k) \wedge l\ (\theta)$, thus we have (where f' is defined) $f'((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l) \equiv f'((y_1 \vee k) \wedge l, \ldots, (y_n \vee k) \wedge l)\ (\theta \restriction K(L))$

as f' is compatible on $K(L)$ (where defined), whence $f(x_1,\dots,x_n) \equiv f(y_1,\dots,y_n)$ (θ). Therefore by 3.1 we can write for $x_1,\dots,x_n \in K(L)$ (where f' is defined)

(a) $f'(\tilde{x}) = f(\tilde{x}) = \bigvee_{\tilde{i},\tilde{j}\in\{-2,-1,0,1,2,3\}^n,\ \tilde{i}<\tilde{j}} (\alpha(i_1,j_1,\dots,i_n,j_n) \wedge x_1^{i_1} \wedge x_1^{j_1} \wedge \cdots \wedge x_n^{i_n} \wedge x_n^{j_n})$.

By 3.2 and the fact that $f'(\tilde{x}) \in K(L)$, the terms x_i^+ can be omitted in (a). Further, the polynomial obtained can be, with use of the distributive laws and the relations $x^\circ \le x^+$ and $x^{++} \le x \le x^{\circ\circ}$, rewritten in the form

(b) $\bigwedge_{\tilde{i},\tilde{j}\in\{-2,-1,0,2,3\}^n,\ \tilde{i}<\tilde{j}} (\beta(i_1,j_1,\dots,i_n,j_n) \vee x_1^{i_1} \vee x_1^{j_1} \vee \cdots \vee x_n^{i_n} \vee x_n^{j_n})$.

Now again from 3.2 and $f'(\tilde{x}) \in K(L)$ it follows that the terms x_i° can be omitted in (b). So we get

(c) $f'(\tilde{x}) = \bigwedge_{\tilde{i},\tilde{j}\in\{-2,0,2,3\}^n,\ \tilde{i}<\tilde{j}} (\beta(i_1,j_1,\dots,i_n,j_n) \vee x_1^{i_1} \vee x_1^{j_1} \vee \cdots \vee x_n^{i_n} \vee x_n^{j_n})$.

Now we see that f' is order-preserving (where defined). The assertion follows from 2.2 (in local version from 2.10). □

3.4 Proposition. *If a double K_2-algebra L is (locally) affine complete then the Kleene algebra $S(L) = L^{\circ\circ}$ is (locally) affine complete, too.*

Proof. Let L be a (locally) affine complete double K_2-algebra. Let f' be an n-ary (finite partial) compatible function on $S(L)$. Define an n-ary (finite partial) function f on L by $f(x_1,\dots,x_n) = f'(x_1^{\circ\circ},\dots,x_n^{\circ\circ})$. Obviously, f is compatible since f' is compatible (where defined), so by hypothesis f can be represented (where defined) by a polynomial $p(x_1,\dots,x_n)$ of L. Hence for all $\tilde{x} = (x_1,\dots,x_n) \in (L^{\circ\circ})^n$ (where f' is defined) we have $f'(\tilde{x}) = f(\tilde{x}) = p(\tilde{x}) = p(\tilde{x})^{\circ\circ}$ as $f'(\tilde{x}) \in L^{\circ\circ}$. Clearly, in $p(x_1,\dots,x_n)^{\circ\circ}$ all constants are elements of $L^{\circ\circ}$, thus f' can be represented (where defined) by a polynomial of $S(L)$. □

3.5 Lemma. *Local affine completeness of the Kleene algebra $S(L)$ yields local affine completeness of the lattices $S(L)^\wedge$ and $S(L)^\vee$.*

Proof. We know that in a Kleene algebra $S(L)$, the ideal $S(L)^\wedge$ and the filter $S(L)^\vee$ are isomorphic. Let $f : F \subseteq (S(L)^\wedge)^n \to S(L)^\wedge$ be a finite partial compatible function of the lattice $S(L)^\wedge$. We claim that f also preserves the congruences of the Kleene algebra $S(L)$ where defined. Indeed, if θ is a congruence of $S(L)$ and $(x_1,\dots,x_n),(y_1,\dots,y_n) \in F$, $x_i \equiv y_i$ (θ), $i = 1,\dots,n$, then $f(x_1,\dots,x_n) \equiv f(y_1,\dots,y_n)$ $(\theta \restriction S(L)^\wedge)$ as f preserves the lattice congruence $\theta \restriction S(L)$. Now local affine completeness of $S(L)$ yields that for all $\tilde{x} = (x_1,\dots,x_n) \in F$, $f(\tilde{x})$ can be written as in (a) of the proof of 3.3. However, now in (a) we have only terms x_j and x_i^+ because $x^{\circ\circ} = x^{++} = x$ and $x^\circ = x^+$ hold in $S(L)$. Since $x_i \in S(L)^\wedge$ are of the form $x_i = a^+ \wedge a^{++}$, we have $x_i^+ = a^+ \vee a^{++} = a \vee a^+ \ge a^+ \wedge a^{++} = x_i$ by $(4^d),(5^d)$, consequently the terms x_i^+ can be omitted. Hence f is order-preserving (where defined). By 2.10, $S(L)^\wedge$ is locally affine complete. □

3.6 Corollary. *A double K_2-algebra with a non-empty bounded core $K(L)$ is locally affine complete if and only if it is a Post algebra of order 3.*

Proof. If $K(L)$ is locally affine complete then by 3.3 and 2.10, $|K(L)| = 1$. Further, by 3.4, 3.5 and 2.10 again, $|S(L)^\wedge| = |S(L)^\vee| = 1$. Hence for any $x \in L$, $0 = x^\circ \wedge x^{\circ\circ} = x \wedge x^\circ$ and

$1 = x^+ \vee x^{++} = x \vee x^+$. Hence L is a double Stone algebra ($S(L)$ is a Boolean algebra) and consequently L is a Post algebra of order 3. The converse follows from 2.12. $\square$

From now we shall deal with the affine completeness only.

3.7 Corollary. *Let L be a double K_2-algebra such that the ideal $L^\wedge = \{x \wedge x^\circ;\ x \in L\} \subseteq S(L)$ is finite. Then L is affine complete if and only if L is an affine complete double Stone algebra.*

Proof. If L is an affine complete double K_2-algebra then by 3.4 the Kleene algebra $S(L)$ is affine complete, too. Since $S(L)^\wedge \subseteq L^\wedge$ and $L^\wedge$ is finite, $S(L)$ is a Boolean algebra by 2.7. Therefore for any $x \in L$, $x^\circ \vee x^{\circ\circ} = 1$ and $x^+ \wedge x^{++} = 0$, hence (as in 3.6) $x \wedge x^\circ = x^\circ \wedge x^{\circ\circ} = 0$ and $x \vee x^+ = x^{++} \vee x^+ = 1$, i.e. L is a double Stone algebra. $\square$

3.8 Corollary. *A double K_2-algebra with a finite skeleton is affine complete if and only if it is an affine complete double Stone algebra .*

3.9 Corollary. *A double K_2-algebra L with a finite skeleton and a finite non-empty core (in particular, a finite double K_2-algebra) is affine complete if and only if L is a (finite) Post algebra of order 3.*

Proof. Let L be affine complete . By 3.8, L is affine complete double Stone algebra and by 3.3 and 2.9, $|K(L)| = 1$. Hence, L is a Post algebra of order 3. $\square$

Next, by L we always mean an (infinite) double K_2-algebra with a non-empty bounded core $K(L) = [k, l]$. Obviously, a mapping $\varphi : L \to K(L)$, $\varphi(x) = (x \vee k) \wedge l$ is a lattice homomorphism. We abbreviate by $\varphi(\tilde{x})$ the n-tuple $(\tilde{x} \vee k) \wedge l = ((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l)$.

3.10 Lemma. *Every element of L can be decomposed in the form*

(d) $\quad x = x^{++} \vee (x^{\circ\circ} \wedge (x \vee k) \wedge l)$.

Proof. We shall show that for any $a \in L$, $a = a^{\circ\circ} \wedge (a \vee k)$. By the distributivity of L, $a^{\circ\circ} \wedge (a \vee k) = a \vee (a^{\circ\circ} \wedge k)$. It suffices to show that $a^{\circ\circ} \wedge k = a \wedge k$. Suppose on the contrary that $a \wedge k < a^{\circ\circ} \wedge k$. Then for $x = a^{\circ\circ} \wedge k$, $y = a^\circ \wedge k$, $z = a \wedge k$ we have

$$x \wedge y = a^{\circ\circ} \wedge a^\circ \wedge k = a \wedge a^\circ \wedge k = a \wedge a^\circ = z \wedge y$$

as $k = c \vee c^\circ$ for some $c \in L$ and (4), (5) hold,

$$x \vee y = (a^{\circ\circ} \vee a^\circ) \wedge k = k = (a \vee a^\circ) \wedge k = z \vee y.$$

Hence, $\{a \wedge a^\circ,\ z,\ y,\ x,\ k\}$ is a five-element non-modular sublattice of L (pentagon), which contradicts to the distributivity of L.

Hence in L we have

$$x = x^{\circ\circ} \wedge (x \vee k)$$

and dually,

$$x = x^{++} \vee (x \wedge l).$$

These two equations imply (d). $\square$

We recall that $(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})$ $(x \in L^n)$ in the next definition is an abbreviation for the $5n$-tuple $((x_1 \vee k) \wedge l, \ldots, (x_n \vee k) \wedge l, (x_1^\circ \vee k) \wedge l, \ldots, (x_n^\circ \vee k) \wedge l, (x_1^{\circ\circ} \vee k) \wedge l, \ldots, (x_n^{\circ\circ} \vee k) \wedge l, (x_1^+ \vee k) \wedge l, \ldots, (x_n^+ \vee k) \wedge l, (x_1^{++} \vee k) \wedge l, \ldots, (x_n^{++} \vee k) \wedge l)$.

3.11 Definition. *We shall say that L satisfies an 'extension property'*

(EC) *if for any compatible function $f : L^n \to L$, the partial function $f'_K : K(L)^{5n} \to K(L)$ defined on the core such that for all $\tilde{x} \in L^n$*
$f'_K(\varphi(\tilde{x}), \varphi(\tilde{x}^{\circ}), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^{+}), \varphi(\tilde{x}^{++})) = \varphi(f(\tilde{x}))$
and f'_K is undefined elsewhere can be extended to a total compatible function of the lattice $K(L)$.

3.12 Lemma. *The partial function f'_K in the preceding definition is a (well-defined) partial compatible function of the lattice $K(L)$.*

Proof. We associate to any congruence θ_K of the lattice $K(L)$ an equivalence relation θ_L on L defined by the rule
(*e*) $x \equiv y\ (\theta_L)$ iff $\varphi(x^i) \equiv \varphi(y^i)\ (\theta_K)$ for all $i \in \{-2, -1, 0, 1, 2\}$,
where $x^0 = x,\ x^1 = x^+,\ x^2 = x^{++},\ x^{-1} = x^{\circ},\ x^{-2} = x^{\circ\circ}$. One can easily verify that θ_L is a congruence on L. Let $\varphi(x_i^j) \equiv \varphi(y_i^j)\ (\theta_K)$ for some elements $x_i, y_i \in K(L),\ i = 1, \ldots, n$ and all $j \in \{-2, -1, 0, 1, 2\}$. Then $x_i \equiv y_i\ (\theta_L)$, thus $f(\tilde{x}) \equiv f(\tilde{y})\ (\theta_L)$ as f is compatible on L. Now by (e) again $\varphi(f(\tilde{x})) \equiv \varphi\ (f(\tilde{y}))\ (\theta_K)$, i.e. f'_K preserves the congruences of $K(L)$ where defined. To show that f'_K is well-defined, it suffices to use $\theta_K = \triangle_{K(L)}$, the smallest congruence of $K(L)$. □

3.13 Proposition. *Let one of the following conditions hold in L:*
(i) L is affine complete;
(ii) $K(L)$ is simple (i.e. has only trivial congruences).
Then (EC) is fulfilled in L.

Proof. (i) For the function f'_K associated to a compatible function $f : L^n \to L$ we define a function $f_1 : L^n \to L$ by $f_1(\tilde{x}) = \varphi(f(\tilde{x}))$. This is compatible on L, hence it can be represented by a polynomial $p(x_1, \ldots, x_n)$ of L. Using the rules of computation for $^{\circ}$ and $^{+}$, $p(\tilde{x})$ can be rewritten as $l(\tilde{x}, \tilde{x}^{\circ}, \tilde{x}^{\circ\circ}, \tilde{x}^{+}, \tilde{x}^{++})$ for some lattice polynomial $l(x_1, \ldots, x_{5n})$, i.e. as a lattice polynomial in which terms $x_i, x_i^{\circ}, x_i^{\circ\circ}, x_i^{+}, x_i^{++}$ stand for variables. Further, using the homomorphism φ, one can show that for all $\tilde{x} \in L^n$
$f'_K(\varphi(\tilde{x}), \ldots, \varphi(\tilde{x}^{++})) = f_1(\tilde{x}) = p(\tilde{x}) = \varphi(p(\tilde{x})) = l'(\varphi(\tilde{x}), \ldots, \varphi(\tilde{x}^{++}))$,
where all constants in l' are of the form $(a \vee k) \wedge l$, i.e. $l'(x_1, \ldots, x_{5n})$ is a polynomial of the lattice $K(L)$. Now, of course, l' can be chosen as the required total compatible extension of the partial function f'_K.
(ii) The statement is obvious as any total extension of f'_K is compatible. □

3.14 Lemma. *Let $f : L^n \to L$ be a compatible function on L. Let $f_s^{\circ},\ f_s^{+} : S(L)^{2n} \to S(L)$ be partial functions such that for all $(x_1, \ldots, x_n) \in L^n$*
$f_s^{\circ}(x_1^{\circ}, \ldots, x_n^{\circ}, x_1^{+}, \ldots, x_n^{+}) = f(x_1, \ldots, x_n)^{\circ\circ}$,
$f_s^{+}(x_1^{\circ}, \ldots, x_n^{\circ}, x_1^{+}, \ldots, x_n^{+}) = f(x_1, \ldots, x_n)^{++}$
and $f_s^{\circ},\ f_s^{+}$ are undefined elsewhere. Then $f_s^{\circ},\ f_s^{+}$ are well-defined and preserve the congruences of $S(L)$ where defined.

Proof. Obviously, $f_s^{\circ},\ f_s^{+}$ are well-defined since $x_i^{\circ} = y_i^{\circ},\ x_i^{+} = y_i^{+},\ i = 1, \ldots, n$ yield $x_i \equiv y_i(\Phi_{+}^{\circ})$ (the determination congruence), which follows $f(\tilde{x}) \equiv f(\tilde{y})(\Phi_{+}^{\circ})$, thus $f(\tilde{x})^{\circ\circ} =$

$f(\tilde{y})^{\circ\circ}$, $f(\tilde{x})^{++} = f(\tilde{y})^{++}$. Further, for any congruence $\theta_{S(L)}$ of $S(L)$ we define an equivalence relation θ_L on L by $x \equiv y\ (\theta_L)$ iff $x^\circ \equiv y^\circ\ (\theta_{S(L)})$ and $x^+ \equiv y^+\ (\theta_{S(L)})$. Since $S(L)$ is a subalgebra of L, θ_L is obviously a congruence of L containing $\theta_{S(L)}$. Similarly as in Lemma 3.12 one can now easily show that $f_s^\circ,\ f_s^+$ preserve the congruences of $S(L)$ where defined. □

3.15 Definition. *We shall say that L satisfies an 'extension property'*

(ES) *if for any compatible function $f : L^n \to L$, the partial functions $f_s^\circ,\ f_s^+ : S(L)^{2n} \to S(L)$ defined as in 3.14 can be extended to total compatible functions of the Kleene algebra S(L).*

3.16 Proposition. *The extension property (ES) concerning the skeleton is fulfilled in L whenever one of the following conditions holds:*
(i) L is affine complete ;
(ii) $S(L)$ is a Boolean algebra;
(iii) $S(L)$ is simple.

Proof. (i) Let L be affine complete and $f : L^n \to L$ be a compatible function on L. We proceed similarly as in 3.13. Concerning the function f_s° associated to f, we define a function $f_1 : L^n \to S(L)$ by $f_1(\tilde{x}) = f(\tilde{x})^{\circ\circ}$. Clearly, f_1 is compatible on L, thus it can be represented by a polynomial $p(x_1, \ldots, x_n)$ of the algebra L. Since for any $\tilde{x} \in L^n$, $f_1(\tilde{x}) \in L^{\circ\circ}$, we have $p(\tilde{x}) = p(\tilde{x})^{\circ\circ} = p(\tilde{x})^{++}$ and using the laws for $^\circ$ and $^+$, $p(\tilde{x})^{\circ\circ}$ can be rewritten as $k(x_1^\circ, \ldots, x_n^\circ, x_1^+, \ldots, x_n^+)$ for some polynomial $k(x_1, \ldots, x_{2n})$ of the Kleene algebra S(L). Hence for all $\tilde{x} \in L^n$

$$f_s^\circ(\tilde{x}^\circ, \tilde{x}^+) = f(\tilde{x})^{\circ\circ} = p(\tilde{x})^{\circ\circ} = k(\tilde{x}^\circ, \tilde{x}^+)$$

showing that $k(x_1, \ldots, x_{2n})$ can serve as the required total compatible extension of the partial function f_s°. The case of f_s^+ is analogical.

(ii) If $S(L) = L^{\circ\circ} = L^{++}$ is a Boolean algebra, then for any $x \in L$, $0 = x^{\circ\circ} \wedge x^\circ = x \wedge x^\circ$ by (4), and dually, $1 = x^{++} \vee x^+ = x \vee x^+$ by (4^d). Thus L is a double Stone algebra. Let S be the domain of f_s°, i.e.

$$S = \{(\tilde{x}^\circ, \tilde{x}^+);\ \tilde{x} \in L^n\} \subset S(L)^{2n}.$$

One can easily verify that the function f_s° can be interpolated on the set $S \cap \{0,1\}^{2n}$ by a Boolean polynomial function $b : S(L)^{2n} \to S(L)$ defined as follows:

$$b(x_1, \ldots, x_{2n}) = \bigvee_{(\tilde{a},\tilde{b}) \in S \cap \{0,1\}^{2n}} (f_s^\circ(\tilde{a}, \tilde{b}) \wedge x_1^{a_1} \wedge \cdots \wedge x_n^{a_n} \wedge x_{n+1}^{b_1} \wedge \cdots \wedge x_{2n}^{b_n})$$

where $x_i^0 = x_i,\ x_i^1 = x_i^+ = x_i^\circ = x_i'$.

We shall verify that the assumptions of Lemma 2.13 are satisfied for the Boolean algebra $S(L) = (S(L), \vee, \wedge, ', 0, 1)$ and the functions f_s° and b. It is easy to see that for $x_i = 0$, $(x_i^\circ, x_i^+) = (1,1)$ and for $x_i = 1$, $(x_i^\circ, x_i^+) = (0,0)$. Hence $S \cap \{0,1\}^{2n} \neq \varnothing$ and we claim that for any Boolean homomorphism $h : S(L) \to \{0,1\}$, $h(S) = h(S \cap \{0,1\}^{2n})$.

If for $(\tilde{x}^\circ, \tilde{x}^+) \in S$ we have $h(x_i^\circ) = 1$, then also $h(x_i^+) = h(x_i^\circ \vee x_i^+) = h(x_i^\circ) \vee h(x_i^+) = 1$. In this case $(h(x_i^\circ), h(x_i^+)) = (1,1) = (h(1), h(1))$. If $(h(x_i^\circ), h(x_i^+)) = (0,0)$ then trivially $(h(1^\circ), h(1^+)) = (h(0), h(0)) = (0,0) = (h(x_i^\circ), h(x_i^+))$. The remaining case is $(h(x_i^\circ), h(x_i^+)) = (0,1)$. Since we assume that L has a non-empty core $K(L) = [k, l]$, we have $(h(k^\circ), h(k^+)) = (h(0), h(1)) = (0,1) = (h(x_i^\circ), h(x_i^+))$. We have showed that $h(S) = h(S \cap \{0,1\}^{2n})$.

Applying Lemma 2.13 we get that $f_s^\circ \equiv b$ identically on the whole set S, hence the polynomial function b is the required compatible extension of f_s° on the skeleton $S(L)$. For f_s^+ one can proceed in the same way.

(iii) If $S(L)$ is simple, then both total extensions of $f_s^\circ,\ f_s^+$ are compatible. $\square$

3.17 Theorem. *Let L be an (infinite) double K_2-algebra with a non-empty bounded core $K(L)$. The following conditions are equivalent.*
(1) L is affine complete ;
(2) (i) $K(L)$ is an affine complete distributive lattice and
(ii) $S(L)$ is an affine complete Kleene algebra (cf. [11]*) and*
(iii) (EC) and
(iv) (ES).

Proof. The necessity follows from Theorem 3.3 and Propositions 3.4, 3.13 and 3.16. To prove the converse, let $f : L^n \to L$ be a compatible function on L. By Lemma 3.10 we can write

(f) $\quad f(\tilde{x}) = f(\tilde{x})^{++} \vee (f(\tilde{x})^{\circ\circ} \wedge (f(\tilde{x}) \vee k) \wedge l)$ for all $\tilde{x} \in L^n$.

To replace $(f(\tilde{x}) \vee k) \wedge l,\ f(\tilde{x})^{\circ\circ}$ and $f(\tilde{x})^{++}$ in (f) by polynomials of L, we take the partial functions f'_K and $f_s^\circ,\ f_s^+$ associated to f as in 3.11 and 3.14. By (EC) and (ES), these partial compatible functions can be extended to total compatible functions $f_1(x_1, \ldots, x_{5n})$ and $f_2(x_1, \ldots, x_{2n}),\ f_3(x_1, \ldots, x_{2n})$ of $K(L)$ and S(L), respectively, which by hypothesis can be represented by polynomials $p_1(x_1, \ldots, x_{5n})$ and $p_2(x_1, \ldots, x_{2n}),\ p_3(x_1, \ldots, x_{2n})$ of $K(L)$ and S(L), respectively. Therefore in (f),

$f(\tilde{x}) = p_3(\tilde{x}^\circ, \tilde{x}^+) \vee [p_2(\tilde{x}^\circ, \tilde{x}^+) \wedge p_1(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \ldots, \varphi(\tilde{x}^{++})]$

for all $\tilde{x} \in L^n$, thus f can be represented by a polynomial of the algebra L. $\square$

Now we derive the Beazer characterization of affine complete double Stone algebras with a non-empty bounded core (Proposition 2.3). The equivalence of (2) and (3) in 2.3 is known from 2.2, thus we show the equivalence of (1) and (2).

(1) $\Longrightarrow$ (2) of 2.3 is immediate from 3.3. Now let in a double Stone algebra L the core $K(L) = [k, l]$ is affine complete, i.e. $K(L)$ does not contain a proper Boolean interval. Since the skeleton $S(L)$ is a Boolean algebra, by 3.16 and 3.17 it only remains to show (EC). We first prove the following two lemmas.

3.18 Lemma. *Let L be a double Stone algebra with a non-empty bounded core $K(L) = [k, l]$ and $x, y \in L$. Then*

$\varphi(x^\circ) = \varphi(y^\circ)$ *iff* $\varphi(x^{\circ\circ}) = \varphi(y^{\circ\circ})$ *and*
$\varphi(x^+) = \varphi(y^+)$ *iff* $\varphi(x^{++}) = \varphi(y^{++})$.

Proof. Let $\varphi(x^+) = \varphi(y^+)$. The identities $x^+ \wedge x^{++} = 0$ and $x^+ \vee x^{++} = 1$ imply $\varphi(x^+) \wedge \varphi(x^{++}) = k,\ \varphi(x^+) \vee \varphi(x^{++}) = l$ for any $x \in L$. Hence

$\varphi(x^{++}) = (\varphi(y^+) \wedge \varphi(y^{++})) \vee \varphi(x^{++}) = (\varphi(x^+) \vee \varphi(x^{++})) \wedge (\varphi(y^{++}) \vee \varphi(x^{++})) = \varphi(y^{++}) \vee \varphi(x^{++})$.

In the same way one can show $\varphi(y^{++}) = \varphi(y^{++}) \vee \varphi(x^{++})$. The converse statement as well as the proof of the first one are analogical. $\square$

3.19 Lemma. *Let L be a double Stone algebra with a non-empty bounded core $K(L) = [k,l]$, $k < l$ and let $x \in L$ such that $\varphi(x^\circ), \varphi(x^{\circ\circ}), \varphi(x^+), \varphi(x^{++}) \in \{k,l\}$. Then $\varphi(x^\circ) = l$ implies $\varphi(x^{\circ\circ}) = k$ and analogously, $\varphi(x^+) = l$ implies $\varphi(x^{++}) = k$.*

Proof. Let $\varphi(x^\circ) = l$. It is obvious that $\varphi(x^{\circ\circ}) = l$ would yield $l = \varphi(x^\circ) \wedge \varphi(x^{\circ\circ}) = \varphi(0) = k$, a contradiction. Analogously, if $\varphi(x^+) = l = \varphi(x^{++})$, then $l = \varphi(x^+) \wedge \varphi(x^{++}) = \varphi(0) = k$, a contradiction, using the identity $x^+ \wedge x^{++} = 0$. $\square$

Now we are ready to prove the final result.

3.20 Proposition. *Let L be a double Stone algebra with a non-empty bounded core $K(L) = [k,l]$ such that $K(L)$ contains no proper Boolean interval. Then (EC) is fulfilled in L.*

Proof. If $k = l$, then L is a Post algebra of order 3 and trivially, (EC) is fulfilled in L. So we can further assume that $k < l$.

Let $f' = f'_K : K(L)^{5n} \to K(L)$ be the partial compatible function associated to a compatible function $f : L^n \to L$ as in 3.11. Let $S = \{(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++}));\ \tilde{x} \in L^n\}$ be the domain of f', $S \subset K(L)^{5n}$. We shall show that f' can be interpolated on the set $S \cap \{k,l\}^{5n}$ by the following polynomial of the lattice $K(L)$:

$$(g)\quad q(x_1, \dots, x_{5n}) = \bigvee_{\tilde{b} \in S \cap \{k,l\}^{5n}} (f'(b_1, \dots, b_{5n}) \wedge y_1 \wedge \dots \wedge y_{5n}),$$

where $y_i = \begin{cases} x_i, & \text{if } b_i = l \\ l, & \text{if } b_i = k. \end{cases}$

Let $\tilde{x}$ be any (fixed) vector from $S \cap \{k,l\}^{5n}$. If $\tilde{a} \neq \tilde{x}$ and $b_j \neq x_j$ for some j, $n < j \leq 5n$, then either $b_j = l$, $x_j = k$ and then $f'(\tilde{b}) \wedge y_1 \wedge \dots \wedge y_{5n} = k$ or $b_j = k$, $x_j = l$ and then by Lemmas 3.18, 3.19 there exists $s \in \{j-n, j+n\}$ such that $x_s = k, b_s = l$, thus again $f'(\tilde{b}) \wedge y_1 \wedge \dots \wedge y_{5n} = k$. Hence it suffices to take into account in (g) only conjunctions $f'(\tilde{b}) \wedge y_1 \wedge \dots \wedge y_{5n}$ such that $b_i = x_i$ for all i, $n < i \leq 5n$ and moreover, $b_i \leq x_i$ for all i, $1 \leq i \leq n$. So

$$q(x_1, \dots, x_{5n}) = \bigvee_{\tilde{b} \in S \cap \{k,l\}^{5n},\ \tilde{b} \leq \tilde{x}} (f'(b_1, \dots, b_n, x_{n+1}, \dots, x_{5n}).$$

Next, we show that $f'(\tilde{b}) \leq f'(\tilde{x})$ for any $\tilde{b} \in S \cap \{k,l\}^{5n}$ such that $b_i = x_i$ for $i = n+1, \dots, 5n$ and $b_i \leq x_i$ for $i = 1, \dots, n$. Denote $u_s = b_s$ if $b_s = x_s$, otherwise $u_s = u$, $1 \leq s \leq n$. We get a unary compatible function $g : K(L) \to K(L)$, $g(u) = f'(u_1, \dots, u_n, x_{n+1}, \dots, x_{5n})$ and we have to show that $g(k) \leq g(l)$. Since $g(k) \equiv g(u)\ (\theta_{\text{lat}}(k,u))$ and $g(u) \equiv g(l)\ (\theta_{\text{lat}}(u,l))$ for any $u \in K(L)$ ($\theta_{\text{lat}}(k,u)$ and $\theta_{\text{lat}}(u,l)$ denote the principal lattice congruences generated by the pairs (k,u) and (u,l), respectively), we get

$$g(u) \vee u = g(k) \vee u \text{ and}$$
$$g(u) \wedge u = g(l) \wedge u.$$

This means that for any $u \in [g(l), g(k) \vee g(l)]$, $g(u)$ is the relative complement of u in this interval, which is therefore Boolean. By the assumption of Proposition 3.20 this implies $g(k) \leq g(l)$, what was to be proved. Hence

$$q(x_1, \dots, x_{5n}) = f'(x_1, \dots, x_{5n}) \quad \text{for any } \tilde{x} \in S \cap \{k,l\}^{5n}.$$

We shall show that the assumptions of Lemma 2.13 are satisfied for the lattice $K(L)$ and the functions f' and q. If in the $5n$-tuple $(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})) \in S$ we take $x_i = 0$ then $(\varphi(x_i), \varphi(x_i^\circ), \varphi(x_i^{\circ\circ}), \varphi(x_i^+), \varphi(x_i^{++})) = (k, l, k, l, k)$ and if $x_i = 1$ then $(\varphi(x_i), \varphi(x_i^\circ), \varphi(x_i^{\circ\circ}), \varphi(x_i^+), \varphi(x_i^{++})) = (l, k, l, k, l)$. Hence $S \cap \{k, l\}^{5n} \neq \varnothing$ and we claim that $h(S) = h(S \cap \{k, l\}^{5n})$ for any $(0,1)$-lattice homomorphism $h : K(L) \to \{k, l\}^{5n}$.

Let $(\varphi(\tilde{x}), \varphi(\tilde{x}^\circ), \varphi(\tilde{x}^{\circ\circ}), \varphi(\tilde{x}^+), \varphi(\tilde{x}^{++})) \in S$ and $i \in \{1, \dots, n\}$. Since h and φ are lattice homomorphisms and L is a double Stone algebra, we have

$$h(\varphi(x_i^\circ)) \vee h(\varphi(x_i^{\circ\circ})) = h(\varphi(x_i^\circ \vee x_i^{\circ\circ})) = h(\varphi(1)) = h(l) = l,$$
$$h(\varphi(x_i^\circ)) \wedge h(\varphi(x_i^{\circ\circ})) = h(\varphi(x_i^\circ \wedge x_i^{\circ\circ})) = h(\varphi(0)) = h(k) = k$$

and analogously,

$$h(\varphi(x_i^+)) \vee h(\varphi(x_i^{++})) = h(\varphi(x_i^+ \vee x_i^{++})) = h(\varphi(1)) = h(l) = l,$$
$$h(\varphi(x_i^+)) \wedge h(\varphi(x_i^{++})) = h(\varphi(x_i^+ \wedge x_i^{++})) = h(\varphi(0)) = h(k) = k.$$

Hence

$$h(\varphi(x_i^\circ)) = k \text{ if and only if } h(\varphi(x_i^{\circ\circ})) = l \text{ and}$$
$$h(\varphi(x_i^+)) = k \text{ if and only if } h(\varphi(x_i^{++})) = l.$$

This yields that each 5-tuple $(h(\varphi(x_i)), h(\varphi(x_i^\circ)), h(\varphi(x_i^{\circ\circ})), h(\varphi(x_i^+)), h(\varphi(x_i^{++})))$ can only be one of the 5-tuples (k, l, k, l, k), (k, k, l, k, l), (l, k, l, k, l) or (l, l, k, l, k). Since moreover,

$$h(\varphi(x_i)) \wedge h(\varphi(x_i^\circ)) = h(\varphi(x_i \wedge x_i^\circ)) = h(\varphi(0)) = h(k) = k,$$

the last 5-tuple (l, l, k, l, k) is impossible. Hence we get

$$\{(h(\varphi(x_i)), h(\varphi(x_i^\circ)), h(\varphi(x_i^{\circ\circ})), h(\varphi(x_i^+)), h(\varphi(x_i^{++}))); \ x_i \in L\} =$$
$$\{(k, l, k, l, k), (k, k, l, k, l), (l, k, l, k, l)\} =$$
$$\{(h(\varphi(x_i)), h(\varphi(x_i^\circ)), h(\varphi(x_i^{\circ\circ})), h(\varphi(x_i^+)), h(\varphi(x_i^{++}))); \ x_i \in \{0, k, 1\} \}.$$

Note that for $x_i \in \{0, k, 1\}$ we have $(\varphi(x_i), \varphi(x_i^\circ), \varphi(x_i^{\circ\circ}), \varphi(x_i^+), \varphi(x_i^{++})) \in \{k, l\}^5$. Consequently, $h(S) = h(S \cap \{k, l\}^{5n})$.

By Lemma 2.13, $f' \equiv q$ on S, thus $q(x_1, \dots, x_{5n})$ is the required total compatible extension of the partial function f'. $\square$

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Dept. of Mathematics
Matej Bel University
Zvolenská 6
974 01 Banská Bystrica
SLOVAKIA

E-mail address: mhaviar@pdf.umb.sk